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Large-time behavior of the difference of solutions of two evolution equation

Andrei Perjan, Galina Rusu

Abstract

In a real Hilbert space H we consider a linear self-adjoint positive definite operator $A : V = D(A) \subset H \to H$ and investigate the behavior of the difference $u - v$ of solutions to the problems

$$\begin{cases} u''(t) + u'(t) + Au(t) = f(t), \quad t > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

$$\begin{cases} v'(t) + Av(t) = f(t), \quad t > 0, \\ v(0) = u_0, \end{cases}$$

where $u_0, u_1 \in H, f : [0, +\infty) \to H$.

Keywords: large-time behavior, abstract first order differential equation, abstract second order differential equation, a priori estimate.

Let H be a real Hilbert space endowed with the scalar product $(\cdot, \cdot)$ and the norm $|\cdot|$, and V be a real Hilbert space endowed with the norm $||\cdot||$, densely and continuously embedded in H i. e. there exists $\gamma_0 > 0$ such that

$$\gamma_0 |u| \leq ||u||, \quad \forall u \in V. \tag{\textbf{HV}}$$

Let $A : V \subset H \to H$ be a linear, self-adjoint and positive definite operator, i.e. there exists $\gamma > 0$ such that

$$(Au, u) \geq \gamma_1 ||u||^2, \quad \forall u \in V. \tag{\textbf{HA}}$$

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Consider the following Cauchy problems:

$$\begin{cases} u''(t) + u'(t) + Au(t) = f(t), \quad t > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{1}$$

$$\begin{cases} v'(t) + Av(t) = f(t), \quad t > 0, \\ v(0) = u_0, \end{cases} \tag{2}$$

where $u_0, u_1 \in H$, $f : [0, +\infty) \to H$.

We investigate the behavior of the difference $u - v$ of solutions to the problems (1) and (2) when $t \to +\infty$. The main result is established in Theorem 3. This result improves the results from [2] and [3] in the sense that we skip the condition of separability of the space H and take a nonzero right hand part therm.

For $k \in \mathbb{N}^\star$, $1 \leq p < +\infty$, $(a, b) \subset (-\infty, +\infty)$ and Banach space X we denote by $W^{k,p}(a, b; X)$ the Banach space of all vectorial distributions $u \in D'(a, b; X)$, $u^{(j)} \in L^p(a, b; X)$, $j = 0, 1, \ldots, k$, endowed with the norm $\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)}\right)^{1/p}$. For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$, we define the Banach space $W^{k,p}_s(a, b; X) = \{f : (a, b) \to H; f^{(l)}(\cdot)e^{s\,t} \in L^p(a, b; X),\ l = 0, \ldots, k\}$, with the norm $\|f\|_{W^{k,p}_s(a,b;X)} = \|fe^{s\,t}\|_{W^{k,p}(a,b;X)}$.

The results concerning the solvability of problems (1) and (2) are established in the following two theorems.

Theorem 1 [1]. *Let $t > 0$. Let us assume that the conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}(0, t; H)$, then there exists the unique function $u \in W^{2,\infty}(0, t; H) \bigcap W^{1,\infty}(0, t; V)$, (strong solution) which satisfies the equation a.e. on $(0, t)$ and the initial conditions from* (1).

Theorem 2 [1]. *Let $t > 0$. Let us assume that the conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$ and $f \in W^{1,1}(0, t; H)$, then there exists the unique function $v \in W^{1,2}(0, t; V)$ which satisfies a. e. on $(0, t)$ the equation and the initial conditions from* (2).

Lemma 1. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$ and $f \in W^{1,1}(0, t; H)$, for every $t > 0$, then there exists the*

constant $C > 0$ such that for the strong solution v to the problem (2) the following estimate is valid:

$$|v(t)| \le C(\gamma)\,\Big(|u_0| + \int_0^t e^{\gamma\tau}|f(\tau)|d\tau\Big)e^{-\gamma t}, \forall t \ge 0,\ \gamma = \gamma_0\,\gamma_1.$$

Lemma 2. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}(0,t;H)$, for every $t > 0$, then there exists the constant $C = C(\gamma) > 0$ and $\delta = \delta(\gamma) \in (0,1/2)$, such that for the strong solution u to the problem (1) the following estimate is valid:*

$$|u'(t)| + |A^{1/2}u(t)| \le C\,\Big(||u_0|| + |u_1| + \int_0^t e^{\delta\tau}|f(\tau)|d\tau\Big)e^{-\delta\, t}, \forall t \ge 0.$$

In what follows, denote by

$$K(t,\tau) = \frac{1}{2\sqrt{\pi}}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau) - 2K_3(t,\tau)\Big),$$

$K_1(t,\tau) = \exp\Big\{\frac{3t-2\tau}{4}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{t}}\Big)$, $K_2(t,\tau) = \exp\Big\{\frac{3t+6\tau}{4}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{t}}\Big)$, $K_3(t,\tau) = \exp\big\{\tau\big\}\lambda\Big(\frac{t+\tau}{2\sqrt{t}}\Big)$, $\lambda(s) = \int_s^\infty e^{-\eta^2}d\eta$.

The properties of kernel $K(t,\tau)$ are collected and proved in [4] and they allow to obtain the following lemma.

Lemma 3. *Assume that conditions* **(HV)** *and* **(HA)**. *If $u_0 \in V$, $u_1 \in H$ and $f \in W^{1,1}_\alpha(0,\infty;H)$, with some $\alpha > 0$, then the function*

$$w(t) = \int_0^\infty K(t,\tau)\,u(\tau)\,d\tau,$$

where u is the strong solution to the problem (2), verifies the system

$$\begin{cases} w'(t) + Aw(t) = F_0(t), \quad t > 0, \quad in \quad H, \\ w(0) = \displaystyle\int_0^\infty e^{-\tau}\,u(2\tau)\,d\tau, \end{cases}$$

$$F_0(t) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4}\Big\}\lambda\Big(\sqrt{t}\Big) - \lambda\Big(\frac{1}{2}\sqrt{t}\Big)\Big]u_1 + \int_0^\infty K(t,\tau)\,f(\tau)\,d\tau.$$

The main result of the paper is presented in the following theorem and it follows from the Lemmas 1-3.

Theorem 3. *Assume that conditions* **(HV)** *and* **(HA)** *are fulfilled. If* $u_0 \in V$, $u_1 \in H$ *and* $f \in W^{1,1}_{\alpha}(0,\infty;H)$, *with some* $\alpha > 0$, *then there exists the constant* $C(\gamma,\alpha) > 0$ *and* $\delta(\gamma,\alpha) > 0$, *such that for the strong solutions* u *and* v *to the respective problems (1) and (2) the estimate*

$$|u(t) - v(t)| \leq C\Big(||u_0|| + |u_1| + ||f||_{W^{1,1}_{\alpha}(0,\infty;H)}\Big)e^{-\delta t}, \quad \forall t \geq 0,$$

is true.

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Andrei Perjan[1], Galina Rusu[2]

[1]Moldova State University
Email: aperjan1248@gmail.com

[2]Moldova State University
Email: rusugalinamoldova@gmail.com