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Some Analytical Properties of the Laplace Transform of the Characteristic Function of the Three-Dimensional Markov Random Flight

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Abstract

The analytical properties of the Laplace transform of the characteristic function of the three-dimensional symmetric Markov random flight, are studied. The only singular point (ordinary pole) and the residue at this point are evaluated, which gives the first coefficient (by the negative power of complex variable) in the Laurent decomposition of the Laplace-Fourier transform of the transition density of the process.

Keywords: Markov random flight, Laplace-Fourier transform, characteristic function, singular point, ordinary pole, residue, Laurent series

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t)),\ t > 0$, be the three-dimensional symmetric Markov random flight represented by the stochastic motion of a particle that moves with constant speed c in the Euclidean space $\mathbb{R}^3$ and changes its direction at λ-Poisson distributed random time instants, choosing the initial and each new direction at random according to some probability distribution on the surface of the unit sphere.

Consider the characteristic function $H(\boldsymbol{\alpha}, t)$ of the process $\mathbf{X}(t)$ defined by $H(\boldsymbol{\alpha}, t) = \mathbb{E}\left[e^{\langle \boldsymbol{\alpha}, \mathbf{X}(t)\rangle}\right], \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3, \quad t > 0$, where $\mathbb{E}$ means the expectation and $\langle \boldsymbol{\alpha}, \mathbf{X}(t)\rangle$ is the inner product of three-dimensional vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

It is known (see [1, formula (4.6.4)], [2, page 1054]), [4, formula(45)], [5, formulas (1.6) and (5.8) for $c = 1$] that the Laplace transform of

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characteristic function $H(\boldsymbol{\alpha}, t)$ with respect to time variable t is given by the formula:

$$f(s) = \mathcal{L}_t \left[H(\boldsymbol{\alpha}, t) \right] (s) = \frac{\operatorname{arctg} \left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda} \right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg} \left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda} \right)}, \tag{1}$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, $\quad \operatorname{Re} s > 0$.

The main goal of this article is to briefly outline some analytical properties of function (1). The results presented here are a part of a more general research related to the asymptotic behaviour of the characteristic function $H(\boldsymbol{\alpha}, t)$. The following lemma yields the singular point of this function, which is the ordinary pole of first order.

Lemma 1. *Function (1) has the single singular point (ordinary pole)*

$$s^* = -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg} \left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda} \right). \tag{2}$$

Proof. To find the singular point of function (1), one needs to solve with respect to s the equation

$$\operatorname{arctg} \left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda} \right) = \frac{c\|\boldsymbol{\alpha}\|}{\lambda}. \tag{3}$$

In the inverse tangent function on the left-hand side of equation (3), we take its principal branch. Under such a choice, this function is holomorphic (analytical) and single-valued and, therefore, equation (3) has the unique solution. We also see that the right-hand side of (3) is a real-valued positive number. Hence, one can conclude that the inverse tangent function on the left-hand side of (3) must also take real values and, therefore, the variable s solving this equation must be a real number (that is, its imaginary part is zero). Then solving equation (3) with real values of variable s, we easily obtain the unique solution given by (2). □

Lemma 1 enables us to evaluate the residue of function (1) at the pole s^* given by (2).

Theorem 1. *The residue of function (1) at the ordinary pole s^* is given by the formula:*

$$\operatorname{res} f(s^*) = \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right). \tag{4}$$

Proof. Since s^* is the ordinary pole, then residue (4) can be evaluated by means of the standard relation (see [3, page 79, formula (4)]):

$$\begin{aligned}
\operatorname{res} f(s^*) &= \lim_{s\to s^*} [(s-s^*)\, f(s)] \\
&= \lim_{s\to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\left(s+\lambda - c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)\right)\right. \\
&\qquad \left. \times \frac{\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}\right] \\
&= \lim_{s\to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)\right] \\
&\qquad \times \lim_{s\to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\frac{s+\lambda - c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}\right].
\end{aligned}$$

The first limit is obviously equal to $c\|\boldsymbol{\alpha}\|/\lambda$. By applying L'Hôpital's rule to the second limit, we get:

$$\begin{aligned}
\operatorname{res} f(s^*) &= \frac{c\|\boldsymbol{\alpha}\|}{\lambda} \lim_{s\to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\frac{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}{\lambda\, c\|\boldsymbol{\alpha}\|}\right] \\
&= \frac{1}{\lambda^2} \lim_{s\to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2\right] \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \left[\operatorname{ctg}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + 1\right] \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right),
\end{aligned}$$

proving (4).

This formula can also be obtained by other way. Taking into account that

$$\left(c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)\right)'_s = \frac{\lambda\, c\|\boldsymbol{\alpha}\|}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}$$

and applying [3, page 79, formula (5)], we get:

$$\begin{aligned}
\operatorname{res} f(s^*) &= \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{-\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + \lambda}\right) \\
&\qquad \times \left[\frac{\lambda\, c\|\boldsymbol{\alpha}\|}{\left(-\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + \lambda\right)^2 + (c\|\boldsymbol{\alpha}\|)^2}\right]^{-1} \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2}\left[\operatorname{ctg}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + 1\right] \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right),
\end{aligned}$$

and we again arrive at (4). The theorem is proved. □

Corollary 1. Theorem 1 yields the first coefficient

$$\xi_{-1} = \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) \tag{5}$$

in the Laurent decomposition $f(s) = \sum\limits_{k=-1}^{\infty} \xi_k\, (s-s^*)^k$ of function $f(s)$ given by (1), which is the Laplace transform of the characteristic function of the three-dimensional symmetric Markov random flight $\mathbf{X}(t)$.

Corollary 2. Let D be an arbitrary domain in complex plane $\mathbb{C}$ such that the pole s^* is the interior point of D, and let C be the boundary of D. Since s^* is the only singular point of function $f(s)$ and it lies strictly inside D, then $f(s)$ is holomorphic (analytical) in all

the interior points of D, except s^*, and continuous on its boundary C. Then, according to the residues theorem,

$$\int_C f(s)\, ds = 2\pi i\, \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2}\, \mathrm{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right),$$

where the boundary C is circuiting in positive direction (that is, counterclock-wise).

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