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Homogeneous Linear Recurrent Games

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Abstract

Games defined on deterministic systems with homogeneous linear recurrent dynamic are studied. It is proved that they represent homogeneous linear recurrent systems and an efficient method for determining the generating vector is proposed. This enables the usage of the advanced properties of homogeneous linear recurrences for their algebraic characterization.

Keywords: dynamical process, homogeneous linear recurrence, game.

1 Introduction and Problem Formulation

Let L be a discrete system with the set of states $V \subseteq \mathbb{C}$. At every discrete moment of time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The state $v(t)$ of the system for every t, $0 \leq t < m$, is known and, at every moment of time $t \geq m$, the dynamic of the system satisfies the relation $v(t) = \sum_{k=0}^{m-1} q_k v(t-1-k)$, where $q_k \in K$, $k = \overline{0, m-1}$, and $K \subseteq \mathbb{C}$. The system L represents a discrete system with homogeneous linear recurrent dynamic over K, generated by generating vector $q = (q_k)_{k=0}^{m-1}$ and initial state $I_m^{[v]} = (v(t))_{t=0}^{m-1}$. These homogeneous linear recurrent dynamics were deeply investigated in [2] and their advanced algebraic properties were studied in [1].

Next, in this paper, a variation is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his stationary strategy, represented by one generating vector $q^{(\ell)} \in K^{m_\ell}$ and one initial state $I_{m_\ell}^{[v]}$, $\ell = \overline{0, r-1}$.

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The game is started by first player $\mathcal{P}_0$. At every moment of time t, the system passes to the state $v(t)$ accordingly to the strategy of the current player $\mathcal{P}_{t \bmod r}$. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way.

Our goal is to prove that the defined game represents a homogeneous linear recurrent system and to provide an efficient approach for determining its generating vector, such that the previously developed methods for its algebraic characterization to be enabled for usage.

2 Homogeneous Linear Recurrences

As mentioned above, the sequence $a = \{a_n\}_{n=0}^{\infty}$ represents a homogeneous linear m-recurrence on the set K if $\exists q = (q_k)_{k=0}^{m-1} \in K^m$ such that $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$, $\forall n \geq m$, where q is the generating vector and $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ is the initial state of the sequence a. It is denoted by $Rol^*[K][m]$ the set of all homogeneous linear m-recurrences on the set K. The set $G^*[K][m](a)$ represents the set of generating vectors of length m of the sequence $a \in Rol^*[K][m]$.

The function $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ is the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$ and the function $G_t^{[a]}(z) = \sum_{n=0}^{t-1} a_n z^n$ is the partial generating function of order t of the sequence a. We consider the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$. For an arbitrary non-zero α, the polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ represents a characteristic polynomial of the sequence a of order m. We denote by $H^*[K][m](a)$ the set of characteristic polynomials of order m of the sequence $a \in Rol^*[K][m]$.

The sequence a is called m-minimal on the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, for all $t < m$. The number m is called the dimension of sequence a on the set K (denoted $dim[K](a) = m$).

Next we will recall several important theorems presented in [2].

Theorem 1. *If $a \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](a)$, then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Theorem 2. *Consider that $G^{[a]}(z) = \frac{A(z)}{B(z)}$ is a rational fraction with $deg(A(z)) = M$, $B(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k$ and $q_k \in K$, $k = \overline{0, m-1}$. Then $a \in Rol^*[K][t]$, $B(z) \in H^*[K][t](a)$, where $t = \max\{m, M+1\}$.*

Theorem 3. *If $a \in Rol^*[\mathbb{C}][m]$ is a sequence with at least one non-zero element, then $dim[\mathbb{C}](a) = R$ and $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, where $R = rank(A_m^{[a]})$, $A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}}$, $f_n^{[a]} = (a_k)_{k=\overline{n,2n-1}}$, $\forall n \geq 1$ and the vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ represents the unique solution of the system $A_R^{[a]} x^T = (f_R^{[a]})^T$.*

3 Homogeneous Linear Recurrent Games

We consider the game defined in Section 1. The dynamic generated by this game is $a = (a_n)_{n=0}^{\infty}$, where $a_n = v(n)$, $n = \overline{0, \infty}$.

The players $\mathcal{P}_0$, $\mathcal{P}_1$, ..., $\mathcal{P}_{r-1}$ act consecutively on the system evolution and the game continues in the loop after that. In this way, every player $\mathcal{P}_\ell$ directly contributes only at establishment of the states $a_{nr+\ell}$, $n = \overline{0, \infty}$, for each $\ell \in \{0, 1, \ldots, r-1\}$.

Let $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$. The contribution of each player $\mathcal{P}_\ell$ to this generating function is the function $g_\ell(z) = \sum_{n=0}^{\infty} a_{nr+\ell}\, z^{nr+\ell}$, $\ell = \overline{0, r-1}$. Based on these contributions, the generating function becomes $G^{[a]}(z) = \sum_{\ell=0}^{r-1} g_\ell(z)$.

The function $g_\ell(z)$ can be written in the following way:

$$g_\ell(z) = \sum_{n:\ rn+\ell<m_\ell} a_{rn+\ell}\ z^{rn+\ell} + \sum_{n:\ rn+\ell\geq m_\ell} z^{rn+\ell} \sum_{k=0}^{m_\ell-1} q_k^{(\ell)} a_{rn+\ell-k-1} =$$

$$= W^{(\ell)}_{m_\ell-1}(z) + \sum_{k=0}^{m_\ell-1} q_k^{(\ell)} z^{k+1} g_{s(\ell,k)}(z),$$

where $deg(W^{(\ell)}_{m_\ell-1}(z)) \leq m_\ell - 1$, $s(\ell,k) = (\ell-k-1)\ mod\ r$, $\ell = \overline{0,r-1}$.

These relations can be rewritten as $A(z)(g(z))^T = (W(z))^T$, where $A(z) = (a_{ij}(z))_{i,j=0}^{r-1}$, $g(z) = (g_j(z))_{j=0}^{r-1}$, $W(z) = (W^{(j)}_{m_j-1}(z))_{j=0}^{r-1}$ and $deg(a_{ij}(z)) \leq m_j$, $deg(W^{(j)}_{m_j-1}(z)) \leq m_j - 1$, $i,j = \overline{0,r-1}$. Using Cramer's rule, $g_\ell(z) = \dfrac{A_\ell(z)}{B(z)}$, where $deg(A_\ell(z)) < m$, $deg(B(z)) \leq m$ and $m = \sum\limits_{j=0}^{r-1} m_j$, $\ell = \overline{0,r-1}$. Calculating $G^{[a]}(z) = \sum\limits_{\ell=0}^{r-1} g_\ell(z)$, and applying Theorem 2, we obtain that $a \in Rol^*[\mathbb{C}][m]$. The minimal generating vector can be determined using Theorem 3.

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