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Combinatorial Symmetry of Twisty Puzzles and Twist Axes of Order 4, 6, 8, 9, 10 and 11

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Abstract

This paper introduces the notion of conbinatorial symmetry of twisty puzzles, presents geometric principles of searching for twisty puzzles with sun mechanism. Additionally, some geometrical and combinatorial properties of such puzzles and their polyhedra are discussed.

Keywords: combinatorial symmetry, twisty puzzles, sun mechanism.

1 Introduction

Ernő Rubik created the puzzle known today as Rubik's Cube to teach his students about algebraic groups. Today, there are many puzzles similar to Rubik's Cube, which are generally named *twisty puzzles*. Such puzzles have usually symmetric form. However, the group of permutations plays much more important role in twisty puzzles. Still the moves of a twisty puzzle have clear geometric interpretation and thus have much in common with puzzle symmetry.

There are two important differences between symmetry and combinatorics of twisty puzzles. First, their moves are twists of one part relative to another rather than rotations. And unlike rotations, the composition of twists is usually not a twist. So, the permutation group of twisty puzzles is richer than their symmetry group. Second, for physical realization of twisty puzzles the exact geometry of parts is not necessary. The puzzles with approximate geometry are named *fudged*.

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If the difference between the exact geometric quantity, say angle, and physically possible is of the order of magnitude of parts tolerance, the puzzle can be constructed.

To emphasize the similarity with symmetry, consider *combinatorial symmetry.*

Definition. *Combinatorial symmetry of a twisty puzzle is the group of moves (the orbit) of its individual parts including all different ways a part can be placed.*

The order of combinatorial symmetry group of each part is usually as large as the order of even symmetry of puzzle polyhedron. However, the combinatorial symmetry can be greatly enriched by introducing partial twists. These twists change the puzzle shape, such puzzles are named *jumbling.* One way to achieve this is to allow half-twists of puzzle faces, such puzzles are named *sun* twisty puzzles. Sun mechanism doubles the order of twist axes.

2 Main results

The article [1] presents two principles that help finding all possible sun twisty puzzles:

Result 1: *The polyhedra suitable for sun twisty puzzles are dual to those which have all faces regular (not necessarily equal) and whose all vertex defects are equal.* They include regular solids, some Catalan solids, bipyramids, trapezohedra and certain Johnson solids duals. Some examples of puzzles with shape from the last category are Cullinan (truncated pentagonal bipyramid), Big Dipper (trapezo-rhombic dodecahedron) and Cygnus (pseudo-deltoidal icositetrahedron).

Such polyhedra have all symmetric vertices. All described until now sun puzzles are based only on polyhedra with 3-fold and 4-fold vertices. Sun puzzles have clusters of four 3-fold vertex pieces in place of 4-fold vertices. If polyhedra with 5-fold vertices can be considered for sun puzzles, they would form 5-fold faces, so their 5-fold vertices would be truncated. Puzzles with only standalone 3-fold vertices have exact geometry, with clusters of 4 vertices have fudged geometry.

Result 2: *When the puzzle is based on polyhedron with only 3-fold and 4-fold vertices, its number of faces f depends only on twist axes order n as:*

$$f = \frac{24}{12 - n}. \tag{1}$$

Despite the fact sun puzzles were initially introduced by doubling the axes order, there exist also sun puzzles with odd axes order. Table 1 presents integer solutions of (1).

Table 1. Face number f of puzzle as function on axes order n

n	f	Complexity c	Examples and comments
0	2	–	No obvious geometrical realization
4	3	1/2	Spherical trigonal prismoid
6	4	1	Tetrahedron, tetragonal prism
8	6	2	Cube, trigonal bipyramid
9	8	3	Octahedron, tetragonal trapezohedron
10	12	5	Dodecahedron, rhombododecahedron, truncated pentagonal bipyramid, trapezo-rhombic dodecahedron
11	24	11	Pentagonal icositetrahedron, deltoidal icositetrahedron, pseudo-deltoidal icositetrahedron
12	∞	–	2D puzzles

3 Conclusion and further work

The attempt to enumerate all sun twisty puzzles together with their geometrical and combinatorial properties are approached in [2].

The inner cut of vertex pieces forms always almost right angle. In the cluster of four vertices, the inner edges of adjacent parts are almost parallel and form almost regular pattern. Are these coincidence or there are geometrical properties that ensure these aspects?

If we divide all sun puzzles in categories by axes order then inside each group, the following numbers seem to be invariant:

Table 2. Apparent invariants of each group of sun puzzles

Invariant	Description and comments
n	Axes order
f	Number of faces
$v + 2e$	Number of vertex parts plus double of number of edges
$v_4 + e$	Number of 4-fold vertices plus number of edges

The last two invariants from Table 2 above are also related with complexity from Table 1 as:

$$c = \frac{v + 2e}{16} = \frac{v_4 + e}{6}. \tag{2}$$

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References

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