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On isohedral tilings for hyperbolic group of genus 2 by tiles with large number of edges

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Abstract

The paper studies isohedral tilings of the hyperbolic plane for the group of translations of genus 2 by 14-gons. It is shown how to enumerate all eligible cycles of valencies for polygon with large number of vertices.

Keywords: isohedral tilings, hyperbolic plane, group of translations, genus two.

On the Euclidean plane there are 2 types as well as 2 Delone classes of isohedral tilings by disks for the group of translations $p1$. Those disks are parallelograms and center-symmetric hexagons (if convex). On the hyperbolic plane there are a countable series of discrete groups of translations with compact fundamental domain, each group of the series is characterized by its genus. The simplest hyperbolic group of translations has genus 2, its Conway's orbifold symbol is $\circ\circ$. So I study isohedral tilings for the hyperbolic translation group of genus 2.

Definition. *Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group with a bounded fundamental domain. The tiling W is called isohedral with respect to the group G if G acts transitively on the set of all disks of the tiling.*

Isohedral tilings of the hyperbolic plane are enumerated up to Delone classes. A Delone class takes in consideration a tiling W, a group G and the action of the group G on the tiles of W (see [1]).

Any translation group admits only fundamental Delone classes of tilings, i. e. tilings of the plane with fundamental domains. To find isohedral tilings for the hyperbolic translation group of genus 2, I have

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solved some Diophantine equations in integers, then use B. Delone's adjacency diagrams and adjacency symbols [2] (see also [1]).

In the present communication I would like to dwell upon some new difficulties that arise when dealing with polygons with a large number of vertices.

The solutions to Diophantine equatons give us $k = 8$, 10, 12, 14, 16, and 18 as the numbers of vertices of a polygon. For $k = 14$ there are two sets of vertex valencies: the set A contains nine 3 and five 5, the set B contains six 3 and eight 4. From the sets of valencies one should form all possible ordered cycles of valencies up to equivalence.

Choose the set B : 33333344444444 and proceed to form possible classes of ordered cycles. In order to avoid superfluous examination we take in consideration that edges of a 14-gon must fall into pairs. The edges of a pair are mapped one into another by translation, so in a pair two edges have ends of the same valencies, in appropriate order. Thus we enumerate only eligible cycles of valencies.

For valencies 3 take into account one more forbidden situation: neighboring vertices of valency 3 cannot be sent one into another. Now examine possible groups of consecutive numbers of valencies from six 3 and eight 4. In an eligible cycle the group 333333 is not admissible because of yielding 5 edges 33 (both ends have valency 3). The arrangement of 33333 and a separate 3 as well as the arrangement of 3333 and 33 (both giving 4 edges 33) is not admissible because of the above mentioned forbidden situation. The arragement of 3333, separate 3, and 3 is not admissible because of yielding 3 edges 33. The arrangement of 333 and 333 yields 4 edges 33 and leads to some eligible ordered cycles of valencies:
$B1$: 33343334444444, $B2$: 33344333444444, $B3$: 33344433344444, $B4$: 33344443334444.

The arrangement of 333, 33, and separate 3 yields 3 edges 33 and is not admissible. The arrangement of 333, seperate 3, 3, and 3 yields 2 neighboring edges 33 and is not admissible. The arrangement of 33, 33, separate 3, and 3 yields 2 edges 33 and can be used in different combinations with eight vertices of valency 4. Now examine different

ways to divide 44444444 into four groups. First verify that the arrangement of 44444, separate 4, 4, and 4 is not admissible because, for 4 consecutive edges 44, no pairing allows to go round a vertex of valency 4. The arrangement of 4444, 44, separate 4, and 4 yields 4 edges 44 and leads to the following eligible ordered cycles of valencies:
$B5$: 33433434434444, $B6$: 33433443434444, $B7$: 33434334434444, $B8$: 33434433434444, $B9$: 33443343434444, $B10$: 33443433434444, $B11$: 33443343444434, $B12$: 33443433444434, $B13$: 33434344334444, $B14$: 33434434334444, $B15$: 33443434334444, $B16$: 33443434444334, $B17$: 33433434444344, $B18$: 33434334444344, $B19$: 33434344443344, $B20$: 33433443444434, $B21$: 33434433444434, $B22$: 33434434444334, $B23$: 33433444434344, $B24$: 33434444334344, $B25$: 33434444343344, $B26$: 33433444434434, $B27$: 33434444334434, $B28$: 33434444344334, $B29$: 33443344443434, $B30$: 33443444433434, $B31$: 33443444434334, $B32$: 33444433434344, $B33$: 33444434334344, $B34$: 33444434343344, $B35$: 33444433434434, $B36$: 33444434334434, $B37$: 33444434344334, $B38$: 33444433443434, $B39$: 33444434433434, $B40$: 33444434434334.

The combinations of the arrangement of 444, 444, separate 4, and 4 with the arrangement of 33, 33, separate 3, and 3 yielding 2 edges 33, 8 edges 34 and 4 edges 44 lead to the following eligible ordered cycles of valencies:
$B41$: 33433434443444, $B42$: 33434334443444, $B43$: 33434344433444, $B44$: 33433444343444, $B45$: 33434443343444, $B46$: 33434443433444, $B47$: 33433444344434, $B48$: 33434443344434, $B49$: 33434443444334, $B50$: 33444334343444, $B51$: 33444343343444, $B52$: 33444343433444, $B53$: 33444334344434, $B54$: 33444343344434, $B55$: 33444343444334, $B56$: 33444334443434, $B57$: 33444344433434, $B58$: 33444344434334.

The combinations of the arrangement of 44, 44, 44, and 44 with the arrangement of 33, 33, separate 3, and 3 leads to the following eligible ordered cycles of valencies:
$B59$: 33443344344344, $B60$: 33443443344344.

If all six vertices of valency 3 are separate, then eight vertices of valency 4 should be divided into six groups. The arrangement of 44, 44, and all other separate 4 is not admissible because it yields 2 consecutive

edges 44. So the arrangement of 44, 44, and all other separate 4 is the only admissible in this case, and it leads to the following eligible ordered cycles of valencies:
$B61 : 34343434344344$, $B62 : 34343434434344$, $B63 : 34343443434344$.

Thus all classes of eligible cycles of valencies are enumerated for the set $14B$. For each class of ordered cycles of valencies, we form all possible adjacency diagrams. Then to every adjacency diagram we apply the procedure of going round all the vertices of a tile and verify whether the condition is satisfied. Altogether for the list of $14B$, there are 36 Delone classes of isohedral tilings. Each Delone class is given by an adjacency symbol.

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