

Experimental results on the number of perfect matchings in graphs of Latin rectangles

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Abstract. Latin rectangles, perfect matchings, and 1-factorizations of regular bipartite graphs are closely related through combinatorial constructions. We give a survey of classical results concerning the existence and enumeration of perfect matchings in bipartite graphs, including König's and Hall's theorems, as well as lower bounds provided by van der Waerden, Voorhoeve, and Schrijver. We discuss the connection between Latin rectangles and k -factorizations of regular bipartite graphs, highlighting that every $k \times n$ Latin rectangle corresponds to a k -regular bipartite graph with $2n$ vertices. In 1986, Bollobás and McKay presented the asymptotic estimate for the expected number of perfect matchings in random k -regular bipartite graphs. Motivated by this probabilistic result, we performed numerical experiments on the average number of perfect matchings in random 3-regular bipartite graphs arising from Latin rectangles. Our computational results, based on calculating matrix permanents for subsets of Latin squares, show strong agreement with the Bollobás-McKay asymptotic formula.

1. Introduction

This paper explores the connection between Latin squares, Latin rectangles, and perfect matchings of bipartite graphs, with a primary focus on bicubic graphs, which are bipartite graphs where each vertex in one vertex set is connected to three vertices in the other set. Classical results by König, Hall, and van der Waerden laid the foundation for this field, while modern advancements have provided sharper lower bounds on the number of perfect matchings. The result by Schrijver established a tight lower bound on the number of perfect matchings in k -regular bipartite graphs. Furthermore, probabilistic approaches have improved these deterministic results.

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In addition, Bollobás and McKay provided asymptotic estimates for the expected number of perfect matchings in random k -regular bipartite graphs, demonstrating that these graphs typically have a large number of perfect matchings.

A k -factorization of a regular bipartite graph corresponds to a $k \times n$ Latin rectangle, and algorithms to find perfect matchings play a central role in their generation. For any $k \times n$ Latin rectangle, one can associate a k -regular bipartite graph on $2n$ vertices in a unique way.

In this article, we begin by outlining key definitions and classical results related to Latin rectangles and perfect matching in bipartite graphs. In Section 2, we formalize the connection between k -regular bipartite graphs through 1-factorizations, providing illustrative examples. Section 3 surveys foundational and modern lower bounds on the number of perfect matchings, including those by van der Waerden, Voorhoeve, and Schrijver. In Section 4, we shift our focus to probabilistic methods, highlighting the asymptotic results of Bollobás and McKay. We conclude this section with our numerical experiments on the average number of perfect matchings in random 3-regular bipartite graphs, which support the theoretical estimates.

2. Preliminaries

A *Latin square* of size n is an $n \times n$ array filled with n symbols, where each symbol is used exactly once in every row and column. A *Latin rectangle* is a $k \times n$ array with n symbols where $k \leq n$, any symbol occurs exactly once in each row and at most once in each column. Two Latin rectangles are *isotopic (or similar)* if one can be obtained from the other by permuting the rows, columns, and symbols. (See [17].)

A *graph* is an ordered pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. In our study, we use G as a *simple graph*: a graph with no multiple edges and no loops. The *degree* of a vertex $v \in G$ of a graph G is the number of edges incident to v . The graph $G = (V, E)$ is called *regular* if each vertex v of G has the same degree. A regular graph G with degree 3 is also called a *cubic graph*.

A graph $G = (U_1 \cup U_2, E)$ is a *bipartite graph* if its set of vertices can be partitioned into two disjoint subsets U_1 and U_2 so that each edge of the graph has one end in U_1 and the other end in U_2 . A bipartite cubic graph is called *bicubic graph*. Let $(U_1 \cup U_2, E)$ be a bipartite graph with parts U_1, U_2 , $\ell = |U_1|$, $r = |U_2|$. Its *bi-adjacency matrix* B is a $\ell \times r$ $(0, 1)$ -matrix

where $b_{ij} = 1$ if and only if $(u_i, u'_j) \in E$. Notice that B depends on the labeling of the vertices of G ; changing the labeling means a permutation of the rows and columns of B .

A k -factor of a graph G , where k is a positive integer, is a set of edges of G such that each vertex of G is incident with exactly k of them. A 1-factor of a graph G is known as a *perfect matching* of the graph. According to König's theorem, the set of edges of a k -regular bipartite graph can be partitioned into k disjoint perfect matchings. This partition is a *1-factorization* of a k -regular bipartite graph.

The *permanent* of an $n \times n$ matrix $A = (a_{ij})$ is defined as

$$\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i,\sigma(i)}$$

A matrix is a $(0, 1)$ -matrix if all its entries are from $\{0, 1\} \subseteq \mathbb{R}$. The *permanent* of a $(0, 1)$ matrix is related to the number of perfect matchings of a bipartite graph.

Theorem 2.1. *Let $G = (U_1 \cup U_2, E)$ be a bipartite graph, $|U_1| = |U_2|$, and B its bi-adjacency matrix. The number of perfect matchings of G is $\text{Per}(B)$.*

The relationship between Latin rectangles and perfect matchings is a fascinating topic of combinatorics and graph theory. Algorithms designed to find perfect matchings in bipartite graphs can be used to construct Latin rectangles and vice versa. Roughly speaking, by interpreting Latin squares in terms of edge-colored bipartite graphs, Latin rectangles, and ordered 1-factorizations of regular bipartite graphs turn out to be the same thing.

We refine this loose formulation for greater precision. Let $A = (a_{ij})$ be a $k \times n$ Latin rectangle with entries $1, \dots, n$. Let $S = \{s_1, \dots, s_n\}$, $C = \{c_1, \dots, c_n\}$ be disjoint sets; we can think of them as the set of symbols and the set of columns of A . For $m \in \{1, \dots, k\}$, define

$$E_m = \{\{s_i, c_j\} \mid a_{mj} = i, \text{ that is, row } m \text{ and column } j \text{ has symbol } i \text{ in } A\}.$$

Write $E = \cup_{m=1}^k E_m$, and $G = (S \cup C, E)$. Then G is a k -regular bipartite graph, and $\mathcal{E} = (E_1, \dots, E_m)$ is an ordered 1-factorization of G .

Conversely, let \mathcal{F} be a 1-factorization of a k -regular bipartite graph $G = (U_1 \cup U_2, E)$. First, we have to label the elements, U_1, U_2 with $1, \dots, n$, and the 1-factors in \mathcal{F} with $1, \dots, k$: $U_1 = \{u_{11}, \dots, u_{1n}\}$, $U_2 = \{u_{21}, \dots, u_{2n}\}$, $\mathcal{F} = \{F_1, \dots, F_k\}$. Then, we write $a_{ij} = m$ if and only if the vertices

u_{1j}, u_{2m} are adjacent and $\{u_{1j}, u_{2m}\} \in F_i$. The matrix $A = (a_{ij})$ is a $k \times n$ Latin rectangle.

Change in the labeling of U_1, U_2 or \mathcal{F} corresponds to permuting the columns, symbols, and rows of A . Hence, different Latin rectangles associated with G are isotopic, and isotopic Latin rectangles correspond to isomorphic 1-factorizations.

For example, the 3×7 Latin rectangle

$$A = \begin{bmatrix} 6 & 2 & 1 & 5 & 3 & 7 & 4 \\ 3 & 6 & 2 & 4 & 1 & 5 & 7 \\ 1 & 4 & 5 & 7 & 2 & 6 & 3 \end{bmatrix} \quad (1)$$

corresponds to the edge-colored bipartite cubic graph G given in Figure 1. The colors of the edges form the factors of the 1-factorization.

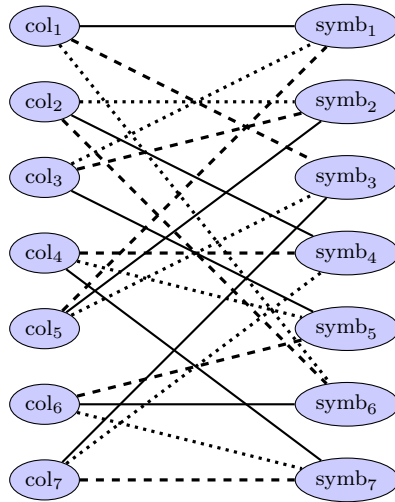


Figure 1: 3-regular bipartite graph constructed from a 3×7 Latin rectangle

The bi-adjacency matrix of G is

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

The entry 1 shows that there is an edge between the vertices c_i and s_j , where $1 \leq i, j \leq 7$, and the entry 0 tells us that there is no edge between the vertices c_i and s_j , where $1 \leq i, j \leq 7$. As we can see in the matrix B , each row and column sums up to 3. Generally, B is an $n \times n$ $(0, 1)$ -matrix so that each row and column sum equals k .

3. The minimum number of perfect matchings

In the early 20th century König’s theorem [8] established the relation between the maximum cardinality of a matching in a bipartite graph and the minimum cardinality of a vertex cover. This result paved the way for the development of algorithms to find perfect matchings. In the mid-20th century, Philip Hall [5] proved Hall’s Marriage Theorem, which provides a necessary and sufficient condition for the existence of perfect matchings in bipartite graphs, fundamentally linking combinatorics and set theory. These results are constructive; their augmenting paths lead to a practical algorithm to find a perfect matching in bipartite graphs. In general, the Hungarian Algorithm ([9, 12]) determines a minimum-weight perfect matching in the edge-weighted complete bipartite graph.

In this section, we are interested in the number of perfect matchings in regular bipartite graphs. For any k, n , let $\mathcal{B}(n, n; k\text{-reg})$ be the set of all k -regular bipartite graphs with n vertices in each part upto isomorphism and let $p(k, n) = \min X(G)$, where $G \in \mathcal{B}(n, n; k\text{-reg})$ and $X(G)$ be the number of perfect matchings in any k -regular bipartite graph with $2n$ vertices. Trivially, $p(1, n) = 1$.

In a 2-regular bipartite graph, each connected component is an even cycle. If a 2-regular graph G has r connected components (or even cycles), then the number of perfect matchings of G is 2^r . Hence, $p(2, n) = 2$. The graphs with exactly two perfect matchings are the cycle graphs on $2n$

vertices. For the $2 \times n$ Latin rectangle A , the associated 2-regular bipartite graph has exactly two perfect matchings if and only if $\alpha\beta^{-1}$ is a cyclic permutation on n points, where α, β are the permutations given by the rows of A .

Doubly stochastic matrices are square matrices of nonnegative real numbers where each row and column sums to 1. If B is the bi-adjacency matrix of a k -regular bipartite graph G , then $\frac{1}{k}B$ is doubly stochastic. Hence, by Theorem 2.1, the value $p(k, n)$ is related to the permanent of doubly stochastic matrices. The Dutch mathematician B. L. van der Waerden proposed the following conjecture in 1926.

Conjecture 1 (van der Waerden 1926 [15]). *The permanent of any doubly stochastic $n \times n$ matrix should be at least $n!/(n^n)$.*

Notice that with the all-one $n \times n$ matrix J , we have $\text{Per}(\frac{1}{n}J) = n!/(n^n)$, hence the lower bound is sharp. As for any bi-adjacency matrix B , $\frac{1}{k}B$ is doubly stochastic, van der Waerden's conjecture would imply

$$p(k, n) \geq \frac{n!k^n}{n^n} \approx \frac{\sqrt{2\pi n} \cdot (n/e)^n \cdot k^n}{n^n} = \sqrt{2\pi n} \left(\frac{k}{e}\right)^n. \quad (3)$$

In 1981, Falikman [4] proved van der Waerden's conjecture, and Ego-rychev [2] refined his result in the same year. Several years earlier, in 1968, Erdős and Rényi [3] extended this line of inquiry by conjecturing bounds for the permanent of matrices with prescribed row and column sums.

Conjecture 2 (Erdős and Rényi 1968 [3]). *For every $\varepsilon > 0$, there exists an $n \times n$ nonnegative integer matrix with all row and column sums equal to k , such that the permanent of the matrix is at least $(1 + \varepsilon)^n$ for sufficiently large n .*

In 1979, Voorhoeve [16] resolved the case where $k = 3$. Voorhoeve's findings demonstrated that in the Erdős-Rényi conjecture, ε can be as small as $1/3$.

Theorem 3.2 (Voorhoeve 1979 [16]). *A 3-regular bipartite graph with $2n$ vertices possesses a minimum of $(4/3)^n$ perfect matchings.*

The method used by Voorhoeve for the proof of case $k = 3$ is considered elementary and is explained in the book by Lovász and Plummer [10]. With considerable effort, this idea was generalized for any value of k by Schrijver [13] in 1998.

Theorem 3.3 (Schrijver 1998 [13]). *The lower bound for the perfect matchings of a k -regular bipartite graph with $2n$ vertices is*

$$((k - 1)^{k-1} / k^{k-2})^n$$

For large k (and n), we have

$$\frac{(k - 1)^{k-1}}{k^{k-2}} = \left(\left(1 - \frac{1}{k} \right)^k \right)^{\frac{k-1}{k}} \cdot k \approx \frac{k}{e}.$$

Hence, for large k , Schrijver’s and van der Waerden’s lower bounds differ only in the factor $\sqrt{2\pi n}$. The best bound known on the 1-factorizations of a k -regular bipartite graph follows Schrijver’s theorem.

Corollary 3.4 (Schrijver 1998 [13]). *The number of ordered 1-factorizations of a k -regular bipartite graph is at least*

$$\left(\frac{(k!)^2}{k^k} \right)^n$$

This expression is also a lower bound for the number of $k \times n$ Latin rectangles.

Proof. As we know, each Latin rectangle A can be associated with a k -regular bipartite graph, and the rows of A determine the 1-factorizations of the graph. The problem of counting the number of Latin rectangles of a given order is equivalent to counting the number of ordered 1-factorizations in the $K_{n,n}$ graph.

Let G be a k -regular bipartite graph on $2n$ vertices. We can choose a perfect matching F_1 in at least $p(k, n)$ ways. A disjoint perfect matching F_2 can be chosen from $G - F_1$ in at least $p(k - 1, n)$ ways, etc. Hence, the number of ordered 1-factorizations is at least

$$\prod_{\ell=1}^k p(\ell, n) \geq \prod_{\ell=2}^k \left(\frac{(\ell - 1)^{\ell-1}}{\ell^{\ell-2}} \right)^n.$$

By straightforward calculation,

$$\begin{aligned} \prod_{\ell=2}^k \frac{(\ell - 1)^{\ell-1}}{\ell^{\ell-2}} &= 1 \cdot \frac{2^2}{3^1} \cdot \frac{3^3}{4^2} \cdot \frac{4^4}{5^3} \cdots \frac{(k - 1)^{k-1}}{k^{k-2}} \\ &= 1 \cdot \frac{2^2}{1} \cdot \frac{3^2}{1} \cdot \frac{4^2}{1} \cdots \frac{(k - 1)^2}{k^{k-2}} \\ &= \frac{(k!)^2}{k^k}. \end{aligned} \quad \square$$

4. Average and exact number of perfect matchings

In addition to the deterministic lower bounds for the number of perfect matchings in k -regular bipartite graphs, probabilistic methods have also been used to study the average behavior of such graphs. In a remarkable result, Bollobás and McKay [1] provided an asymptotic estimate for the expected number of perfect matchings in a random k -regular bipartite graph with $2n$ vertices. Their result shows that, for sufficiently large n , the expected number of perfect matchings grows exponentially with n for a fixed degree of regularity $k \geq 3$. This estimate complements the earlier deterministic bounds and highlights the abundance of perfect matchings in random regular bipartite graphs.

Theorem 4.5 (Bollobás and McKay 1986 [1]). *Let $k = k(n)$ be an integer satisfying $3 \leq k \leq (\log n)^{1/3}$, and let $\mathcal{B}(n, n; k\text{-reg})$ denote the probability space consisting of all k -regular bipartite graphs with n vertices in each part. For a graph G in this space, let $X(G)$ denote the number of perfect matchings in G . Then the expected number of perfect matchings in G is*

$$\mathbb{E}[X] = (1 + o(n^{-3/4})) \cdot e^{-1/2} \cdot \left(\frac{2\pi(k-1)n}{k} \right)^{1/2} \cdot \left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n$$

For $k = 3$, this result means that for sufficiently large n , the average number of perfect matchings is approximately

$$\sqrt{\frac{4\pi n}{3e}} \left(\frac{4}{3} \right)^n. \quad (4)$$

If $k = 3$, then the condition on n is $n > 2^{27} \approx 1.34 \times 10^8$.

For some specific Latin rectangles, we know the exact value of the number of perfect matchings in the associated bipartite graph.

Theorem 4.6 (Minc 1964 [11]). *Let $n \geq 3$ be an integer, and define the $3 \times n$ circulant Latin rectangle*

$$A = \begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & \cdots & 1 \\ 3 & 4 & \cdots & 2 \end{bmatrix}.$$

Then, the number of perfect matchings of the associated bicubic graph G is

$$X(G) = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n + 2.$$

We conclude this section with a straightforward observation on disconnected bicubic graphs. Recall that the disjoint union (also called graph sum) of bipartite graphs $G_i = (C_i \cup S_i, E_i)$, $i = 1, 2$, is

$$G_1 \oplus G_2 = (C_1 \cup C_2 \cup S_1 \cup S_2, E_1 \cup E_2),$$

with parts $C_1 \cup C_2$ and $S_1 \cup S_2$. Clearly, we have $X(G_1 \oplus G_2) = X(G_1)X(G_2)$ for the number of perfect matchings.

Proposition 4.7. *Let A be a $3 \times n$ Latin rectangle, and let $\alpha_1, \alpha_2, \alpha_3$ be the permutations corresponding to the rows of A . Assume that for $i, j \in \{1, 2, 3\}$, the permutations $\alpha_i \alpha_j^{-1}$ are commuting involutions. Then the associated bicubic graph G has $3^{n/2}$ perfect matchings.*

Proof. Let A' be the $3 \times n$ Latin rectangle obtained from A by applying α_1^{-1} to the columns. Then the rows of A' are $1, \alpha_2 \alpha_1^{-1}, \alpha_3 \alpha_1^{-1}$, and the bicubic graphs B, B' of A and A' are isomorphic. Moreover, B' is the disjoint union of bicubic graphs $D_1, \dots, D_{n/4}$ on 8 points. Such a graph D_i is $K_{4,4}$ minus a perfect matching, which means that it has $X(D_i) = 9$ perfect matchings. We obtain $X(B') = 9^{n/4}$. □

We remark that $\frac{1+\sqrt{5}}{2} \approx 1.618$ and $\sqrt{3} \approx 1.732$, hence the number of perfect matchings in Minc's Theorem 4.6 and Proposition 4.7 is significantly higher than that for random 3-regular bipartite graphs (4).

5. Experimental results on the average number of perfect matchings

To complement the theoretical result stated in Theorem 4.5, we conducted numerical experiments to investigate the average number of perfect matchings in random 3-regular bipartite graphs. Since, as in Theorem 2.1, the number of perfect matchings in a bipartite graph equals the permanent of its biadjacency matrix, we approached this by computing the permanents of the corresponding matrices. For the permanent, we used a straightforward recursive algorithm, which works for sparse matrices. For small values of n , we double-checked the results against the permanent functions in SageMath [14]. This approach was computationally efficient due to sparsity and the manageable size of $3 \times n$ matrices. For random Latin squares of orders

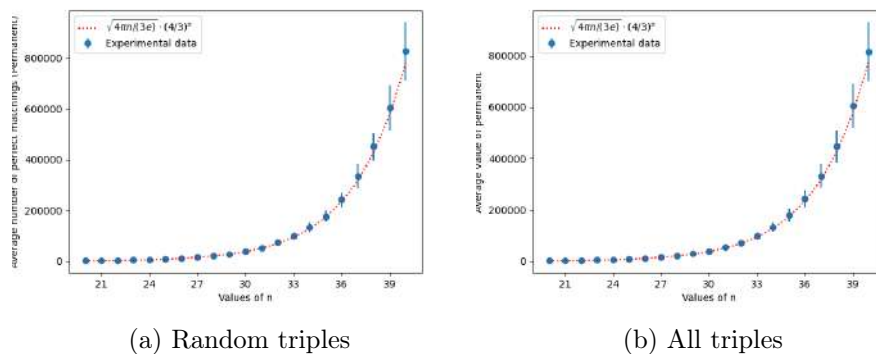


Figure 2: The average number of perfect matchings for triples of rows in random Latin squares

ranging from 10 to 40, we made a C++ implementation of the Jacobson-Matthews method [7], adapted from the Go language program [6] by Paul Hankin.

In the first experiment, for each order $n \in \{10, \dots, 40\}$, we generated 100 random Latin squares. For each square, we constructed the $n \times n$ (0,1)-matrix A associated to the first three rows, and computed its permanent. Figure 2a displays our experimental results.

In the second experiment, for five random Latin squares of order 20 to 40, we performed exhaustive analysis by calculating the permanent of every possible combination of three rows. The resulting data were aggregated by computing mean permanent values for each set of experiments. A graphical visualization, Figure 2b, was prepared to illustrate the growth trends and distribution patterns across different orders and trial sizes. In both plots, the dotted line shows the theoretical values of the Bollobás-McKay estimate.

Finally, we were interested in the $o(n^{-3/4})$ term of the Bollobás-McKay formula. For our data, we computed the $z(n) = \frac{y_{\text{measured}}(n)/y_{\text{estimated}}(n)-1}{n^{-3/4}}$. As shown in Figure 3, the values of $z(n)$ seem to increase in this range.

Conclusion

Our investigation integrates the classical combinatorial theory with experimental analysis, confirming that the average number of perfect matchings in various classes of 3-regular bipartite graphs grows rapidly with the graph

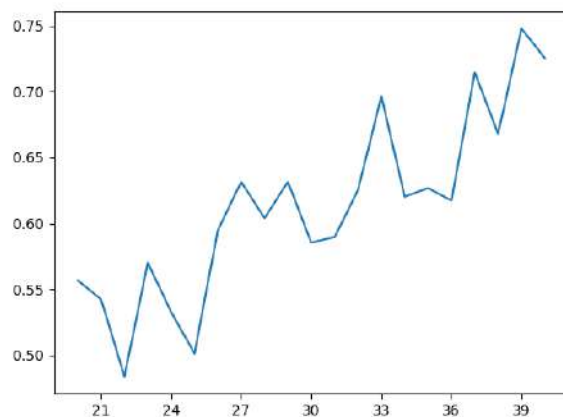


Figure 3: The quotient $\frac{y_{\text{measured}}(n)/y_{\text{estimated}}(n)-1}{n^{-3/4}}$

size. Even for small values of n , the experimental results show strong agreement with the asymptotic estimate given by the Bollobás-McKay formula. Moreover, special constructions such as cyclic permutations or commuting involutions demonstrate how specific structural properties can lead to significantly higher than average matching counts. These observations open up further directions on extremal configurations and the development of efficient enumeration methods for perfect matchings.

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