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Semisymmetric Quasigroups

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Abstract

A quasigroup satisfying the identity $x(yx) = y$ is called semisymmetric. In this article, the isotopy of semisimmetric quasigroups is studied. A condition is found when a loop, isotopic to a semisymmetric quasigroup, is a semisymmetric loop.

Keywords: Quasigroup, loop, isotopy, semisymmetric quasigroups.

1 Introduction

As a class, the semisymmetric quasigroups arguably warrant particular interest due to both their algebraic and combinatorial properties. Commutatively semisymmetric, i.e., totally symmetric quasigroups, have been studied for almost as long as quasigroups themselves [1].

In [2], a bijection between totally symmetric quasigroups and directed graphs meeting certain specifications is established. Alex W. Nowak in [3] studied modules over semisymmetric quasigroups.

2 Preliminaries

Definition 1. A binary groupoid $(Q, \cdot)$ is called a quasigroup, if for any ordered pair $(a, b) \in Q^2$, there exist unique solutions $x, y \in Q$ to the equations $x \cdot a = b$ and $a \cdot y = b$ ([4, 5].

Definition 2. A quasigroup $(K, \cdot)$ is called semisymmetric if it satisfies the semisymmetric law:

$$x(yx) = y, \tag{1}$$

$\forall x, y \in K$ [5].

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3 Results

Theorem 1. In any semisymmetric quasigroup $(K, \cdot)$, the following holds:

a) $e_x = f_x$,$\forall x \in K$, where $f_x x = x e_x = x$.

b) $x \cdot yx = xy \cdot x$, $L_x R_x = R_x L_x$, $L_x = R_x^{-1}$, $R_x = L_x^{-1}$, $xx = f_x = e_x, \forall x, y \in K$.

c) The equations $xa = b$ and $ay = b$ have, respectively, the solutions $x = ab$, $y = ba$.

Theorem 2. A quasigroup $(K, \circ)$, isotopic to a semisymmetric quasigroup $(K, \cdot)$, where $x \circ y = \gamma^{(-1)}(\alpha x \cdot \beta y)$ is a semisymmetric quasigroup if and only if $T = (\gamma\alpha^{-1}, \alpha\beta^{-1}, \beta\gamma^{-1})$ is an autotopy of the quasigroup $(K, \cdot)$.

Example 1. We define the semisymmetric quasigroup $(K, \cdot)$ using the Cayley table. Let $K = \{1, 2, 3, 4\}$.

$\cdot$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

In the quasigroup $(K, \cdot)$, the following holds: 1) $x \cdot xy = y$, 2) $x(yz \cdot t) = t(xz \cdot y)$, 3) $x^2 = x$, 4) $x \cdot xy = yx$, $yx \cdot x = xy$ (left and right Stein identities), 5) $x \cdot yz = xy \cdot xz$, $yz \cdot x = yx \cdot zx$ (distributive laws), 6) $x \cdot yx = xy \cdot x$ (elastic law), 7) $xy \cdot uv = xu \cdot yv$ (medial law). These examples show the way in which semisymmetric quasigroups can be studied.

Example 2. We define a loop $(K, \circ)$, isotopic to the quasigroup $(K, \cdot)$, where the isotopy is of the form $x \circ y = R_4^{-1} x \cdot L_2^{-)} y$. $(K, \circ)$ is an abelian group, $x \circ x = 1$.

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Theorem 3. A loop $(K, \circ)$, isotopic to a semisymmetric quasigroup $(K, \cdot)$, where the isotopy is given by the equality $x \circ y = R_a^{-1}x \cdot L_b^{-1}y$, $\forall x, y, a, b \in K$, a, b – are any fixed elements, will be a semisymmetric loop if and only if, $\forall x, y \in K$, the equality

$$b(xa \cdot y) = x(ya \cdot b), \tag{2}$$

holds in $(K, \cdot)$.

Then, naturally, the problem arises to study a semisymmetric quasigroup $(K, \cdot)$ with the identity

$$x \cdot (yz \cdot t) = t \cdot (xz \cdot y), \tag{3}$$

$\forall x, y, z, t \in K$.

The semisymmetric quasigroup $(K, \circ)$ from Example 2, satisfies the identity 3.

Theorem 4. A semisymmetric quasigroup $(K, \cdot)$, in which the identity 3 is satisfied, is isotopic to the abelian group $(K, \circ)$, where $x \circ x = e$, $\forall x \in K$, the element e is the identity element of the group $(K, \circ)$, and the isotopy is of the form:

$$x \circ y = R_a^{-1} \cdot L_b^{-1}y, \tag{4}$$

$\forall x, y \in K$.

4 Conclusion

In this paper, some properties of semisymmetric quasigroups are studied.

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