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On a generalization of the inner mapping group

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Abstract

We consider the group $GM(Q,\cdot)$, generated by all left, right and middle translations of a loop $(Q,\cdot)$. The generalized inner mapping group J consists of all mappings $\alpha \in GM(Q,\cdot)$, such that $\alpha(e) = e$, where e is the unit of $(Q,\cdot)$. In the present note we give a set of mappings which generates the group J .

Keywords: loop, generalized multiplication group, inner mapping, inner mapping group.

1 Introduction

Let $(Q,\cdot)$ be a quasigroup, $h \in Q$ and let $M(Q,\cdot) =< R_x, L_y | x, y \in Q >$ be the multiplication group of $(Q,\cdot)$, where $R_x(u) = u \cdot x, L_y(u) = y \cdot u, \forall u \in Q$. A mapping $\alpha \in M(Q,\cdot)$ is called an inner mapping, with respect to h, if $\alpha(h) = h$. The group I_h of all inner mappings, with respect to h, is called the inner mapping group of $(Q,\cdot)$, with respect to h. If $h_1, h_2 \in Q$, then the inner mapping groups I_{h_1} and I_{h_2} are isomorphic. It is well known the role of I_h in the study of normality: a subquasigroup H of a quasigroup $(Q,\cdot)$ is normal if and only if $I_h(H) = H$, where h is an arbitrary element of H. A set of generators for the inner mapping group I_h of a quasigroup was given by Belousov [1].

If $(Q,\cdot)$ is a loop and $h = e$ is its unit, then I_e is called the inner mapping group of $(Q,\cdot)$. The multiplication group and the inner mapping group of a loop are important tools when studying the properties and the structure of the loops. These tools were introduced by Bruck, who used them to investigate centrally nilpotent loops. The

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inner mapping group of loops is studied in a wide series of works, by Bruck, Baer, Garrison, Medoch, Niemenmaa, Kepka, Nagy, Drapal, Vojtechovsky, Csorgo, Gagola and others (see, for example, [1-3]).

In the present note we consider the group $GM(Q,\cdot)$, generated by all left, right and middle translations of a loop $(Q,\cdot)$, and give a set of mappings which generates the stabilizer of the neutral element of this loop in $GM(Q,\cdot)$.

2 The generalized multiplication group of a loop and the inner mapping group

A set of generators of the inner mapping group of a quasigroup (with respect to some element) was given by Belousov [1].

Theorem 1.[1] *If $(Q,\cdot)$ is a quasigroup and $h \in Q$, than the set $\{R_{x,y}, L_{x,y}, T_x | x,y \in Q\}$ generates the inner mapping group I_h, where $R_{x,y} = R^{-1}_{x*y} R_y R_x, L_{x,y} = L^{-1}_{x\circ y} L_x L_y, T_x = L^{-1}_{\sigma(x)} R_x$, $(*) = (\cdot)^{(L_h,\varepsilon,L_h)}$, $(\circ) = (\cdot)^{(\varepsilon,R_h,R_h)}$, $\sigma = R_h^{-1} L_h$.*

For a loop $(Q,\cdot)$, we consider the group:

$$GM(Q,\cdot) =< L_x, R_y, I_z | x,y,z \in Q >,$$

where $R_x(u) = u \cdot x$, $L_y(u) = y \cdot u$, $I_z : Q \mapsto Q, I_z(u) = u \setminus z$, $\forall u \in Q$, are the right, left and, respectively, middle translations of $(Q,\cdot)$.

Let e is the the unit of a loop $(Q,\cdot)$. Denote the stabilizer of e in $GM(Q,\cdot)$ by

$$J = \{\alpha \in GM(Q,\cdot) | \alpha(e) = e\}.$$

The group J is a generalization of the inner mapping group of $(Q,\cdot)$.

Theorem 2.[4] *If $(Q,\cdot)$ and $(Q,\circ)$ are two isostrophic loops then $GM(Q,\cdot) \cong GM(Q,\circ)$*

Let $(Q,\cdot)$ be a loop. Consider the mappings: $R_{x,y} = R^{-1}_{x\cdot y} R_y R_x$, $L_{x,y} = L^{-1}_{x\cdot y} L_x L_y$,$T_x = L_x^{-1} R_x$,$P_{x,y} = R_y^{-1} L_x I_y I_x$,$P'_{x,y} = L_x^{-1} R_y I_x^{-1} I_y^{-1}$, $V_x = I_x^{-1} R_x$, $U_x = I_x R_x$, for every $x,y \in Q$.

Lemma 1. *If $(Q,\cdot)$ is a loop, then the mappings $R_{x,y}, L_{x,y}$, T_x, $P_{x,y}, P'_{x,y}$, V_x, U_x belong to its generalized inner mapping group J.*

Proof. It is well known (and easy to verify) that $R_{x,y}(e) = e, L_{x,y}(e) = e$, and $T_x(e) = e$, where e is the unit of the loop $(Q, \cdot)$. For the remaining four mappings we have:

$V_x(e) = I_x^{-1}R_x(e) = I_x^{-1}(x) = x/x = e$; $U_x(e) = I_xR_x(e) = I_x(x) = x \setminus x = e, \forall x \in Q$;

$x = e\backslash x \Leftrightarrow x \setminus y = (e\backslash x) \setminus y \Leftrightarrow x \cdot [(e\backslash x) \setminus y] = y \Leftrightarrow L_xI_yI_x(e) = R_y(e) \Leftrightarrow R_y^{-1}L_xI_yI_x(e) = e \Leftrightarrow P_{x,y}(e) = e, \forall x, y \in Q$;

$y = y/e \Leftrightarrow x/y = x/(y/e) \Leftrightarrow [x/(y/e)] \cdot y = x \Leftrightarrow R_yI_x^{-1}I_y^{-1}(e) = L_x(e) \Leftrightarrow L_x^{-1}R_yI_x^{-1}I_y^{-1}(e) = e \Leftrightarrow P'_{x,y}(e) = e, \forall x, y \in Q$.

Theorem 3. *The generalized inner mapping group J of a loop $(Q, \cdot)$ is generated by the set of mappings $\{R_{x,y}, L_{x,y}, T_x, P_{x,y}, P'_{x,y}, V_x, U_x | x, y \in Q\}$.* *Proof.* Denote $F =< R_{x,y}, L_{x,y}, T_x, P_{x,y}, P'_{x,y}, V_x, U_x | x, y \in Q >$. According to Lemma 1, all generators of F belong to J, hence $F \subseteq J$.

Every mapping $\alpha \in J$ is a products of a finite number of translations (left, right, middle) or their inverses: $S_{a_1}^{\varepsilon_1}S_{a_2}^{\varepsilon_2} \dots S_{a_n}^{\varepsilon_n}$, where $a_i \in Q$, and $\varepsilon_i = 1$ or -1, $\forall i = 1, 2, \dots, n$. The number n is called the length of α. We may consider that $S_{a_n}^{\varepsilon_n} = R_x$, for some $x \in Q$, otherwise we may add the product $R_x^{-1}R_x$ to the right side of α. The proof of the inclusion $J \subseteq F$ is similar to that of Belousov for Theorem 1, and uses the mathematical induction by the length n of α.

If $n = 1$, then $\alpha = R_x$. So as $R_x(e) = e$, we get $x = e$ and, for example, $\alpha = L_e^{-1}R_e \in F$. Suppose that every mapping $\alpha \in J$, of length $\leq n$, belongs to F.

Let $\alpha \in J$ be a mapping of length n. We'll show that α may be represented in the form $\alpha = \alpha'\tau$, where α' is a mapping of length $n-1$ and $\tau \in F$. Then, using the mathematical induction, we have $\alpha' \in F$, hence $\alpha = \alpha'\tau \in F$. So as the last factor in the product equal to α is R_x, we have to consider six possible cases: 1. $\alpha = \alpha''R_yR_x$; 2. $\alpha = \alpha''L_yR_x$; 3. $\alpha = \alpha''R_y^{-1}R_x$; 4. $\alpha = \alpha''L_y^{-1}R_x$; 5. $\alpha = \alpha''I_yR_x$; 6. $\alpha = \alpha''I_y^{-1}R_x$, where $\alpha'' \in J$ is a mapping of length $n-2$.

1. $\alpha = \alpha''R_yR_x = \alpha''R_{x\cdot y}R_{x\cdot y}^{-1}R_yR_x = \alpha''R_{x\cdot y}R_{x,y} \in F$, so as $\alpha''R_{x\cdot y}$ is a mapping of length $n-1$ (we use the inductive assumption) and $R_{x,y} \in F$.

2. $\alpha = \alpha''L_yR_x = \alpha''L_yL_xL_x^{-1}R_x = \alpha''L_yL_xT_x = \alpha''L_{x\cdot y}L_{x\cdot y}^{-1}L_yL_xT_x$

$= \alpha'' L_{x\cdot y} L_{x,y} T_x \in F$, so as $\alpha'' L_{x\cdot y}$ is of length $n-1$ and $L_{x,y}T_x \in F$.

3. $\alpha = \alpha'' R_y^{-1} R_x = \alpha'' R_z R_z^{-1} R_y^{-1} R_x = \alpha'' R_z (R_x^{-1} R_y R_z)^{-1} = \alpha'' R_z R_{z,y}^{-1} \in F$, by the inductive assumption, where $x = z \cdot y$.

4. $\alpha = \alpha'' L_y^{-1} R_x = \alpha'' L_y^{-1} L_x L_x^{-1} R_x = \alpha'' L_y^{-1} L_x T_x = \alpha'' L_z L_z^{-1} L_y^{-1} L_x T_x = \alpha'' L_z (L_x^{-1} L_y L_z)^{-1} T_x = \alpha'' L_z (L_{y,z})^{-1} T_x \in F$, where $x = t \cdot z$.

5. $\alpha = \alpha'' I_y R_x = \alpha'' I_y I_x I_x^{-1} R_x = \alpha'' I_y I_x V_x = \alpha'' L_x^{-1} R_y P_{x,y} V_x = \alpha'' L_x^{-1} L_y L_y^{-1} R_y P_{x,y} V_x = \alpha'' L_x^{-1} L_y T_y P_{x,y} V_x = \alpha'' L_z L_z^{-1} L_x^{-1} L_y T_y P_{x,y} V_x = \alpha'' L_z (L_y^{-1} L_x L_z)^{-1} T_y P_{x,y} V_x = \alpha'' L_z L_{x,z}^{-1} T_y P_{x,y} V_x \in F$, where $y = x \cdot z$.

6. $\alpha = \alpha'' I_y^{-1} R_x = \alpha'' I_y^{-1} I_x^{-1} I_x R_x = \alpha'' I_y^{-1} I_x^{-1} U_x = \alpha'' R_x^{-1} L_y P'_{x,y} U_x = \alpha'' R_x^{-1} R_y R_y^{-1} L_y P'_{x,y} U_x = \alpha'' R_x^{-1} R_y T_y^{-1} P'_{x,y} U_x = \alpha'' R_z R_z^{-1} R_x^{-1} R_y T_y^{-1} P'_{x,y} U_x = \alpha'' R_z (R_y^{-1} R_x R_z)^{-1} T_y^{-1} P'_{x,y} U_x = \alpha'' R_z (R_{z,x})^{-1} T_y^{-1} P'_{x,y} U_x \in F$,

where $y = z \cdot x$, so $J \subseteq F$, which implies $J = F$.

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