Darboux Integrability of a Cubic Differential System with One Invariant Straight Line and One Invariant Cubic

Dumitru Cozma, Angela Matei

Abstract

We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one invariant straight line and one invariant cubic. The presence of a center at $O(0,0)$ is proved by method of Darboux integrability.

Keywords: cubic differential system, invariant algebraic curve, Darboux integrability, the problem of the center.

1 Introduction

We consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -x - gx^2 - dxy - by^2 - sx^3 - qx^2y - nxy^2 - ly^3 \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$, $Q(x,y)$ are real and coprime polynomials in the variables x and y, $\dot{x} = dx/dt$, $\dot{y} = dy/dt$. The origin $O(0,0)$ is a singular point for (1) with purely imaginary eigenvalues, i.e., a focus or a center.

The problem of distinguishing between a center and a focus (the problem of the center) is open for general cubic differential system. It was solved for some families of cubic systems (1) with invariant algebraic curves: four invariant straight lines; three invariant straight lines [1]; two invariant straight lines and one invariant conic [1]; two invariant straight lines and one invariant cubic; two parallel invariant straight lines [2].

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2 First integrals and integrating factors

It is known [1] that a singular point $O(0,0)$ is a center for (1) if and only if the system has in some neighborhood U of $O(0,0)$ a nonconstant analytic first integral $F(x,y)=C$ or an analytic integrating factor

$$\mu(x,y)=1+\sum_{k=1}^{\infty}\mu_k(x,y),$$

where μ_k are homogeneous polynomials of degree k.

An integrating factor for system (1) on some open set U of $\mathbb{R}^2$ is a C^1 function μ defined on U, not identically zero on U such that

$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\equiv 0. \tag{2}$$

The relation $F(x,y)=C$ is a first integral of (1) on U if and only if

$$P(x,y)\frac{\partial F}{\partial x}+Q(x,y)\frac{\partial F}{\partial y}\equiv 0. \tag{3}$$

Conditions for the existence of integrating factors of the form $\mu=\Phi^{\beta}$, $\beta\in\mathbb{R}$, were obtained for system (1) when $\Phi=0$ is an invariant cubic $a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3+x^2+y^2=0$ and when $\Phi=0$ is an invariant conic $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0$ [3].

3 Darboux integrability

In this paper, we determine the center conditions for system (1) by constructing integrating factors and first integrals of the Darboux form

$$(1-x)^{\alpha_1}\Phi^{\alpha_2}, \tag{4}$$

where $x=1$ is an invariant straight line and $\Phi(x,y)=0$ is an irreducible invariant cubic for (1); α_1,α_2 are real exponents, $\alpha_1\alpha_2\neq 0$.

Identities (2) and (3) will be used in finding integrating factors and first integrals of the form (4) for system (1). Identifying the coefficients in (2) ((3)) and solving the obtained algebraic system, we prove

Theorem 1. *The cubic system (1) has an integrating factor of the form (4), where* $\Phi(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2$, *if and only if one of the following nine sets of conditions holds:*

(i) $a = [(2b+3c+3)^2+24fu-3u^2]/(8u)$, $d = [(2b+3c+3)^2-8fu+u^2]/(4u)$, $g = [(2b+3c+3)^3+(2b+3c+3)(8fu-u^2)-8u^2]/(8u^2)$, $k = -a$, $l = r = 0$, $m = -c-1$, $n = [(2b+3c+3)(u-12f)]/(4u)$, $p = -f$, $q = -d$, $s = [(2b+3c+3)(u^2-8fu-(2b+3c+3)^2)]/(8u^2)$, $4f(2b+3c+3) - u(2b+c+1) = 0$;

(ii) $a = 3f$, $c = (-2b-5)/3$, $d = [2f(2b+5)]/(1-2b)$, $g = (b-5)/3$, $k = -a$, $l = r = 0$, $m = (2(b+1))/3$, $n = -b$, $p = -f$, $q = -d$, $s = (2-b)/3$, $(2b-1)^2 - 108f^2 = 0$;

(iii) $b = 1/2$, $c = (-3)/2$, $d = (11a+21f)/3$, $g = -1$, $k = -a$, $l = f/2$, $m = 1/2$, $n = (27-84af-140a^2)/2$, $p = -f$, $q = (-2a-21f)/18$, $r = 0$, $s = (20a^2+12af+9)/36$, $(10a+6f)^2 - 27 = 0$;

(iv) $c = -1$, $d = -2a$, $f = -a$, $g = (2b-5)/2$, $k = -a$, $l = (-3a)/2$, $m = n = r = 0$, $p = a$, $q = (5a)/2$, $s = (3-2b)/2$;

(v) $b = 1/4$, $c = -2$, $d = -2a$, $f = -a$, $g = (-1)/4$, $k = -a$, $l = a/2$, $m = 1$, $n = (-1)/4$, $p = a$, $q = a/2$, $r = s = 0$;

(vi) $b = (2u^4+9uv-9v^2-6au^3)/(18uv)$, $c = -(u^4+9uv+9v^2-3au^3)/(9uv)$, $f = p = r = 0$, $g = (6au^2-9a^2u-u^3-3v)/(6v)$, $k = -a$, $m = -c-1$, $n = [(3a-u)(u-v)u]/(6v)$, $q = (9a^2u^2-6au^3-18av+u^4+9uv)/(18v)$, $s = [(9a^2u^2-6au^3-54av+u^4+9uv)(u-3a)u]/(162v^2)$, $u = 2a-d$, $v = 2a-d+6l$;

(vii) $c = 2(b-1)$, $d = -a$, $f = n = p = r = s = 0$, $g = (-1)/2$, $k = -a$, $l = -ab$, $m = 1-2b$, $q = a/2$;

(viii) $a = [-2(h^4l+h^3+8h^2l+16l)]/[h^2(h^2+4)]$, $b = (-2h^4l+h^3-16h^2l-32l)/[2h(h^2+4)]$, $c = (h^4l-h^3+8h^2l-8h+16l)/[h(h^2+4)]$, $d = -a+(6h)/(h^2+4)$, $f = p = r = 0$, $g = [-2(h^2+1)]/(h^2+4)$, $k = -a$, $m = -c-1$, $n = (3hl)/2$, $q = [-2(2h^4l-h^3+10h^2l+8l)]/[(h^2+4)h^2]$, $s = (h^3+2h^2l+8l)/[(h^2+4)h]$;

(ix) $c = 1-2b$, $d = 2(a+2f)$, $g = [(3a+4f)(1-b)]/(f)$, $k = -a$, $l = 2f$, $m = 2(b-1)$, $n = 4af-3b+4f^2+3$, $p = -f$, $q = 8(1-b)(a+f)$, $r = 0$, $s = [(4ab-3a+4bf-4f)(b-1)]/(f)$.

Theorem 2. *The cubic system (1) has a first integral of the form (4), where* $\Phi(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$*, if and only if one of the following two sets of conditions holds:*

(x) $a = d = k = r = 0$, $g = c + 1 - b$, $l = bf$, $m = -(c+1)$, $n = [b(2c+3-b)]/2$, $p = -f$, $q = -f(c+1+b)$, $s = -b(c+1)$, $b^2 - 2f^2 - b = 0$;

(xi) $d = 2a$, $k = -a$, $l = [f(2b-c-1)]/3$, $m = -(c+1)$, $n = [(2b-c-2)(c+1)]/2$, $p = -f$, $q = a(2b-c-3)$, $r = 0$.

The existence of Darboux first integrals (integrating factors) of the form (4) implies the origin $O(0,0)$ to be a center for system (1).

Theorem 3. *The origin is a center for cubic system* (1) *with the invariant straight line* $1 - x = 0$ *and an invariant cubic* $\Phi(x,y) = 0$ *if one of the conditions* (i)–(xi) *holds.*

References

[1] D. Cozma. *Integrability of cubic systems with invariant straight lines and invariant conics.* Chişinău: Ştiinţa, 2013.

[2] D. Cozma. *Darboux integrability of a cubic differential system with two parallel invariant straight lines.* Carpathian J. Math., vol. 38, no. 1, 2022, pp. 129–137.

[3] D. Cozma and A. Matei. *Center conditions for a cubic differential system having an integrating factor.* Bukovinian Mathematical Journal, vol. 8, no. 2, 2020, pp. 6–13.

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