A NUMERICAL METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS WITH PIECEWISE CONTINUOUS COEFFICIENTS

O METODĂ NUMERICĂ PENTRU REZOLVAREA ECUATIILOR INTEGRALE SINGULARE CU COEFICIENȚI CONTINUI PE PORȚIUNI

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Maria CAPCELEA,
ORCID: 0009-0009-3321-3946
Moldova State University
Titu CAPCELEA,
ORCID: 0009-0002-7178-9291
e-mail: titu.capcelea@usm.md
Moldova State University

Summary. *The present study is dedicated to developing an efficient computational scheme for solving the Cauchy singular integral equation, defined on a closed and smooth contour in the complex plane. The coefficients and the right-hand side of the equation are piecewise continuous functions, numerically defined on a finite set of points along the contour. The approximate solution is constructed as a linear combination of B-spline functions and Heaviside functions, with coefficients determined using the collocation method. This method generates a sequence of approximations that converge almost uniformly to the exact solution of the equation.*

Keywords: *singular integral equation, piecewise continuous coefficients, B-spline functions, computational scheme.*

INTRODUCTION

Let Γ be a closed and smooth contour that forms the boundary of the simply connected domain $\Omega^+ \subset \mathbb{C}$, with the point $z = 0 \in \Omega^+$. We consider the points on the contour Γ to be defined by the Riemann function $z = \psi(w)$, which maps the circle $\Gamma_0 = \{w \in \mathbb{C} : |w| = 1\}$ onto the contour Γ.

Let's consider the Cauchy singular integral equation defined on the contour Γ

$$c(t)\varphi(t) + d(t)(S\varphi)(t) + (K\varphi)(t) = f(t),\ t \in \Gamma, \tag{1}$$

where $(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$ is the singular integral in the sense of the Cauchy principal value, and $(K\varphi)(t) := \frac{1}{2\pi i}\int_\Gamma h(t,\tau)\varphi(\tau)d\tau$ is the regular part, with the kernel $h \in C(\Gamma \times \Gamma)$. We assume that the coefficients c and d of the equation, the right-hand side f, and the solution φ are piecewise continuous functions.

Let $\{t_j\}_{j=1}^{n_B}$, $t_j = \psi(w_j)$, $w_j = e^{i\theta_j}$, $\theta_j = 2\pi(j-1)/n_B$, $j = 1,...,n_B$, be the set of distinct points of the contour Γ where the values of the right-hand side f and the coefficients c and d are defined. Additionally, we assume that the values of the functions f, c, and d are known at the discontinuity points t_r^d, $r = 1,...,n_{pd}$, on the contour Γ. For piecewise continuous functions defined numerically, several numerical algorithms have been proposed in [1] to establish the locations of the discontinuity points on Γ.

We aim to develop a numerical method that determines a sequence of approximations φ_n, which converge uniformly to the solution φ of equation (1) everywhere on Γ, except at a finite number of discontinuity points of φ, where we have pointwise convergence.

Currently, several efficient numerical approximation methods exist for computing the solution of the Cauchy singular integral equation when the integration contour is the unit circle Γ_0 or the interval $(-1,1)$, and the coefficients and the right-hand side of equation (1) are continuous and sufficiently smooth functions (see, for example, [2], [3]). The convergence of these methods has been demonstrated, and estimates for the rate of convergence in Hölder spaces and Lebesgue spaces L_p $(1 < p < \infty)$ have been established.

Additionally, in [4] the theoretical foundation of collocation and finite section methods in Hölder and Lebesgue function spaces was obtained for equation (1) when defined on a smooth and closed contour in the complex plane, with coefficients that are continuous functions satisfying the Hölder condition.

Subsequently, the convergence of collocation and finite section methods for equation (1), defined on the circle Γ_0 or interval $(-1,1)$, with coefficients and the right-hand side being piecewise continuous functions, in the Lebesgue space L_2 of square-integrable functions, has been demonstrated [5], [6].

In this paper, to approximate the solution of equation (1), we apply a modification of the collocation method, where the solution is sought as a linear combination of B-spline functions and Heaviside step functions, resulting in a piecewise continuous approximation function that converges almost uniformly to the exact solution.

A METHOD FOR APPROXIMATING THE SOLUTION OF THE EQUATION

For two points $t_1, t_2 \in \Gamma$ we use the notation $t_1 \prec t_2$ if when traversing the contour Γ in counterclockwise direction we meet first the point t_1, and then t_2. Let $t_1 \prec t_2 \prec \ldots \prec t_n$ $(\prec t_1)$ be a set of distinct points of the contour Γ. We denote by $\Gamma_j := arc[t_j, t_{j+1}]$ the set of points on the contour Γ, located between the points t_j and t_{j+1}, including these points.

The algorithm we propose for approximating the solution φ of equation (1) uses the concept of B-spline functions of order m $(2 \le m \le n_B)$, which are defined at the points t_j on the contour Γ by the recursive formula

$$B_{m,j}(t) := \frac{m}{m-1}\left(\frac{t-t_j^B}{t_{j+m}^B-t_j^B}B_{m-1,j}(t)+\frac{t_{j+m}^B-t}{t_{j+m}^B-t_j^B}B_{m-1,j+1}(t)\right),\ j=1,...,n_B,$$ where $B_{1,j}(t)=\begin{cases}\dfrac{1}{t_{j+1}^B-t_j^B} & if\ \ t\in arc\left[t_j^B,t_{j+1}^B\right)\\ 0 & otherwise\end{cases}$. The set of nodes $\left\{t_j^B\right\}_{j=1}^{n_B+m}$ satisfies the condition $t_j^B=t_j,\ j=1,...,n_B,\ t_{n_B+1}^B=t_1^B, t_{n_B+2}^B=t_2^B,...,t_{n_B+m}^B=t_m^B$ (see [7]). For a fixed $m\geq 2$, the B-spline functions $B_{m,k}(t),\ k=1,...,n_B$, have an explicit representation [7], from which it follows that they have support on the curve arc $\left[t_k^B,t_{k+m}^B\right)$. Thus, the B-spline function $B_{m,k}(t)$ is described by m branches $p_r^{m,k}(t),\ r=1,...,m$, each defined on the arc $arc\left[t_{k+r-1}^B,t_{k+r}^B\right)$.

In [8], it was shown that any continuous function on Γ can be uniformly approximated by a linear combination of B-spline functions. When linear combinations of B-splines are used to approximate piecewise continuous functions, the approximation error decreases as the number of spline knots increases, even in the vicinity of discontinuity points.

With respect to the points of discontinuity $t_r^d,\ r=1,...,n_{pd}$, we define the Heaviside step function H on the contour Γ:

$$H\left(t-t_s^d\right) := \begin{cases}0 & if\ \ t\in\left(\Gamma_1\setminus\left\{t_1^B\right\}\right)\cup...\cup\Gamma_{q-1}\cup arc\left[t_q^B,t_s^d\right)\\ 1 & if\ \ t\in arc\left[t_s^d,t_{q+1}^B\right)\cup\Gamma_{q+1}\cup...\cup\Gamma_{n_B}\end{cases},$$

where $\Gamma_q=arc\left[t_q^B,t_{q+1}^B\right],\ t_s^d\in\Gamma_q$.

Taking into account that the solution φ of equation (1) exhibits jump discontinuities on the contour Γ, we seek to approximate φ in the form

$$\varphi_{n_B}^H(t) := \sum_{k=1}^{n_B}\alpha_k B_{m,k}(t)+\sum_{s=1}^{n_{pd}}\beta_s H\left(t-t_s^d\right),$$

where the coefficients $\alpha_k\in\mathbb{R}$, $k=1,...,n_B$, and $\beta_s\in\mathbb{R}$, $s=1,...,n_{pd}$, are determined by imposing $n := n_B+n_{pd}$ interpolation conditions

$$c\left(t_j^C\right)\varphi_{n_B}^H\left(t_j^C\right)+d\left(t_j^C\right)\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)+\left(K\varphi_{n_B}^H\right)\left(t_j^C\right)=f\left(t_j^C\right),\ \ j=1,...,n. \qquad (2)$$

In relation (2) the first n_B interpolation points t_j^C, $j=1,...,n_B$, are the nodes $t_j^B=t_j$, $j=1,...,n_B$, and the remaining n_{pd} points t_j^C, $j=n_B+1,...,n$, are the discontinuity points t_s^d, $s=1,...,n_{pd}$, of the function f (and also of c and d). If among the interpolation points t_j^C, $j=1,...,n_B$, there are discontinuity points $t_s^d=\psi\left(e^{i\theta_s^d}\right)$ of the function f on Γ, then instead of them, for sufficiently small $\varepsilon_2>0$, we consider points $\tilde{t}_s^d=\psi\left(e^{i\left(\theta_s^d-\varepsilon_2\right)}\right)$ such that $f\left(\tilde{t}_s^d\right)=f\left(t_s^d\right)$ (also for values of c and d).

To calculate the value of $\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)$, we perform the following transformations:

$$\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi_{n_B}^H(\tau)}{\tau-t_j^C}d\tau=\sum_{k=1}^{n_B}\alpha_k I_k^{1,m}\left(t_j^C\right)+\sum_{s=1}^{n_{pd}}\beta_s I_s^2\left(t_j^C\right),$$

where

$$I_k^{1,m}\left(t_j^C\right)=\frac{1}{\pi i}\int_\Gamma\frac{B_{m,k}(\tau)}{\tau-t_j^C}d\tau=\sum_{r=1}^{m}I_r^{3,j,k},\ I_r^{3,j,k}=\frac{1}{\pi i}\int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B}g_{j,k}^r(\theta)d\theta,\ g_{j,k}^r(\theta)=\frac{p_r^{m,k}\left(\psi\left(e^{i\theta}\right)\right)}{\psi\left(e^{i\theta}\right)-t_j^C}\psi'\left(e^{i\theta}\right)ie^{i\theta}$$

,

$$I_s^2\left(t_j^C\right)=\frac{1}{\pi i}\int_\Gamma\frac{H\left(\tau-t_s^d\right)}{\tau-t_j^C}d\tau=\frac{1}{\pi i}\int_{\theta_s^d}^{2\pi}q_j(\theta)d\theta,\ q_j(\theta)=\frac{1}{\psi\left(e^{i\theta}\right)-t_j^c}\psi'\left(e^{i\theta}\right)ie^{i\theta}.$$

Similarly, an expression for $\left(K\varphi_{n_B}^H\right)\left(t_j^C\right)$ can be established:

$$\left(K\varphi_{n_B}^H\right)\left(t_j^C\right)=\frac{1}{2\pi i}\int_\Gamma h\left(t_j^C,\tau\right)\varphi_{n_B}^H(\tau)d\tau=\sum_{k=1}^{n_B}\alpha_k I_k^{4,m}\left(t_j^C\right)+\sum_{s=1}^{n_{pd}}\beta_s I_s^5\left(t_j^C\right),$$

where

$$I_k^{4,m}\left(t_j^C\right)=\frac{1}{2\pi i}\sum_{r=1}^{m}\int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B}\tilde{g}_{j,k}^r(\theta)d\theta,$$

$$\tilde{g}_{j,k}^r(\theta)=h\left(t_j^C,\psi\left(e^{i\theta}\right)\right)p_r^{m,k}\left(\psi\left(e^{i\theta}\right)\right)\psi'\left(e^{i\theta}\right)ie^{i\theta},$$

$$I_s^5\left(t_j^C\right)=\frac{1}{2\pi i}\int_\Gamma h\left(t_j^C,\tau\right)H\left(\tau-t_s^d\right)d\tau=\frac{1}{2\pi i}\int_{\theta_s^d}^{2\pi}\tilde{q}_j(\theta)d\theta,\ \tilde{q}_j(\theta)=h\left(t_j^C,\psi\left(e^{i\theta}\right)\right)\psi'\left(e^{i\theta}\right)ie^{i\theta}.$$

The Riemann integrals $\int_{\theta_{k+r-1}^B}^{\theta_{k+r}^B} \tilde{g}_{j,k}^r(\theta)d\theta$ and $\int_{\theta_s^d}^{2\pi} \tilde{q}_j(\theta)d\theta$ are aproximated using the generalized trapezoidal rule, which is also applicable to functions with complex values:

$$I=\int_{\theta_{in}}^{\theta_f} f(\theta)d\theta \approx I_N=\Delta_\theta\left(0.5\left(f(\theta_{in})+f(\theta_f)\right)+\sum_{j=1}^{N-1} f(\theta_{in}+jh)\right),\ \Delta_\theta=(\theta_f-\theta_{in})/N.$$

As $N\to\infty$, it is known that $I_N\to I$ at the rate of a geometric progression.

APPROXIMATION OF CAUCHY SINGULAR INTEGRALS

We denote by Γ_{el}^B the arc $arc\left[t_{k+r-1}^B, t_{k+r}^B\right]$. As the interpolation points t_j^C, $j=1,...,n$ form a subdivision covering the contour Γ, if $t_j^C\in\Gamma_{el}^B$, the integral $I_r^{3,j,k}$ will exhibit a singularity of type $1/(\tau-t_j^C)$. In such cases, we interpret $I_r^{3,j,k}$ as the Cauchy principal value. Similarly, we handle the integral $I_s^2(t_j^C)$ when $t_j^C\in arc\left[t_s^d,\psi(1)\right]$.

Let θ_j^C be the polar angle corresponding to the point $t_j^C=\psi\left(e^{i\theta_j^C}\right)\in\Gamma$. Since $\frac{1}{\pi i}\int_\Gamma \frac{1}{\tau-t}d\tau=1$ on the closed contour Γ, we can mitigate the strong singularity when computing the singular integral $(S\varphi)(t)=\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)-\varphi(t)}{\tau-t}d\tau+\varphi(t)$. If the function $\varphi(\tau)$ is continuous and sufficiently smooth, then $\frac{\varphi(\tau)-\varphi(t)}{\tau-t}$ also possesses these properties. Specifically, we can write

$$\left(S\varphi_{n_B}^H\right)(t_j^C)=\sum_{k=1}^{n_B}\alpha_k\left\{\tilde{I}_k^{1,m}(t_j^C)+B_{m,k}(t_j^C)\right\}+\sum_{s=1}^{n_{pd}}\beta_s\left\{\tilde{I}_s^2(t_j^C)+H(t_j^C-t_s^d)\right\},$$

where

$$\tilde{I}_k^{1,m}(t_j^C)=\frac{1}{\pi i}\int_\Gamma\frac{B_{m,k}(\tau)-B_{m,k}(t_j^C)}{\tau-t_j^C}d\tau,\ \tilde{I}_s^2(t_j^C)=\frac{1}{\pi i}\int_\Gamma\frac{H(\tau-t_s^d)-H(t_j^C-t_s^d)}{\tau-t_j^C}d\tau.$$

We represent the integral $\tilde{I}_k^{1,m}(t_j^C)$ in the form $\tilde{I}_k^{1,m}(t_j^C)=\sum_{r=1}^m \tilde{I}_r^{3,j,k}$, where $\tilde{I}_r^{3,j,k}=$

$=\frac{1}{\pi i}\int_{\Gamma_{el}^B}\frac{p_r^{m,k}(\tau)-B_{m,k}(t_j^C)}{\tau-t_j^C}d\tau$. When $\tau\to t_j^C$, we have $\frac{B_{m,k}(\tau)-B_{m,k}(t_j^C)}{\tau-t_j^C}\to B'_{m,k}(t_j^C)$, where $B'_{m,k}$ is calculated as $B'_{m,k}(t)=\frac{m}{t_{k+m}^B-t_k^B}\left(B_{m-1,k}(t)-B_{m-1,k+1}(t)\right)$ [8]. For $\varepsilon>0$ suffi-

ciently small, let $\Gamma_\varepsilon := arc\left[\psi\left(e^{i\left(\theta_j^C-\varepsilon\right)}\right),\psi\left(e^{i\left(\theta_j^C+\varepsilon\right)}\right)\right]$ be the arc containing the point t_j^C. We approximate the expression $\dfrac{B_{m,k}(\tau)-B_{m,k}\left(t_j^C\right)}{\tau-t_j^C}$ on the arc Γ_ε by $B'_{m,k}\left(t_j^C\right)$.

If $t_j^C \in \Gamma_{el}^B$, then the singular integral $\tilde{I}_r^{3,j,k}$ will be computed according to the relation

$$\tilde{I}_r^{3,j,k} \approx \frac{1}{\pi i}\int\limits_{\Gamma_{el}^B/\left(\Gamma_\varepsilon\cap\Gamma_{el}^B\right)} \frac{p_r^{m,k}(\tau)-B_{m,k}\left(t_j^C\right)}{\tau-t_j^C}d\tau + \frac{1}{\pi i}\left(p_r^{m,k}\right)'\left(t_j^C\right)\int\limits_{\Gamma_\varepsilon\cap\Gamma_{el}^B} d\tau,$$

where $\int\limits_{\Gamma_\varepsilon\cap\Gamma_{el}^B} d\tau$ is calculated analytically, and the regular integral $\dfrac{1}{\pi i}\int\limits_{\Gamma_{el}^B/\left(\Gamma_\varepsilon\cap\Gamma_{el}^B\right)} \dfrac{p_r^{m,k}(\tau)-B_{m,k}\left(t_j^C\right)}{\tau-t_j^C}d\tau$ using the trapezoidal rule. However, if $t_j^C \notin \Gamma_{el}^B$, then the integral $\tilde{I}_r^{3,j,k}$ is regular and will be computed using the trapezoidal rule. Similarly, the integral $\tilde{I}_s^2\left(t_j^C\right)$ is computed.

THE COMPUTATIONAL SCHEME AND A NUMERICAL EXAMPLE

Considering the relationships established for $\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)$ and $\left(K\varphi_{n_B}^H\right)\left(t_j^C\right)$, we express relation (2) in the form of the linear algebraic system

$$\sum_{k=1}^{n_B} a_{jk}\alpha_k + \sum_{s=1}^{n_{pd}} b_{js}\beta_s = f\left(t_j^C\right),\ \ j=1,\ldots,n,$$

where $a_{j,k} := c\left(t_j^C\right)B_{m,k}\left(t_j^C\right) + d\left(t_j^C\right)I_{jk} + I_k^{4,m}\left(t_j^C\right)$,

$$b_{j,s} := c\left(t_j^C\right)H\left(t_j^C - t_s^d\right) + d\left(t_j^C\right)I_{sj} + I_s^5\left(t_j^C\right), \text{ and}$$

$$I_{jk} = \tilde{I}_k^{1,m}\left(t_j^C\right) + B_{m,k}\left(t_j^C\right),\ I_{sj} = \tilde{I}_s^2\left(t_j^C\right) + H\left(t_j^C - t_s^d\right).$$

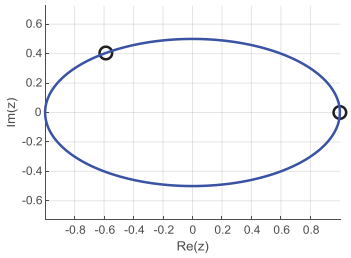


Figure 1. Contour and discontinuity points

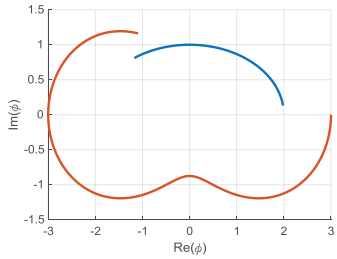


Figure 2. Graph of the function $\varphi(t)$

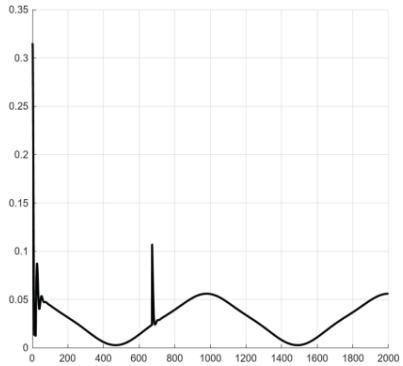


Figure 3. The approximation error for $n_B = 175$

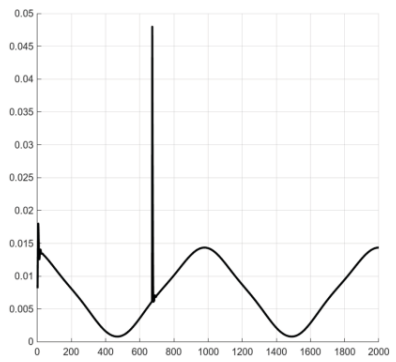


Figure 4. The approximation error for $n_B = 350$

Let the Riemann function $\psi(w) = 0.25\left(3w + w^{-1}\right)$ map the circle Γ_0 onto the ellipse Γ (see Figure 1) and $m = 4$. To test the algorithm, we consider the following parameters:

$$c(t) = \begin{cases} t & if \quad t \in arc\left(\psi\left(e^{0i}\right), t_1^d\right] \\ 2 - t & if \quad t \in arc\left(t_1^d, \psi\left(e^{2\pi i}\right)\right] \end{cases}, \quad d(t) = \begin{cases} 1 - t & if \quad t \in arc\left(\psi\left(e^{0i}\right), t_1^d\right] \\ 1 + t & if \quad t \in arc\left(t_1^d, \psi\left(e^{2\pi i}\right)\right] \end{cases},$$

where $t_1^d = \psi\left(e^{i\theta_1^d}\right)$, $\theta_1^d = 0.7\pi$ and $t_2^d = \psi\left(e^{i\theta_2^d}\right)$, $\theta_2^d = 2\pi$ represent jump discontinuity points on Γ. The kernel function is $h(t,\tau) = t^2 + \tau^2$, and the exact solution (see Figure 2) is

$$\varphi(t) = \begin{cases} \varphi_1(t) = 2t & if \quad t \in arc\left(\psi\left(e^{0i}\right), t_1^d\right] \\ \varphi_2(t) = t^3 + 2t & if \quad t \in arc\left(t_1^d, \psi\left(e^{i\theta_2^d}\right)\right] \end{cases}.$$

The right-hand side $f(t)$ is numerically defined on Γ by the values

$$f\left(t_j^C\right) = c\left(t_j^C\right)\varphi\left(t_j^C\right) + d\left(t_j^C\right)(S\varphi)\left(t_j^C\right) + (K\varphi)\left(t_j^C\right), \; j = 1, \ldots, n.$$

For values $n_B = 175$ and $n_B = 350$ in Figure 3 and Figure 4 the error obtained in approximating the solution φ by $\varphi_{n_B}^H$ is presented. It can be seen that the maximum error decreases significantly for $n_B = 350$.

CONCLUSION

This paper presents a new method for solving Cauchy singular integral equations. The coefficients and the right-hand side of the equation are piecewise continuous functions defined on a smooth and closed contour in the complex plane. The method constructs a sequence of approximations that converge almost uniformly to the exact solution of the integral equation.

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