# On Geodesic Points of 6-Dimensional Hermitian Submanifolds of Cayley Algebra

Mihail Banaru, Galina Banaru

**Abstract**

Six-dimensional Hermitian submanifolds of Cayley algebra defined by means of three-fold vector cross products are considered. It is proved that the Weyl tensor of a 6-dimensional Hermitian submanifold of Cayley algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.

**Keywords:** geodesic point, Hermitian manifold.

## 1 Introduction

The geometry of 6-dimensional almost Hermitian submanifolds of Cayley algebra has been developing since the 60s of the last century. Among the many famous mathematicians who worked in this field, we can especially highlight the outstanding American specialist Alfred Gray and the Russian geometer Vadim Feodorovich Kirichenko. V.F. Kirichenko, among other results, obtained a complete classification of 6-dimensional Kählerian submanifolds of the octave algebra [1]. Note that the survey [2] contains a significant part of the achievements in the field of geometry of 6-dimensional almost Hermitian submanifolds of octave algebra (naturally, without results obtained over the past 10 years).

In the present communication, we consider geodesic points of 6-dimensional Hermitian submanifolds of Cayley algebra, i.e. the points at which the configuration tensor vanishes. We prove that the Weyl tensor of conformal curvature of a 6-dimensional Hermitian submanifold of the octave algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.

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## 2 Preliminaries

Let us consider an almost Hermitian manifold, i.e. an even-dimensional manifold $M^{2n}$ with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure $J$. Moreover, the following condition must hold $\langle JX,\ JY \rangle = \langle X,\ Y \rangle, \quad X, Y \in \aleph(M^{2n})$, where $\aleph(M^{2n})$ is the module of smooth vector fields on $M^{2n}$.

We recall that the fundamental form of an almost Hermitian manifold is determined by the relation $F(X,\ Y) = \langle X,\ JY \rangle, \quad X, Y \in \aleph(M^{2n})$.

As it is known, an almost Hermitian manifold is Hermitian, if its almost complex structure is integrable [3].

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. It is known [4], two non-isomorphic three-fold vector cross products are defined on it by means of the relations:

$$P_1(X,\ Y,\ Z) = -X(\bar{Y}Z) + \langle X,\ Y \rangle Z + \langle Y,\ Z \rangle X - \langle Z, X \rangle Y,$$

$$P_2(X,\ Y,\ Z) = -(X\bar{Y})Z + \langle X,\ Y \rangle Z + \langle Y,\ Z \rangle X - \langle Z, X \rangle Y,$$

where $X,\ Y,\ Z \in \mathbf{O}$, $\langle \cdot,\ \cdot \rangle$ is the scalar product in $\mathbf{O}$ and $X \to \bar{X}$ is the conjugation operator. If $M^6 \subset \mathbf{O}$ is a 6-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_t, g = \langle \cdot,\ \cdot \rangle\}$ is determined by the relation $J_t(X) = P_t(X,\ e_1,\ e_2),\ t = 1,\ 2$, where $\{e_1,\ e_2\}$ is an arbitrary orthonormal basis of the normal space of $M^6$ at the point $p$, $X \in T_p(M^6)$ [1]. We recall that the point $p \in M^6$ is called general [4], [5] if $e_0 \notin T_p(M^6)$, where $e_0$ is the unit of Cayley algebra.

## 3 The main result

In the above mentioned paper [1], V.F. Kirichenko obtained the Cartan structural equations of a Hermitian structure on a 6-dimensional submanifold of the octave algebra:

$$d\omega^a = \omega_b^a \wedge \omega^b + \frac{1}{\sqrt{2}} \varepsilon^{abh} D_{hc} \omega^c \wedge \omega_b;$$

$$d\omega_a = -\omega_a^b \wedge \omega_b + \frac{1}{\sqrt{2}} \varepsilon_{abh} D^{hc} \omega_c \wedge \omega^b; \tag{1}$$

$$d\omega_b^a = \omega_c^a \wedge \omega_b^c - \left( \frac{1}{2} \delta_{bg}^{ah} D_{hd} D^{gc} + \sum_{\phi} T_{\hat{a}\hat{c}}^{\phi} T_{bd}^{\phi} \right) \omega_c \wedge \omega^d.$$

Here $\varepsilon_{abc} = \varepsilon_{abc}^{123}$, $\varepsilon^{abc} = \varepsilon_{123}^{abc}$ are the components of the 3-order Kronecher tensor; $D^{hc} = D_{\hat{h}\hat{c}}$, $D_{cj} = \mp T_{cj}^8 + iT_{cj}^7$, $D_{\hat{c}j} = \mp T_{\hat{c}j}^8 - iT_{\hat{c}j}^7$, where $\left\{ T_{kj}^{\phi} \right\}$ are the components of the configuration tensor (in the Gray–Kirichenko notation); $\phi = 7,\ 8;\ a,\ b,\ c,\ d,\ g,\ h = 1,\ 2,\ 3;$ $\hat{a} = a + 3;\ k,\ j = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$

A point $p \in M^6$ is called geodesic if the configuration tensor at this point vanishes.

Let us consider the Weyl tensor (also known as the tensor of conformal curvature) of a 6-dimensional Hermitian submanifold of Cayley algebra. This tensor is defined as follows [3]:

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2} \left( ric_{ik}\, g_{jl} + ric_{jl}\, g_{ik} - ric_{il}\, g_{jk} - ric_{jk}\, g_{il} \right) +$$

$$+ \frac{K}{(n-1)(n-2)} \left( g_{jk}\, g_{il} - g_{jl}\, g_{ik} \right).$$

Here $R$ is the tensor of Riemannian curvature, $ric$ is the Ricci tensor, $K$ is the scalar curvature and $n$ is the dimension of the manifold.

From Cartan structural equations (1) we can deduce all components of the tensor of Riemannian curvature that determine this tensor (taking into account its symmetry properties):

$$R_{abcd} = 0\ ; R_{\hat{a}bcd} = 0\ ; R_{\hat{a}\hat{b}cd} = 0\ ; R_{\hat{a}b\hat{c}d} = -\sum_{\phi} T_{\hat{a}\hat{c}}^{\phi}\, T_{bd}^{\phi}.$$

Similarly, we obtain the components of the Ricci tensor of 6-dimensional Hermitian submanifold of the octave algebra

$$ric_{ab} = 0; \quad ric_{\hat{a}b} = -\sum_{\phi} T_{\hat{a}\hat{c}}^{\phi}\, T_{cb}^{\phi}.$$

So, we can state our result

**Theorem 1.** *The Weyl tensor of a 6-dimensional Hermitian submanifold of Cayley algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.*

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