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Isotopic classes of transversals in finite simple groups

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Abstract. Let G be a finite group and H be a subgroup of G. In this paper, we give a short proof of the main theorem of Kharvar and Shukla [6, Main Theorem]. Further, we prove that if H is a non-trivial proper subgroup of a finite non-abelian simple group G, then the number of isotopic classes of NRTs of H in G is greater than 1.

1. Introduction and tatement of main results

Let H be a subgroup of group G. A subset S of G is defined as a normalized right transversal (NRT) of H in G if it is formed by selecting one and only one element from each right coset of H in G, with the condition that $1 \in S$, where 1 represents the identity element of the group G. Let S be an NRT of H in G. Then the binary operation of G induces a binary operation on S defined by $\{x \circ y\} = S \cap Hxy,\ x,\ y \in S$. With respect to this binary operation, S is a right loop with identity 1, that is, a right-quasigroup with both-sided identity (see [11, Proposition 4.3.3]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [7, Theorem 3.4]).

Let S be an NRT of H in G. Let $\langle S\rangle$ be the subgroup of G generated by S and H_S be the subgroup $\langle S\rangle \cap H$. Then $H_S = \langle\{xy(x \circ y)^{-1} | x, y \in S\}\rangle$ and $H_S S = \langle S\rangle$ (see [7, Corollary 3.7]). Identifying S with the set $H\backslash G$ of all right cosets of H in G, we get a transitive permutation representation $\chi_S : G \longrightarrow Sym(S)$ defined by $\{\chi_S(g)(x)\} = Hxg \cap S$, $g \in G$, $x \in S$. The Kernel $\mathrm{Ker}\chi_S$ of this action is $\mathrm{Core}_G(H)$, the core of H in G. Our covention for the product in the symmetric group $Sym(S)$ is given as $(rs)(x) = s(r(x))$ for $r, s \in Sym(S)$ and $x \in S$. Let $G_S = \chi_S(H_S)$, the group torsion of the right loop S (see [7, Definition 3.1, p.75]). Since χ_S is injective on S and if we identify S with $\chi_S(S)$, then $\chi_S(\langle S\rangle) = G_S S$.

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Let $(L, \circ)$ and $(L', \circ')$ be two groupoids. A triple (f, g, h), where f, g and h are bijective maps from L to L', is said to be *isotopy* from L to L' if $f(x) \circ' g(y) = h(x \circ y)$, for all x, y in L. Two groupoids $(L, \circ)$ and $(L', \circ')$ are said to be *isotopic* if there exists an isotopy from L to L'. If $f = g = h$, then isotopy (f, g, h) is an isomorphism from L to L'. Let $(L, \circ)$ and $(L, \circ')$ be two groupoids. An isotopy (f, g, I), where I is identity map on L, from $(L, \circ)$ to $(L, \circ')$ is said to be a principal isotope. A non-empty subset T of a groupoid L is called a *subgroupoid* of L if the binary operation of L induces the binary operation on T.

Let $\mathcal{T}(G, H)$ denote the set of all NRTs of H in G. We say that $S, T \in \mathcal{T}(G, H)$ are isomorphic (resp. isotopic) if their induced right loop structures are isomorphic (resp. isotopic). Let $Iso(G, H)$ (resp. $Itp(G, H)$) denote the set of isomorphic (resp. isotopic) classes of NRTs of H in G respectively.

In [7], it is proved that if H is a normal subgroup of finite group G, then $|Iso(G, H)| = 1$. It has been proved in [8, Main Theorem] that if H is a subgroup of finite group G such that $|Iso(G, H)| = 1$, then H is a normal subgroup of G. By [3, 4], $|Iso(G, H)| \neq 2, 4$. Also, it has been proved in [9, Main Theorem] that if G is a finite non-abelian nilpotent group non-isomorphic to D_8, the dihedral group of order 8 and H is a non-normal subgroup of G, then $|Iso(G, H)| > 16$. It is easy to prove that if H is a normal subgroup of finite group G, then $|Itp(G, H)| = 1$. A natural question that arises is whether the converse of this problem is true. In [5, Proposition 3.8], it is proved that if H is a subgroup of finite nilpotent group G such that $|Itp(G, H)| = 1$, then H is normal in G. It has been shown in [5, Proposition 3.9] that if H is a subgroup of finite solvable group G. If $(|H|, [G : H]) = 1$ and $|Itp(G, H)| = 1$, then H is normal in G. In [6, Main Theorem], it is proved that if G is a finite solvable group and H is a subgroup of G such that $|Itp(G, H)| = 1$, then H is normal in G. In this paper, we give a short proof of the main theorem in [6]

Theorem 1.1. [6, Main Theorem] *Let G be a finite solvable group and H be a subgroup of G such that $|Itp(G, H)| = 1$. Then H is normal in G.*

and prove

Theorem 1.2. *Let G be a finite non-abelian simple group and H be a non-trivial proper subgroup of G. Then $|Itp(G, H)| > 1$.*

2. Preliminaries

In this section, we give some basic definitions and results that are needed to prove results of this paper.

Definition 2.1. [1] Let $(L, \circ)$ be a groupoid. Then the subset

$A = \{a \in L : \text{for all } x, y \in L,\ (x \circ a) \circ y = x \circ (a \circ y)\}$ of L is called the *middle-associators* of $(L, \circ)$.

Definition 2.2. [1] Let $(L, \circ)$ be a groupiod. Then $a \in L$ is called a *left non-singular* (resp. *right non-singular*) element of L if the equation $a \circ X = b$ (resp. $X \circ a = b$) has unique solution for all $b \in L$.

If a is a left non-singular (resp. right non-singular) element of a groupiod $(L, \circ)$, then the map $L_a : L \longrightarrow L$ (resp. $R_a : L \longrightarrow L$) defined by $L_a(x) = a \circ x$ (resp. $R_a(x) = x \circ a$) is a bijective map.

Theorem 2.3. [8, Proposition 1.10] *Let G be a finite group, H be a subgroup of G and $S \in \mathcal{T}(G, H)$. Then there exists a surjective homomorphism χ from $\langle S \rangle$ to $G_S S$. Further, if H is corefree, then χ is an isomorphism.*

Let G be a finite group and H be a subgroup of G. Let $Aut_H G$ denote the subgroup of $Aut(G)$, the automorphism group of G, consisting of those automorphisms of G which take H onto H.

Theorem 2.4. [8, Proposition 2.7] *Let H be a corefree subgroup of a finite group G. Let $S \in \mathcal{T}(G, H)$ such that $\langle S \rangle = G$. Then $Aut_H G$ acts transitively on the set $\{T \in \mathcal{T}(G, H) | T \cong S\}$.*

Theorem 2.5. [8, Lemma 3.4] *Let G be finite non-abelian simple group. Then $|G|^2 > |Aut(G)|$, where $Aut(G)$ is the group of automorphisms of G.*

Theorem 2.6. [1, Theorem 1B] *Let $(L, \circ)$ be a groupoid. Then the middle-associators of $(L, \circ)$ is a subgroupoid of $(L, \circ)$ and is a semigroup. If $(L, \circ)$ has identity 1, and $(L, \circ')$ is a groupoid with identity isotopic to $(L, \circ)$, then the middle-associators of $(L, \circ)$ is isomorphic to the middle-associators of $(L, \circ')$.*

Remark 2.7. Let G be a finite group, H be a subgroup of G and $S_1, S_2 \in \mathcal{T}(G, H)$. Let A_1, A_2 be middle-associators of S_1, S_2 respectively. If S_1 is isotopic to S_2, then by Theorem 2.6, A_1 is isomorphic to A_2.

Theorem 2.8. [6, Proposition 2.3] *Let G be a finite group, H be a corefree subgroup of G and $S \in \mathcal{T}(G,H)$. Let A be the middle-associators of $(S,\circ)$, where $\circ$ is the binary operation on S induced by the binary operation of G. Then $a \in A$ if and only if $ax \in S$ for all $x \in S$, that is, if and only if $a \circ x = ax$ for all $x \in S$.*

Remark 2.9. Let G be a finite group, H be a corefree subgroup of G and $S \in \mathcal{T}(G,H)$. Let A be the middle-associators of $(S,\circ)$. Then by Theorem 2.8, it follows that the middle-associators A of S is a subgroup of G.

Let $(L_1,\circ_1)$ and $(L_2,\circ_2)$ be isotopic groupoids. Then there exists a groupoid $(L_1,\circ_3)$ isomorphic to $(L_2,\circ_2)$ and isotopic to $(L_1,\circ_1)$. In particular $(L_1,\circ_3)$ may be chosen to be a principal isotope of $(L_1,\circ_1)$ [1, Lemma 1A]. Let $(L_1,\circ_1)$ be a groupoid. Then a necessary and sufficient condition that there exists a groupoid $(L_2,\circ_2)$, with an identity, isotopic to $(L_1,\circ_1)$ is that $(L_1,\circ_1)$ possess at least one left non-singular element a and at least one right non-singular element b. Every principal isotope $(L_1,\circ_3)$ of the type is given by

$$x \circ_3 y = R_b^{-1}(x) \circ_1 L_a^{-1}(y)$$

for some such fixed pair a,b of $(L_1,\circ_1)$ [1, Theorem 1A]. The proof of the following proposition immediately from argument given in [5, Second paragraph of Section 3, p.413].

Proposition 2.10. *Let G be a finite group, H be a subgroup of G and $S_1, S_2 \in \mathcal{T}(G,H)$. If S_1 and S_2 are isotopic, then $G_{S_1}S_1 \cong G_{S_2}S_2$.*

3. Proof of the Theorems 1.1 and 1.2

In order to prove our theorems, we need the following propositions and lemma.

Proposition 3.1. *Let G be a finite group and H be a corefree subgroup of G such that $|Itp(G,H)| = 1$. Let $S \in \mathcal{T}(G,H)$ and A be a middle-associators of S. Then $A = \{1\}$.*

Proof. We prove this by the method of contradiction. If possible assume that $A \neq \{1\}$. Then by Remark 2.9, A is a subgroup of G. Let $p > 2$ be a prime and p divide $|A|$. Then there exists a subgroup K of A of order p. Let $k(\neq 1) \in K$, $S_1 = (S \setminus \{k\}) \cup \{hk\}$ for some $h(\neq 1) \in H$ and

A_1 be a middle-associators of S_1. Since $|K| = p$ and $k \notin S_1$, by Remark 2.9, k^2, k^3, ..., k^{p-1} are not in A_1. Since $k \neq k^{-1}$ and $(hk)k^{-1} = h \notin S_1$, by Theorem 2.8, $hk \notin A_1$. Since $|Itp(G,H)| = 1$ and p divides $|A|$, p divides $|A_1|$. Therefore there exists a subgroup K_1 of A_1 of order p. Let $k_1(\neq 1) \in K_1$, $S_2 = (S_1 \setminus \{k_1\}) \cup \{h_1k_1\}$ for some $h_1(\neq 1) \in H$ and A_2 be a middle-associators of S_2. By constructions of S_2 and by Theorem 2.8, elements hk and k^2, k^3, ..., k^{p-1} are not in A_2. Since $|K_1| = p$ and $k_1 \notin S_2$, by Remark 2.9, k_1^2, k_1^3, ..., k_1^{p-1} are not in A_2. Since $k_1 \neq k_1^{-1}$, $k_1^{-1} \in S_2$. Since $(h_1k_1)k_1^{-1} = h_1$ are not in S_2, by Theorem 2.8, h_1k_1 is not in A_2. In each step we get an NRT of H in G with middle associators such that $p-1$ elements of NRT are not member of middle associators. Since index of H in G is finite, by continuing this process we get an NRT $S' \in \mathcal{T}(G,H)$ having middle-associators A' and $|A'| < p$. Therefore by Remark 2.7, $|Itp(G,H)| > 1$, a contradiction. This implies that only a prime divides the order of A, which is 2.

Assume that 2 divides $|A|$. Then there exists a subgroup L of A of order 2. Let $S = \{1, x_1, x_2, ..., x_{n-1}\}$. Without loss of generality we may assume that $L = \{1, x_1\}$. Let $h(\neq 1) \in H$ be fixed and $T_1 = (S \setminus \{x_1\}) \cup \{hx_1\}$ and B_1 be a middle-associators of T_1. Since $x_1 \in A$, by Theorem 2.8, $x_1x_2 \in S$. By costruction of T_1, $x_1x_2 \in T_1$. If $hx_1 \in B_1$, then by Theorem 2.8, $hx_1x_2 \in T_1$, a contradiction. Hence $hx_1 \notin B_1$. Since $|Itp(G,H)| = 1$ and 2 divides $|A|$, 2 also divides $|B_1|$. Therefore there exists a subgroup L_1 of B_1 of order 2. Without loss of generality we may assume that $L_1 = \{1, x_2\}$. Let $h(\neq 1) \in H$ be fixed and $T_2 = (T_1 \setminus \{x_2\}) \cup \{hx_2\}$ and B_2 be a middle-associators of T_2. Since $x_1 \in A$ and $x_2 \in B_1$, by Theorem 2.8, $x_1x_3 \in S$ and $x_2x_3 \in T_1$. By construction of T_2, x_1x_3, $x_2x_3 \in T_2$. If hx_1, $hx_2 \in B_2$, then by Theorem 2.8, hx_1x_3, $hx_2x_3 \in T_2$, a contradiction. Hence hx_1, $hx_2 \notin B_2$. Continuing this process we get $x_1^2 = x_2^2 = x_3^2 = ... = x_{n-1}^2 = 1$. Hence by [5, Proposition 2.1], S is a both sided transversal. Let K be an NRT of H in G and K_1 be a middle-associators of K. Since $|Itp(G,H)| = 1$, $|K_1| > 1$. Hence by previous argument K is also both sided transversals. Therefore by [5, Theorem 2.6], H is normal subgroup of G, a contradiction. This complete the proof. □

Proposition 3.2. *Let G be a finite group and H be a corefree subgroup of G such that $|Itp(G,H)| = 1$. Then $\langle S \rangle = G$, for all $S \in \mathcal{T}(G,H)$.*

Proof. Since H is a corefree subgroup of G, by [2], there exists $S \in \mathcal{T}(G,H)$ such that $\langle S \rangle = G$. Then by Theorem 2.3, $G_S S \cong \langle S \rangle = G$. Let $T \in$

$\mathcal{T}(G,H)$. Since $|Itp(G,H)| = 1$, then by Theorem 2.3 and Proposition 2.10, $\langle T\rangle \cong G_T T \cong G_S S \cong \langle S\rangle = G$. This complete the proof. □

Proposition 3.3. *Let G be a finite group, H be a corefree subgroup of G such that $|Itp(G,H)| = 1$ and $[G:H] = n$. Then $|Iso(G,H)| \leqslant n(n-1)$.*

Proof. Let $S \in \mathcal{T}(G,H)$ and L denote the set of left non-singular elements in S. Since $[G:H] = n$ and $|Itp(G,H)| = 1$, then $|L| < n$ otherwise by [6, Proposition 2.5], H is a normal subgroup of G, a contradiction. Hence $|L| \leqslant n-1$. Let X be a set of principal isotopes of S. By the discussion in the last paragraph of section 2, $|X| \leqslant n(n-1)$. Let S_1 be an NRT of H in G. Since $|Itp(G,H)| = 1$, S_1 is isomorphic to some principal isotope of S. Hence $|Iso(G,H)| \leqslant n(n-1)$. □

Proposition 3.4. *Let G be a finite group, H be a corefree subgroup of G such that $|Itp(G,H)| = 1$. Then there exists an isomorphism class $\mathcal{A}$ of NRTs of H in G such that $|\mathcal{A}| \geqslant \frac{m^{n-1}}{n(n-1)}$, where $|H| = m$, $[G:H] = n$.*

Proof. Since $|H| = m$ and $[G:H] = n$, $|\mathcal{T}(G,H)| = m^{n-1}$. By Proposition 3.3, $|Iso(G,H)| \leqslant n(n-1)$. By Proposition 3.2, $\langle S\rangle = G$ for all $S \in \mathcal{T}(G,H)$. Hence there exists isomorphism class $\mathcal{A}$ of NRTs of H in G such that $|\mathcal{A}| \geqslant \frac{m^{n-1}}{n(n-1)}$. □

Proposition 3.5. *Let G be a finite group, H be a corefree subgroup of G such that $|Itp(G,H)| = 1$. Then $|Aut_H(G)| \geqslant \frac{m^{n-1}}{n^2}$, where $|H| = m$, $[G:H] = n$.*

Proof. By Proposition 3.4, there exists an isomorphism class $\mathcal{A}$ of NRTs of H in G such that $|\mathcal{A}| \geqslant \frac{m^{n-1}}{n(n-1)} \geqslant \frac{m^{n-1}}{n^2}$. By Proposition 3.2 and by Theorem 2.4, $|Aut_H(G)| \geqslant |\mathcal{A}| \geqslant \frac{m^{n-1}}{n^2}$. □

Lemma 3.6. *Let $n \in \boldsymbol{N}$, $n \geqslant 21$. Then $2^{n-3} \geqslant n^4$.*

Proof. Assume that $n = 21$. Then $2^{n-3} = 2^{18} = 2,62,144$ and $n^4 = 21^4 = 1,94,481$. Hence $2^{n-3} \geqslant n^4$ for $n = 21$. Suppose that $2^{k-3} \geqslant k^4$ for some $k \in \mathbf{N}$, $k > 21$. Now $2^{k+1-3} = 2.(2^{k-3})$ and since $2^{k-3} \geqslant k^4$, then $2^{k+1-3} = 2.k^4 = k^4 + k^4 = k^4 + k.k^3 \geqslant k^4 + 21.k^3 = k^4 + 7.k^3 + 5.k^3 + 7.k^3 + 2.k^3 \geqslant k^4 + 4.k^3 + 6.k^2 + 4.k + 1 = (k+1)^4$. Thus, by induction, $2^{n-3} \geqslant n^4$ for all $n \in \mathbf{N}$, $n \geqslant 21$. □

PROOF OF THE THEOREM 1.1: If possible, assume that H is not normal. Let $K = Core_G(H)$. Since $|Itp(G,H)| = |Itp(G/K, H/K)|$ (see [5, Proposition 3.4]), G/K is sovable and H/K is corefree subgroup of G/K, we may assume that H is a corefree subgroup of G. Since G is a finite solvable group, there exists a non-trivial minimal normal subgroup N of G. Then by [10, 5.4.3], N is an elementary abelian p-group for some prime p. If $N \cap H = \{1\}$, then by [6, Proposition 3.1], $|Itp(G,H)| > 1$, a contradiction. Hence $N \cap H \neq \{1\}$. Since H is corefree subgroup of G, $N \cap H \neq N$. Hence $N \cap H$ is a non-trivial proper subgroup of N. Since N is an elementary abelian p-group, there exists a subgroup $K^{'}$ of N such that $K^{'} \in \mathcal{T}(N, H \cap N)$. Hence $K^{'} \in \mathcal{T}(HN, H)$. If $HN = G$, then by [6, Proposition 2.5], H is a normal subgroup of G, a contradiction. Hence $HN \neq G$. Let $S_1 = K^{'}L \in \mathcal{T}(G,H)$, where $L \in \mathcal{T}(G, NH)$, and A_1 be the middle-associators of S_1. Since $K^{'}$ is a subgroup of N, by [6, Proposition 2.4], $K^{'} \subseteq A_1$ and so $|A_1| \geqslant |K^{'}| > 1$. By Proposition 3.1, this is a contradiction. Therefore H is a normal subgroup of G. □

PROOF OF THE THEOREM 1.2: If possible, assume that $|Itp(G,H)| = 1$. Since G is a finite non-abelian simple group, $Core_G(H) = \{1\}$. Let $|H| = m$ and $[G : H] = n$. Then $m \neq 1$ and $n \neq 1, 2$. Let $S \in \mathcal{T}(G,H)$. Then by Theorem 2.3 and Proposition 3.2, $G = HS \cong G_S S \subseteq Sym(S \setminus \{1\})S$ and hence $|G| \leqslant n!$. Since G is a non-abelian simple group, $|G| \geqslant 60$. Hence $n \geqslant 5$.

Suppose that $n = 5$. Then $m \geqslant 12$. Since $Core_G(H) = \{1\}$, we can identify G with a subgroup of $Sym(5)$. Since G is a non-abelian simple group and $|G| \geqslant 60$, G is isomorphic to A_5, alternating group of degree five. Hence $m = 12$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \dfrac{12^4}{5^2} > 144$. But by [12, 2.17], $|Aut(A_5)| = |Aut(G)| = 120$, a contradiction. Hence $n \geqslant 6$.

Suppose that $n = 6$. Then $m \geqslant 10$. Let $m = 10$ and so $G \cong A_5$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \dfrac{10^5}{6^2} > 1000$. But by [12, 2.17], $|Aut(A_5)| = |Aut(G)| = 120$, a contradiction. Hence $m \geqslant 11$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \dfrac{m^5}{6^2} = \dfrac{m^2.m^3}{6^2} \geqslant \dfrac{m^2.11^3}{6^2}$. Since $11^3 \geqslant 6^4$, $|Aut(G)| \geqslant \dfrac{m^2.6^4}{6^2} = m^2.6^2 = |G|^2$. By Theorem 2.5, this is a contradiction. Hence $n \geqslant 7$.

Let $n = 7$. Then $m \geqslant 9$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant$

$\frac{m^6}{7^2} = \frac{m^2.m^4}{7^2} \geqslant \frac{m^2.7^4}{7^2} = m^2.7^2 = |G|^2$. By Theorem 2.5, this is a contradiction. Hence $n \geqslant 8$.

Let $n = 8$. Then $m \geqslant 8$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \frac{m^7}{8^2} = \frac{m^2.m^5}{8^2} \geqslant \frac{m^2.6^5}{8^2}$. Since $6^5 \geqslant 8^4$, $|Aut(G)| \geqslant \frac{m^2.8^4}{8^2} = m^2.8^2 = |G|^2$. By Theorem 2.5, we get this is a contradiction. Hence $n \geqslant 9$.

Let $n = 9$. Then $m \geqslant 7$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \frac{m^8}{9^2} = \frac{m^2.m^6}{9^2} \geqslant \frac{m^2.5^6}{9^2}$. Since $5^6 \geqslant 9^4$, $|Aut(G)| \geqslant \frac{m^2.9^4}{9^2} = m^2.9^2 = |G|^2$. By Theorem 2.5, we get a contradiction. Hence $n \geqslant 10$.

Let $10 \leqslant n \leqslant 15$. Then $m \geqslant 4$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \frac{m^{n-1}}{n^2} = \frac{m^2.m^{n-3}}{n^2} \geqslant \frac{m^2.4^{n-3}}{n^2}$. Since $4^{n-3} \geqslant n^4$, $|Aut(G)| \geqslant \frac{m^2.n^4}{n^2} = m^2.n^2 = |G|^2$. By Theorem 2.5, this is a contradiction. Hence $n \geqslant 16$.

Let $16 \leqslant n \leqslant 29$. Then $m \geqslant 3$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \frac{m^{n-1}}{n^2} = \frac{m^2.m^{n-3}}{n^2} \geqslant \frac{m^2.3^{n-3}}{n^2}$. Since $3^{n-3} \geqslant n^4$, $|Aut(G)| \geqslant \frac{m^2.n^4}{n^2} = m^2.n^2 = |G|^2$. By Theorem 2.5, we get this is a contradiction. Hence $n \geqslant 30$.

Since $n \geqslant 30$, $m \geqslant 2$. By Proposition 3.5, $|Aut(G)| \geqslant |Aut_H(G)| \geqslant \frac{m^{n-1}}{n^2} = \frac{m^2.m^{n-3}}{n^2} \geqslant \frac{m^2.2^{n-3}}{n^2}$. By Lemma 3.6, $|Aut(G)| \geqslant \frac{m^2.n^4}{n^2} = m^2.n^2 = |G|^2$. By Theorem 2.5, we get a contradiction. This complete the proof. □

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