The Regular Perturbations of the Nonviscous Semilinear 1D Canh-Hilliard Equation

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Abstract

We study the behavior of solutions to the relaxation nonviscous semilinear one space dimension Canh-Hilliard equation when the relaxations tend to zero.

Keywords: Initial boundary problem, nonviscous semilinear Canh-Hilliard equation, regular perturbations.

1 Introduction

Let $I := (0, l) \subset \mathbb{R}$ and $L^2(I)$ be the usual real Hilbert space endowed with scalar product $(u, v) = \int_I u(x)\, v(x)\, dx$ and with the norm $||u||^2 = \int_I u^2(x)\, dx$. $H^k(I), k \in \mathbb{N}$ is the usual real Sobolev space endowed with the scalar product $(u, v)_{H^k(I)} = \int_I \sum_{s=0}^{k} u^{(s)}(x)\, v^{(s)}\, dx$. Denote by $V =: H_0^1(I)$ the closure of $C_0^\infty(I)$ in the norm of the space $H^1(I)$ which is endowed with the scalar product $(u, v) = \int_I u_x(x) \cdot v_x(x)\, dx$, and by $V' = H^{-1}(I)$ – the dual space for V. Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; X)$ the usual Sobolev spaces of the vectorial distributions $W^{k,p}(a, b; X) = \{f \in D'(a, b, X); f^{(l)} \in L^p(a, b; X), l = 0, 1, \ldots, k\}$ equipped with the norm $||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;X)} \Big)^{1/p}$.

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Consider the following abstract initial boundary value problem for the relaxation Canh-Hilliard equation, which in what follows will be called (P_ε):

$$\begin{cases} \varepsilon\, u_\varepsilon''(t) + u_\varepsilon'(t) + A\big(Au_\varepsilon(t) + a\, u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t),\ t > 0, \\ u_\varepsilon\big|_{t=0} = u_{0\,\varepsilon}, \quad u_\varepsilon'\big|_{t=0} = u_{1\,\varepsilon}, \end{cases} \tag{P_ε}$$

where $A : D(A) = H^2(I) \bigcap H_0^1(I) \subset L^2(I) \mapsto L^2(I), \quad A = -\frac{d^2}{dx^2}$. $B : D(B) \subseteq L^2(I) \to H^{-1}(I)$, $B(u) = b\,|u|^q\, u$, $u(t) =: u(\cdot, t)$, and by $'$ is denoted the derivative in t, $a, b \in \mathbb{R}$, and ε is a small positive parameter.

We will study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the unperturbed problem (P_0):

$$\begin{cases} v'(t) + A\left(Av(t) + a\, v(t) + B(v(t))\right) = f(t), \quad t > 0, \\ v\big|_{t=0} = u_0, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0$, where $u_{0\,\varepsilon} \to u_0$, $u_{1\,\varepsilon} \to u_1$ and $f_\varepsilon \to f$ in a certain sense, which will be specified below.

2 Existence and a *priori* estimates of solutions to the problems (P_ε) and (P_0)

We shall remind the definitions of solutions to the problems (P_ε) and (P_0) and also the existence theorems for solutions to these problems.

Definition 1. *Let $T > 0$, $f \in L^2(0, T; V')$ and $B : D(B) \subseteq L^2(I) \to V'$. We say a function $u_\varepsilon \in W^{2,2}\big(0, T; V'\big) \bigcap W^{1,2}\big(0, T; L^2(I)\big) \bigcap L^2\big(0, T; D(A) \bigcap D(B)\big)$ is a solution (quasi-strong solution) to the problem (P_ε) if u_ε satisfies the equation (P_ε) in V' a.e. $t \in (0, T)$ and the conditions from (P_ε).*

Definition 2. *Let $T > 0$, $f \in L^2(0, T; V')$ and $B : D(B) \subseteq L^2(I) \to V'$. We say a function $v \in L^2(0, T; V') \bigcap L^2(0, T; D(A) \cap$*

$D(B))$ *is a solution to the problem* (P_0) *if* v *satisfies the equation* (P_0) *in* V' *a.e.* $t \in (0,T)$ *and the conditions from* (P_0).

Using methods developed in [1], the following theorems are proved:

Theorem 1. *Let* $T > 0$ *and* $q \geq 0$. *If* $u_{0\,\varepsilon} \in H^3 \bigcap V$, $u_{1\,\varepsilon} \in V$, *and* $f_\varepsilon \in L^2(0,T;V')$. *Then there exits a unique solution* $u_\varepsilon \in W^{2,\infty}\big(0,T;V'\big) \bigcap W^{1,\infty}\big(0,T;V\big) \bigcap L^\infty\big(0,T;H^3 \bigcap V)\big)$ *to the problem* (P_ε).

Theorem 2. *Let* $T > 0$ *and* $q \geq 0$. *If* $u_0 \in H^4(I) \cap V$ *and* $f \in W^{1,1}(0,T;L^2(I))$. *Then there exits a unique solution to the problem* (P_0), *for which the estimate*

$$||v||_{C([0,t];L^2(I))} + ||v||_{L^2(0,t;H^2(I))} \leq C\,\tilde{M}_0(t), \quad \forall t \in [0,T],$$

is true with $C > 0$ *and* $\gamma > 0$, *depending on* a, b, q, l, *and* $\tilde{M}_0(t) = \Big(||u_0||_{H^2(I)} + ||f||_{L^2(0,t;L^2(I))}\Big)\, e^{\gamma\, t}$.

Lemma 1. *Suppose that* $q \geq 0$, $b > 0$ *and* $\mu = 1 + a\left(\frac{l}{\pi}\right)^2 > 0$. *If* $u_{0\,\varepsilon} \in H^3 \bigcap V$, $u_{1\,\varepsilon} \in V$, $f_\varepsilon \in L^2_{loc}(0,\infty;L^2(I))$, *then there exist the positive constants* $C = C(a,b,q,l)$ *such that for any solution* u_ε *to the problem* (P_ε), *the following estimate*

$$||u_\varepsilon(t)||^2_V + \frac{b}{q+2}\,||u_\varepsilon(t)|^{q+2}_{L^{q+2}(I)} \leq C\,\mathcal{M}_0(t), \quad t \geq 0,\ \varepsilon \leq 1$$

is valid, where $\mathcal{M}_0(t) = ||u_{0\,\varepsilon}||^2_V + ||u_{1\,\varepsilon}||^2 + ||A^{-1/2} f_\varepsilon||^2_{L^2(0,t;L^2(I))}$. *If in addition* $q \geq 1$, then there exist the positive constants $c_0 = c_0(b,q,\mu)$ and $C = C(b,q,\mu)$ such that

$$||u_\varepsilon(t)||_{H^2(I)} + ||u'_\varepsilon||_{L^2(0,t;I)} \leq C\,\mathcal{M}(t), \quad t \geq 0,\ \varepsilon \leq 1. \tag{1}$$

$$\mathcal{M}(t) = ||u_{0\,\varepsilon}||_{H^2(I)} + ||u_{1\,\varepsilon}||_V + ||f_\varepsilon||_{L^2(0,t;L^2(I))} + \exp\big\{c_0 \int_0^t \mathcal{M}_0^{2q}(\tau)\,d\tau\big\}.$$

3 The relationship between solutions to the problems (P_ε) and (P_0)

Now, we will give the relationship between solutions to the problems (P_ε) and (P_0) in the linear case, which was inspired by the work [3]. For each $\varepsilon > 0$, the kernel $K(t,\tau,\varepsilon)$ of transformation which realizes this connection is the following:

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\,\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3\,K_2(t,\tau,\varepsilon) - 2\,K_3(t,\tau,\varepsilon)\Big),$$

where $\lambda(s) = \int\limits_s^\infty e^{-\eta^2}\,d\eta$, $\quad K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t-2\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t-\tau}{2\sqrt{\varepsilon\,t}}\Big)$,

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t+6\,\tau}{4\,\varepsilon}\Big\}\,\lambda\Big(\frac{2\,t+\tau}{2\sqrt{\varepsilon\,t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon\,t}}\Big).$$

The main properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 2 ([2]). *The function $K(t,\tau,\varepsilon)$ is a solution to the problem*

$$\begin{cases} K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t>0, \quad \forall \tau>0,\\ \varepsilon\,K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0\\ K(0,\tau,\varepsilon = \dfrac{1}{2\,\varepsilon}\exp\Big\{-\dfrac{\tau}{2\,\varepsilon}\Big\}, \quad \forall \tau \geq 0, \end{cases}$$

from $C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$. This kernel possesses the following properties:

(i) $K(t,\tau,\varepsilon) > 0, \quad \displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,d\tau = 1, \quad \forall t \geq 0, \quad \forall \tau \geq 0;$

(ii) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2, and ε_0, all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}|t-\tau|^q\,d\tau \leq C_1\,e^{C_2 t}\,\varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t>0;$$

Theorem 3. ([2]) *Suppose that* $e^{\gamma t} f_\varepsilon \in L^\infty(0,\infty;L^2(I))$, $e^{\gamma t} u_\varepsilon \in W^{2,\infty}(0,\infty;L^2(I)) \cap L^\infty(0,\infty;V)$ *and* $e^{\gamma t} A u_\varepsilon \in L^{2,\infty}(0,\infty;V')$, *where* u_ε *is a solution to the problem* (P_ε). *Then, for* $0 < \varepsilon < (4\gamma)^{-1}$, *the function* $w_\varepsilon(t) = \int\limits_0^\infty K(t,\tau,\varepsilon) u_\varepsilon(\tau) d\tau$ *is a solution to the problem*

$$\begin{cases} w_\varepsilon'(t)) + a\, A w_\varepsilon(t) + A^2 w_\varepsilon(t) = F_0(t,\varepsilon), \quad \text{in } V', \quad \text{a.e.} \quad t \in [0,\infty), \\ w_{0\varepsilon} = \varphi_\varepsilon =: \int\limits_0^\infty e^{-\tau} u(2\,\varepsilon\,\tau) d\tau, \end{cases}$$

where $f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\,\exp\Big\{\frac{3\,t}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]$,

$F_0(t,\varepsilon) = f_0(t,\varepsilon)\, u_{1\varepsilon} + \int\limits_0^\infty K(t,\tau,\varepsilon)\, f_\varepsilon(\tau) d\tau$.

Moreover, $w_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,\infty;V)$.

4 The main results

Now, let us present the main result contained in the following theorem.

Theorem 4. *Let* $T > 0$, $q \geq 1$, $b > 0$ *and* $\mu > 0$. *If* $u_0, u_{0\varepsilon} \in H^4(I) \cap V$, $u_{1\varepsilon} \in V$ *and* $f, f_\varepsilon \in W^{1,p}(0,T;L^2(I))(p \geq 1)$, *then there exist constants* C, c_0, *and* ε_0, *depending on* p, q, μ *and* T, *such that*

$$||u_\varepsilon - v||_{C([0,T];L^2(I))} \leq C\left[\mathcal{M}^{q+1}(T)\,\varepsilon^\beta + D_\varepsilon\right] e^{c_0\,\mathcal{M}^{2\,q}(T)}, \quad \varepsilon \in (0,\varepsilon_0],$$

where u_ε *and* v *are solutions to the problems* (P_ε) *and* (P_0), *respectively,*

$$D_\varepsilon = ||u_{0\varepsilon} - u_0|| + ||f_\varepsilon - f||_{L^2(0,T;L^2(I))}. \tag{2}$$

$\mathcal{M}(T)$ *is defined in (1) and* $\beta = \begin{cases} 1/2, & \text{if } f = 0, \\ (p-1)/(2\,p), & \text{if } f \neq 0. \end{cases}$

The proof of this result is based on the following two estimates, which are obtained using Theorem 3 as well as Lemmas 1, 2, and 3:

$$||u_\varepsilon(t) - w_\varepsilon(t)|| \leq C\,\mathcal{M}(T)\,\varepsilon^{1/2}, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0],$$

$$||v(t) - w_\varepsilon(t)|| \leq C\left[\mathcal{M}^{q+1}(T)\,\varepsilon^{\beta} + D_\varepsilon\right] e^{c_0\,\mathcal{M}^{2\,q}(T)},\ t \in [0,T],\ \varepsilon \in (0,\varepsilon_0],$$

are true with $\mathcal{M}(T)$ from (1) and D_ε from (2).

5 Conclusion

The main result from Theorem 4 shows that the the small relaxations of IBV for nonviscous semilinear 1D space Canh-Hilliard equation are the regular character. Also we indicate a rate of speed of convergence of solutions of the perturbed problem to the corresponding solutions of the unperturbed problem when the relaxations tend to zero.

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