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Weakly Hyperbolic IFS's with Condensation of Special Type

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Abstract

We show that for any weakly hyperbolic Iterated Function System with condensation in $\mathbb{R}^n$, whose condensation set is a union of a finite collection of convex compact sets, there exists a standard weakly hyperbolic IFS with the same attractor.

Keywords: Iterated Function System, weak contraction, attractor, convex set.

1 Introduction

The term fractal is usually associated with the attractor of a hyperbolic Iterated Function System (IFS) (see, e.g., [1]).

M. Barnsley [1] has introduced the idea of an Iterated Function System with condensation (IFSC), which means a hyperbolic IFS, accompanied by a constant compact-valued multi-function (condensation). This idea has led to new fractals as attractors of IFS's. However, the computer simulations of such IFS's create more problems than for standard hyperbolic IFS's.

For these reasons, and not only, the authors formulated some questions related to these topics (see [2, 3, 4, 5]):

1. Which compacta can (or can not) serve as attractors of hyperbolic IFS's?

2. Under which conditions the attractor of a hyperbolic IFS with condensation can be represented as the attractor of a standard hyperbolic IFS?

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The second question is motivated, in particular, by the fact that for standard hyperbolic IFS's there are more tools to study and to justify various simulations by computer (e.g., the so-called "Chaos game") than for hyperbolic IFS's with condensation.

Moreover, the questions remain open for weakly hyperbolic IFS's as well.

One of the authors has partially answered the first question above for a special type of compact sets in the Euclidian space $\mathbb{R}^n$ (see [5]). He also answered the second question for the case when the condensation set is a compact convex set or a finite union of such sets in $\mathbb{R}^n$. Following these results, an algorithm to construct some type of plane fractals was proposed in [4, 6].

Here we purpose a generalization of these results to the case when the IFS consists of weakly contracting mappings and a condensation, whose condensation set is a compact convex set or a finite union of such sets. More precisely, we purpose a construction, which replaces the condensation mapping by a finite collection of contractions in such a way, that the attractor of the initial weakly hyperbolic IFS with condensation coincides with the attractor of the new (standard) weakly hyperbolic IFS.

2 Preliminaries

Let consider the Euclidean space $(\mathbb{R}^n, d)$.

Denote by $\mathcal{P}_{cp}(\mathbb{R}^n)$ the set of all nonempty compact subsets of $\mathbb{R}^n$, endowed with the Pompeiu-Hausdorff metric H.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [7, 8] if:

1) φ is monotonically increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

2) the iterations $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$.

It is known [8] that, if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.

Following [7, 8], we call a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ as a *weak contrac-*

tion, if there exists a comparison function φ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \qquad \forall\, x, y \in \mathbb{R}^n.$$

In this case, one says also that f is a *weak contraction with respect to the comparison function* φ or that it is a φ*-contraction* (see [8]).

Remark 1. For any finite collection of weak contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ there exists a common comparison function.

Recall that a *hyperbolic Iterated Function System (IFS)* is a finite collection of contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq m)$, and it is denoted by $\{\mathbb{R}^n; f_1, \ldots, f_m\}$.

Similarly, a *weakly hyperbolic Iterated Function System (wIFS)* $\{\mathbb{R}^n; f_1, \ldots, f_m\}$ is a finite collection of weak contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$.

Associated with a (weakly) hyperbolic Iterated Function System $\mathcal{F} = \{\mathbb{R}^n; f_1, \ldots, f_m\}$, there is the mapping $F_* : \mathcal{P}_{cp}(\mathbb{R}^n) \to \mathcal{P}_{cp}(\mathbb{R}^n)$, defined by the equality $F_*(C) = \bigcup_{i=1}^{m} \bigcup_{x \in C} f_i(x)$, $C \in \mathcal{P}_{cp}(\mathbb{R}^n)$, and reffered to as the *Barnsley-Hutchinson operator*.

Remark 2. A weakly hyperbolic Iterated Function System is a particular case of a weakly contracting relation (see [9]), so we can apply all respective results.

Remark 3. Any weakly hyperbolic Iterated Function System, as a weakly contracting relation, admits a comparison function (e.g., a common comparison function for all components of wIFS).

Theorem 1. *Let* $\{X; f_1, \ldots, f_m\}$ *be a weakly hyperbolic IFS with respect to a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*. Then the corresponding Barnsley-Hutchinson operator* F_* *is also a* φ*-contraction, i.e., for any* $A, B \in \mathcal{P}_{cp}(\mathbb{R}^n)$*, the following inequality holds*

$$H(F_*(A), F_*(B)) \leq \varphi(H(A, B)).$$

A nonempty compact set $A \subset \mathbb{R}^n$ is said to be *attractor* of the weakly hyperbolic IFS $\mathcal{F}$, if $F_*(A) = A$, where F_* is the corresponding Barnsley-Hutchinson operator.

Theorem 2. *For any weakly hyperbolic IFS* $\{\mathbb{R}^n; f_1, \ldots, f_m\}$*, there exists a unique compact set* A*, such that* $F_*(A) = A$*, where* F_* *is the corresponding Barnsley-Hutchinson operator.*

3 Sum of convex compacta as attractor of hyperbolic IFS

M. Barnsley (1988) introduced the concept of Iterated Function System with condensation.

A constant compact-valued function $f_0 : \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$, $f_0(x) \equiv K$ for some $K \in \mathcal{P}_{cp}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, is called a *condensation* with K as *condensation set.*

Theorem 3. *[5] Any convex compact set in* $\mathbb{R}^n$ *can be represented as the attractor of a hyperbolic IFS, consisting only of contractions.*

Theorem 4. *[5] Given a finite family of convex compacta* $\{K_1, \ldots, K_p\}$ *in* $\mathbb{R}^n$*, let* $\mathcal{F}_i = \{\mathbb{R}^n; \xi_{i1}, \ldots, \xi_{iq_i}\}$ *stand for the corresponding hyperbolic IFS, having* K_i *as attractor.*

Then $K = \bigcup_{i=1}^{p} K_i$ *is the attractor of the hyperbolic IFS*

$$\mathcal{G} = \{\mathbb{R}^n; \psi_{11}, \ldots, \psi_{1q_1}, \ldots, \psi_{p1}, \ldots, \psi_{pq_p}\},$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ *and* Pr_i *is the metric projection onto* K_i*.*

4 Attractors of weakly hyperbolic IFS with condensation

A *weakly hyperbolic Iterated Function System with condensation* (*wIFSC*) $\{X; f_0, f_1, \ldots, f_m\}$ consists of a condensation f_0 and of certain weak contractions $f_1, \ldots, f_m$.

The following result generalizes a similar one for hyperbolic IFS with condensation [5].

Theorem 5. *Let the compact set $K \subset \mathbb{R}^n$ be a union of convex compacta $K = \bigcup_{i=1}^{p} K_i$, and let $\mathcal{F}_i = \{\mathbb{R}^n; \xi_{i1}, \ldots, \xi_{iq_i}\}$ stand for the corresponding hyperbolic IFS, having K_i as attractor.*

Let $\mathcal{G} = \{\mathbb{R}^n; f_0, f_1, \ldots, f_m\}$ be a weakly hyperbolic IFS with condensation, having K as the condensation set.

Then the standard weakly hyperbolic IFS

$$\mathcal{T} = \{\mathbb{R}^n; \psi_{11}, \ldots, \psi_{1q_1}, \ldots, \psi_{p1}, \ldots, \psi_{pq_p}, f_1, \ldots, f_m\},$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ and Pr_i is the metric projection onto K_i, has the same attractor as the given weakly hyperbolic IFS with condensation $\mathcal{G}$.

5 Conclusion

These results show that in some conditions weakly hyperbolic IFS's with condensation can be studied by reducing them to standard weakly hyperbolic IFS's.

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