About Multiple Prolongations of Latin Cubes

Fedir Sokhatsky, Daria Moroz

Abstract

We propose an algorithm of prolongation of a Latin cube with k new elements ($k \geq 1$).

Key words: Latin cube, prolongation of Latin cube, two-dimensional transversal, one-dimensional transversal, ternary quasigroup, prolongation of ternary quasigroup, multiple prolongation of Latin cube.

1 Introduction

Consider a cube in the coordinate system $Oxyz$. Each cell is denoted by (x, y, z, u), where (x, y, z) are coordinates of the cell and u is the element (the *forth coordinate*) located in this cell. So, the Latin cube C is denoted by

$$C := \{(x, y, z, u) \mid 0 \leqslant x, y, z, u < m\}. \tag{1}$$

The row plane, column plane, and string plane of the cube defined by an element a is denoted by the equations $x = a$, $y = a$, $z = a$ respectively. And the row, column and string is denoted by a pair of the planes, i.e., by their intersection:

$$y = a, z = b; \qquad x = a, z = b; \qquad x = a, y = b.$$

If any cell of a set $T \subset C$ is uniquely defined by: 1) an arbitrary pair of its coordinates, then T is called a *two-dimensional transversal*; 2) an arbitrary coordinate, then T is called the *one-dimensional transversal* of this Latin cube. The two-dimensional transversal of a Latin cube is a Latin square, and an arbitrary transversal of the square is the one-dimensional transversal of the Latin cube.

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2 Prolongation algorithm on k elements

The purpose of this algorithm is to prolong a Latin cube C of order m on k new elements that are denoted by u_0, ..., u_{k-1}. Let the cube has k two-dimensional transversals τ_i and so each of them has k one-dimensional transversals $\theta_{ij} \subset \tau_i$, where i, $j = 0$, ..., $k-1$.

Step 1. Add $3k$ new planes with empty cells to C:

$$x = m+i, \quad y = m+i, \quad z = m+i, \qquad i = 0, ..., k-1.$$

Step 2. We transfer all elements of two-dimensional transversals, except the elements of the one-dimensional transversals, to the corresponding added planes. Namely, let $i \in \{0, \dots, k-1\}$ and the cell (a, b, c, d) belong to the two-dimensional transversal τ_i, but it does not belong to all of the one-dimensional transversals $\theta_{i,j}$, $j = 0, \dots, k-1$. We transfer element d to cells $(m+i, b, c)$, $(a, m+i, c)$, $(a, b, m+i)$ and we put the new element u_i into the cell (a, b, c).

Step 3. Let the cell (a, b, c, d) belong to a one-dimensional transversal $\theta_{i,j}$. We copy the element d into the cells of the added planes and so d belongs to the following cells:

$$\begin{array}{ll} (a, b, c, d), & (a, m+i, m+j, d), \\ (m+i, b, m+j, d), & (m+i, m+j, c, d). \end{array}$$

We fill the cells $(m+i, b, c)$, $(a, m+i, c)$, $(a, b, m+i)$ with the new element u_j.

Step 4. $\{(m+i, m+j, m+s) \mid i, j, s = 0, \dots, k-1\}$ is an arbitrary Latin cube of the order k filled by the elements u_0, ..., u_{k-1}.

Step 5. The prolongation is over.

3 Example

We prolong the initial cube C on the set $\{0, 1, 2, 3, 4\}$ to the final cube D on the set $\{0, 1, 2, 3, 4, 5\}$ (Fig. 1).

The two-dimensional transversal τ (pink and red circles) and the one-dimensional transversal θ (red circles) are selected. The new element is denoted by 5 (light blue circles).

Step 1. Add three new planes $x = 5$, $y = 5$, $z = 5$ to the cube C.

Step 2. We transfer all elements of two-dimensional transversals τ, except the elements of the one-dimensional transversals θ, to the corresponding added planes. Namely, let $(a, b, c, d) \in \tau$, but $(a, b, c, d) \notin \theta$. We transfer element d to cells $(5, b, c)$, $(a, 5, c)$, $(a, b, 5)$ and we put the new element 5 into the cell (a, b, c):

$$(5, b, c, d), \quad (a, 5, c, d), \quad (a, b, 5, d), \quad (a, b, c, 5) \in D.$$

Step 3. Let $(a, b, c, d) \in \theta$. We copy the element d into the cells of the added planes:

$$(a, b, c, d), \quad (a, 5, 5, d), \quad (5, b, 5, d), \quad (5, 5, c, d) \in D.$$

The cells $(5, b, c)$, $(a, 5, c)$, $(a, b, 5)$ is filled with a new element 5:

$$(5, b, c, 5), \quad (a, 5, c, 5), \quad (a, b, 5, 5) \in D.$$

Step 4. Put a new element 5 in cell $(5, 5, 5)$: $(5, 5, 5, 5) \in D$.

Step 5. Prolongation is over. The cube D of order 6 has been formed.

4 Conclusion

In this work for the first time, the algorithm for prolonging Latin cubes of order m on k new elements has been found and it is illustrated when $m = 5$ and $k = 1$.

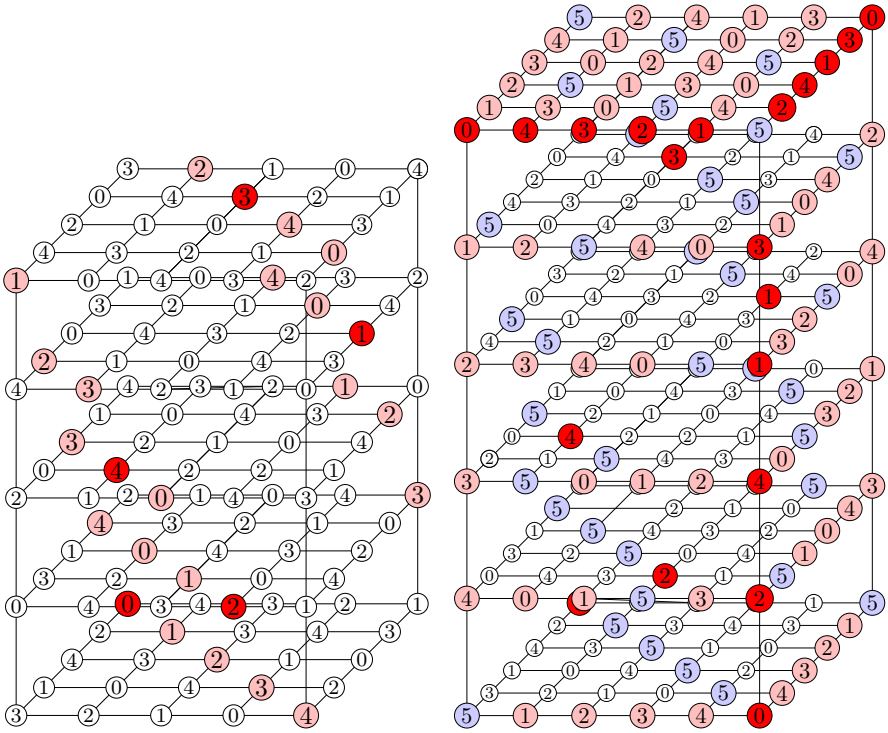


Figure 1. A prolongation of the cube C of order 5 to the cube D of the order 6

Fedir Sokhatsky[1], Daria Moroz[2]

[1]Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NANU, Lviv, Ukraine
E–mail: fmsokha@ukr.net
ORCID: https://orcid.org/0000-0003-4969-5651

[2]Vasyl' Stus Donetsk National University, Vinnytsia, Ukraine
E–mail: dashamoroz2303@gmail.com
ORCID: https://orcid.org/0009-0001-6911-3076