Quasigroups and Related Systems **34** (2026), 141 – 152

https://doi.org/10.56415/qrs.v34.11

Algebras of associative and autodistributive hyperoperations

Thodsaporn Kumduang

Abstract. This work continues the results established by W. A. Dudek and V. S. Trokhimenko [W. A. Dudek, V. S. Trokhimenko, Menger algebras of associative and self-distributive n-ary operations, Quasigroups Relat. Syst., 26 (2018), 45-52]. The first objective is to address a problem concerning superassociative operations. The second aim is to extend the study of Menger algebras of associative and autodistributive n-ary operations to the broader context of hypercompositional algebras.

1. Introduction

The representation of algebraic structures by functions plays a vital role in abstract algebra, as it enables the transformation of abstract operations into concrete, manipulable forms. This translation not only enhances clarity and understanding but also facilitates effective computation and supports practical applications. By modeling operations as functions-such as permutations, transformations, or endomorphisms, we gain valuable tools to analyze the behavior of algebraic structures, investigate their properties through concrete examples, and apply these insights to areas such as logic, theoretical computer science, and universal algebra. According to Cayley's theorem for semigroups, every semigroup can be embedded into a full transformation semigroup, where the semigroup operation is interpreted as the composition of functions. This functional representation renders abstract semigroup operations more accessible and computable, thereby enabling deeper structural analysis.

2020 Mathematics Subject Classification: 08A35, 20M75, 20N15

Keywords: Superassociativity, representation, Menger algebra

This research was supported by National Science, Research and Innovation Fund (NSRF) and Rajamangala University of Technology Rattanakosin (RMUTR) (Ref. No. FRB68018/2568).

More generally, a Menger algebra is an algebraic structure that extends the concept of a semigroup by considering multi-place (i.e., n-ary) operations rather than just binary ones. A representation of a Menger algebra typically involves mapping each element to an n-ary operation (function) on a nonempty set, such that the abstract Menger superposition corresponds to functional composition. This type of representation preserves a key property of Menger algebras known as superassociativity, an associative-like law applicable to the composition of multi-place operations. In recent years, the theory of Menger algebras has been explored in several directions, including their internal structure, functional representations, logical applications, and generalizations. For example, the studies in [10, 13, 15] construct Menger algebras of terms and formulas to classify algebras of a fixed type n into subclasses. More broadly, the concept of Menger hyperalgebras has emerged as a generalization of both Menger algebras and semihypergroups. These have been investigated in works such as [12, 14, 18, 19], where hyperoperations take the place of traditional operations, and outcomes may be sets rather than single values. Additionally, K. Denecke, in publications such as [2, 3], has contributed significantly to the study of partial Menger algebras, focusing on their algebraic properties, representations, and potential applications in theoretical computer science.

To provide representation of Menger algebras, let A be a nonempty set and let A^n denote the Cartesian product of n copies of A. A full n-ary function on A is defined as any mapping from A^n to A that is total, i.e., defined for every element of A^n. The collection of all such functions is denoted by $T(A^n, A)$. This set forms a Menger algebra under the Menger superposition operation $\mathcal{O}$, an $(n+1)$-ary operation defined as follows:

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)),$$

for all $f, g_1, \ldots, g_n \in T(A^n, A)$ and $a_1, \ldots, a_n \in A$ because the operation $\mathcal{O}$ satisfies the superassociative law. For convenient, a_1^n stands for $a_1, \ldots, a_n$. If $a_1 = a_2 = \cdots = a_n = a$, we write $\overset{n}{a}$ instead of a_1^n. Normally, the above equation can be written as $\mathcal{O}(f, g_1^n)(a_1^n) = f(g_1(a_1^n), \ldots, g_n(a_1^n))$. Thus $(T(A^n, A), \mathcal{O})$ is called a Menger algebra of all full n-ary functions on A. Foundational work on such representations has been extensively developed by W. A. Dudek and V. S. Trokhimenko, see for example, [4, 7, 8, 9]. The reader is also refered to [1, 17] for representation by partial functtions. For the abstract characterization of different classes of Menger algebras of multiplace functions partially ordered by the inclusion of domains and by

nonempty intersection of domains, we adhere to the framework presented in the monograph [5].

In [14], the notion of a Menger hyperalgebra is introduced, extending the framework of Menger algebras by replacing the superposition operation with a hyperoperation. In this setting, the composition of elements may yield a nonempty set of results instead of a single value. Formally, a Menger hyperalgebra consists of a set equipped with a hyperoperation mapping an $(n+1)$-tuple of elements to a nonempty subset of the set, and satisfying a generalized form of superassociativity suited to the hyperstructure context. Such a framework is useful in modeling multi-valued or nondeterministic compositions, with applications in fields such as fuzzy logic, automata theory, and the study of non-classical algebraic systems. The fact that every Menger hyperalgebra is embeddable into the algebra of multivalued full n-ary functions is stated and proved. In fact, by a multivalued full n-ary functions we mean a function $f : A^n \to P^*(A)$. On the set $T(A^n, P/(A))$ of all multivalued full n-ary functions, one consider the operation $\bullet$ defined by

$$\bullet(f, g_1^n)(x_1^n) = \bigcup_{\substack{y_i \in g_i(x_1^n) \\ i \in \{1,\dots,n\}}} f(y_1^n),$$

for all $i = 1, \dots, n$ and $f, g_1, \dots, g_n \in T(A^n, P^*(A))$, $x_1, \dots, x_n \in A$. It is proved that $\bullet$ is superassociative and hence $(T(A^n, P^*(A)), \bullet)$ is a Menger algebra.

Given the potential of this framework, it is both natural and important to study representation theorems for hypercompositional structures. For instance, [11] presents necessary and sufficient conditions under which a Menger hyperalgebra can be isomorphically represented by medial n-ary hyperoperations. Furthermore, Menger hyperalgebras defined by k-commutative hyperoperations are studied in [12], while applications of semiabelian hyperoperations to Menger hyperalgebras are discussed in [16].

Building upon the foundational work of W. A. Dudek and V. S. Trokhimenko in [6], the present paper investigates representation theorems for associative and autodistributive hyperoperations defined on an arbitrary set A. In Section 2, we address an open problem from that paper by providing a characterization of Menger algebras that can be represented by superassociative n-ary operations. Section 3 introduces the notions of associative and autodistributive hyperoperations, and establishes representation theorems for Menger hyperalgebras based on these hyperoperations.

2. A representation via superassociative operations

This section begins by recalling an open problem posed by W. A. Dudek and V. S. Trokhimenko in [6].

Problem 1 (W. A. Dudek and V. S Trokhimenko, 2018)**.** *Find necessary and suicient conditions under which a Menger algebra of rank n can be isomorphically represented by superassociative n-ary operations defined on some set.*

Our main purpose of this section is to solve this problem and give a proof. For this, an $(n+1)$-ary operation o on A is called quasi superassociative if $o(a, o(a, b, c_1^{n-1}), d_1^{n-1}) = o(a, b, o(a, c_1, d_1^{n-1}), \ldots, o(a, c_{n-1}, d_1^{n-1}))$.

Theorem 2.1. *A Menger algebra (G, o) of rank n is homomorphically represented by superassociative n-ary operations if and only if it is quasi superassociative.*

Proof. Assume that there exists a homomorphism α from a Menger algebra (G, o) to an algebra $(T(A^n, P^*(A)), \mathcal{O})$ of superassociative n-ary operations defined on the set A. Then, for all $f, g, h_1, \ldots, h_{n-1}, p_1, \ldots, p_{n-1}$ be superassociative n-ary operations on A and all elements $a_1, \ldots, a_n \in A$, by the homomorphism of α, we have

$$\begin{aligned}
&\mathcal{O}(f, \mathcal{O}(f, g, h_1^{n-1}), p_1^{n-1})(a_1^n)\\
&= f(\mathcal{O}(f, g, h_1^{n-1})(a_1^n), p_1(a_1^n), \ldots, p_{n-1}(a_1^n))\\
&= f(f(g(a_1^n), h_1(a_1^n), \ldots, h_{n-1}(a_1^n)), p_1(a_1^n), \ldots, p_{n-1}(a_1^n))\\
&= f(g(a_1^n), f(h_1(a_1^n), p_1(a_1^n), \ldots, p_{n-1}(a_1^n)), \ldots, f(h_{n-1}(a_1^n), p_1(a_1^n), \ldots, p_{n-1}(a_1^n)))\\
&= f(g(a_1^n), \mathcal{O}(f, h_1, p_1^{n-1})(a_1^n), \ldots, \mathcal{O}(f, h_{n-1}, p_1^{n-1})(a_1^n))\\
&= \mathcal{O}(f, g, \mathcal{O}(f, h_1, p_1^{n-1}), \ldots, \mathcal{O}(f, h_{n-1}, p_1^{n-1}))(a_1^n).
\end{aligned}$$

For this, it follows from an injectivity of α, we obtain our claimed.

To prove an opposite direction, suppose first that a Menger algebra (G, o) is quasi superassociative, i.e., it satisfies the identity (6). Actually, for each $g \in G$, the inner left translation $\xi_g : G \to G$ is defined by $\xi_g(x_1^n) = o(g, x_1^n)$ for every $x_1, \ldots, x_n \in G$. Our next aim is to show that the n-ary operation ξ_g is superassociative. Since (G, o) is quasi superassociative, we have

$$\begin{aligned}
\xi_g(\xi_g(a, b_1^{n-1}), c_1^{n-1}) &= \xi_g(o(g, a, b_1^{n-1}), c_1^{n-1})\\
&= o(g, o(g, a, b_1^{n-1}), c_1^{n-1})\\
&= o(g, a, o(g, b_1, c_1^{n-1}), \ldots, o(g, b_{n-1}, c_1^{n-1}))\\
&= \xi_g(a, o(g, b_1, c_1^{n-1}), \ldots, o(g, b_{n-1}, c_1^{n-1}))\\
&= \xi_g(a, \xi_g(b_1, c_1^{n-1}), \ldots, \xi_g(b_{n-1}, c_1^{n-1})).
\end{aligned}$$

Thus, the inner left translation ξ_g is superassociative.

We now show that $g \mapsto \xi_g$ is a homomorphism $(G, o) \to (\{\xi_g \mid g \in G\}, \mathcal{O})$. In fact, for all $a, a_1, \ldots, a_n, x_1, \ldots, x_n \in G$ we obtain

$$\begin{aligned}\xi_{o(a,a_1,\ldots,a_n)}(x_1^n) &= o(o(a, a_1, \ldots, a_n), x_1, \ldots, x_n)\\ &= o(a, o(a_1, x_1, \ldots, x_n), \ldots, o(a_n, x_1, \ldots, x_n))\\ &= \xi_x(\xi_{a_1}(x_1, \ldots, x_n), \ldots, \xi_{a_n}(x_1, \ldots, x_n))\\ &= \mathcal{O}(\xi_a, \xi_{a_1}, \ldots, \xi_{a_n})(x_1^n),\end{aligned}$$

which shows that $g \mapsto \xi_g$ is a homomorphism. □

Remark from Theorem 2.1 that an isomorphic representation of a general Menger algebra by superassociative n-ary operations is not possible, owing to specific structural constraints. In what follows, we elaborate on these challenges. First, we try construct an isomorphism from (G, o) to the set of superassociative n-ary operations on some set. To do this, the set G is extended to the set $G_0 = G \cup \{e, c\}$ where both e and c are elements which are different and not included into the set G. Thus, for every $g \in G$ and $x_1, \ldots, x_n \in G_0$, we define the n-ary operation ξ_g on G_0 by the following items:

(1) if $x_1^n \in G$, then $\xi_g(x_1^n) = o(g, x_1^n)$,
(2) if $x_1 = \cdots = x_n = e$, then $\xi_g(x_1^n) = g$,
(3) in all other cases, we define $\xi_g(x_1^n) = c$.

It is not difficult to see that ξ_g is not a superassociative n-ary operation on G_0. Indeed, we consider a few cases. Assume that a, b_1^{n-1}, c_1^{n-1} are arbitrary elements in G. If all of them belong to the set G, the proof is done in Theorem 2.1 completely. However, the difficulties occur in the following two cases:

$a = b_1 = \cdots = b_{n-1} = e, c_1, \ldots, c_{n-1} \in G$ and
$a \in G, b_1 = \cdots = b_{n-1} = c_1 = \cdots = c_{n-1} = e$.

In the first case, $\xi_g(\xi_g(a, b_1^{n-1}), c_1^{n-1}) = \xi_g(g, c_1^{n-1}) = o(g, g, c_1^{n-1})$, but $\xi_g(a, \xi_g(b_1, c_1^{n-1}), \ldots, \xi_g(b_{n-1}, c_1^{n-1})) = \xi_g(a, c, \ldots, c) = c$, which are not equal. In the second, on one hand, $\xi_g(\xi_g(a, b_1^{n-1}), c_1^{n-1}) = \xi_g(c, e, \ldots, e) = c$. On the other hand, $\xi_g(a, \xi_g(b_1, c_1^{n-1}), \ldots, \xi_g(b_{n-1}, c_1^{n-1})) = \xi_g(a, g, \ldots, g) = o(g, a, g, \ldots, g)$, which shows that the equality does not hold.

Another solution to Problem 1 is to include the condition that every Menger algebra is reductive as part of Theorem 2.1. This leads to an isomorphic representation; however, the theorem remains valid only in the necessary direction.

3. Associative and autodistributive hyperoperations

In this section, a Menger hyperalgebra $(G, \diamond)$ as the main algebraic structure. We discuss representation theorems for such algebras using specific types of n-ary hyperoperations. In particular, we study a hypercompositional approach involving associative and autodistributive n-ary operations as presented in [6]. In particular, we aim to determine the necessary and sufficient conditions under which a Menger hyperalgebra of rank n can be isomorphically represented by (i, j)-associative n-ary hyperoperations on a suitable set.

To this end, we begin with the following definition.

Definition 3.1. An n-ary hyperoperation $\xi \in T(A^n, P^*(A))$ is said to be

(1) (i, j)-associative, where $1 \leqslant i < j \leqslant n$, if it satisfies the identity

$$\bigcup_{b \in \xi(a_i^{i+n-1})} \xi(a_1^{i-1}, b, a_{i+n}^{2n-1}) = \bigcup_{b \in \xi(a_j^{j+n-1})} \xi(a_1^{j-1}, b, a_{j+n}^{2n-1}), \tag{1}$$

(2) associative, if it is (i, j)-associative for all $1 \leqslant i < j \leqslant n$,

(3) autodistributive, if for all $1 \leqslant i \leqslant n$, if it satisfies the equation

$$\bigcup_{b \in \xi(b_1^n)} \xi(a_1^{i-1}, b, a_{i+1}^n) = \bigcup_{\substack{c_k \in \xi(a_1^{i-1}, b_k, a_{i+1}^n) \\ k=1,\ldots,n}} \xi(c_1^n), \tag{2}$$

(4) superassociative, if it satisfies the identity

$$\bigcup_{d \in \xi(a, b_1^{n-1})} \xi(d, c_1^{n-1}) = \bigcup_{\substack{d_i \in \xi(b_i, c_1^{n-1}) \\ i=1,\ldots,n-1}} \xi(a, d_1^{n-1}). \tag{3}$$

It is clear that the hyperoperation $f \in T(A^n, P^*(A))$ is associative if and only if it is $(1, j)$-associative for all $j = 2, 3, \ldots, n$.

Definition 3.2. An $(n+1)$-ary hyperoperation $\diamond \in T(A^{n+1}, P^*(A))$ is

(1) quasi-(i, j)-associative, $1 \leqslant i < j \leqslant n$, if it satisfies the identity

$$\bigcup_{c \in \diamond(a, b_i^{i+n-1})} \diamond(a, b_1^{i-1}, c, b_{i+n}^{2n-1}) = \bigcup_{c \in \diamond(a, b_j^{j+n-1})} \diamond(a, b_1^{j-1}, c, b_{j+n}^{2n-1}), \tag{4}$$

(2) quasi-associative, if it is quasi (i, j)-associative for all $1 \leqslant i < j \leqslant n$,

(3) quasi-autodistributive, if for all $1 \leqslant i \leqslant n$, if it satisfies the equation

$$\bigcup_{c\in\diamond(a,c_1^n)} \diamond(a, b_1^{i-1}, c, b_{i+1}^n) = \bigcup_{\substack{d_k\in\diamond(a,b_1^{i-1},c_k,b_{i+1}^n)\\ k=1,\dots,n-1}} \diamond(a, d_1^n), \tag{5}$$

(4) quasi-superassociative, if it satisfies the identity

$$\bigcup_{c\in\diamond(a,b,c_1^{n-1})} \diamond(a, c, d_1^{n-1}) = \bigcup_{\substack{c_i\in\diamond(a,c_i,d_1^{n-1})\\ i=1,\dots,n-1}} \diamond(a, b, c_1^{n-1}). \tag{6}$$

Note that a Menger hyperalgebra $(G, \diamond)$ is called quasi-(i, j)-associative, quasi-associative, quasi-autodistributive and quasi-superassociative if $\diamond$ is quasi-(i, j)-associative, quasi-associative, quasi-autodistributive and quasi-superassociative, respectively.

For each a in a Menger hyperalgebra $(G, \diamond)$, the inner left hypertranslation $\rho_a : G^n \to G$ is defined by $\rho_a(x_1^n) = \diamond(a, x_1^n)$ for every $x_1, \dots, x_n \in G$. Actually, we note that an $(n + 1)$-ary hyperoperation $\diamond \in T(A^{n+1}, P^*(A))$ is quasi-superassociative if and only if all inner left hypertranslations ρ_a are associative. Hence, we can give a characterization of any hyperalgebra $(A, \diamond)$ with one $(n + 1)$-ary quasi-associative hyperoperation $\diamond$ through the class of n-ary semihypergroup (A, ρ_a) where $a \in A$. Analogously, a quasi-superassociative (quasi-autodistributive) hyperalgebra $(A, \diamond)$ can be characterized by the n-ary superassociative (autodistributive) algebras $(A, \mathcal{F})$ where the class $\mathcal{F} = \{\rho_a \mid a \in A\}$.

We now give necessary and sufficient under which any Menger hyperalgebra can be represented by some type of hyperoperations.

Theorem 3.3. *A Menger hyperalgebra $(G, \diamond)$ of rank n is isomorphically represented by (i, j)-associative n-ary hyperoperations if and only if it is quasi-(i, j)-associative.*

Proof. Suppose first that there is an isomorphism, say β, taking any element g of a Menger hyperalgebra $(G, \diamond)$ to each (i, j)-associative n-ary hyperoperation. Them, for all $f, g_1, \dots, g_{2n-1}$ in the set of (i, j)-associative n-ary hyperoperations on a nonempty set A, we obtain

$$\begin{aligned}\bullet(f, g_1^{i-1}, \bullet(f, g_i^{i+n-1}), g_{i+n}^{2n-1})(a_1^n) &= \bigcup_{b\in\bullet(f,g_i^{i+n-1})(a_1^n)} f(g_1^{i-1}(a_1^n), b, g_{i+n}^{2n-1}(a_1^n))\\ &= \bigcup_{b\in f(g_i^{i+n-1}(a_1^n))} f(g_1^{i-1}(a_1^n), b, g_{i+n}^{2n-1}(a_1^n))\\ &= \bigcup_{b\in f(g_j^{j+n-1}(a_1^n))} f(g_1^{j-1}(a_1^n), b, g_{j+n}^{2n-1}(a_1^n))\\ &= \bigcup_{b\in\bullet(f,g_j^{j+n-1})(a_1^n)} f(g_1^{j-1}(a_1^n), b, g_{j+n}^{2n-1}(a_1^n))\\ &= \bullet(f, g_1^{i-1}, \bullet(f, g_i^{i+n-1}), g_{i+n}^{2n-1})(a_1^n).\end{aligned}$$

It follows from an injectivity β that the identity of quasi (i,j)-associative holds.

Conversely, assume that $(G, \diamond)$ is a Menger hyperalgebra of rank $n > 1$. For elements $x, y \notin G$ and $x \neq y$. We extend the set G to $G \cup \{x, y\}$. Thus, for each g in $G \cup \{x, y\}$, we define an n-ary hyperoperation λ_g on $G \cup \{x, y\}$ by the following

$$\lambda_g(a_1^n) = \begin{cases} \diamond(g, a_1^n) & \text{if } a_1^n \in G,\\ \{g\} & \text{if } a_j = x \text{ for all } j = 1, \ldots, n,\\ \{y\} & \text{otherwise.}\end{cases}$$

For any nonempty set A of $G \cup \{x, y\}$, an n-ary extended function λ_A on $G \cup \{x, y\}$ can be defined by

$$\lambda_A(a_1^n) = \begin{cases} \diamond(A, a_1^n) & \text{if} a_1^n \in G,\\ A & \text{if } a_j = x \text{ for all } j = 1, \ldots, n,\\ \{y\} & \text{in all other cases.}\end{cases}$$

To show that an n-ary hyperoperation λ_g is $(i,j)-$associative, we divide the proof into a few cases. If $a_1, \ldots, a_{2n-1}$ contain in G, then by the fact that $(G, \diamond)$ is quasi-associative, we have

$$\begin{aligned}\lambda_g(a_1^{i-1}, \lambda_g(a_i^{i+n-1}), a_{i+n}^{2n-1}) &= \bigcup_{b\in\diamond(g,a_i^{i+n-1})} \diamond(g, a_1^{i-1}, b, a_{i+n}^{2n-1})\\ = \bigcup_{b\in\diamond(g,a_j^{j+n-1})} \diamond(g, a_1^{j-1}, b, a_{j+n}^{2n-1}) &= \lambda_g(a_1^{j-1}, \lambda_g(a_j^{j+n-1}), a_{j+n}^{2n-1}).\end{aligned}$$

For the case when $a_1 = \cdots = a_{2n-1} = x$, then by the definition, we obtain $\bigcup_{b\in\lambda_g(e_i^{i+n-1})} \lambda_g(e_1^{i-1}, b, e_{i+n}^{2n-1}) = \bigcup_{b\in\{g\}} \lambda_g(e_1^{i-1}, b, e_{i+n}^{2n-1}) = \lambda_g(e_1^{i-1}, g, e_{i+n}^{2n-1}) =$

$\{y\}$ and $\bigcup_{b\in\lambda_g(e_j^{j+n-1})}\lambda_g(e_1^{j-1},b,e_{j+n}^{2n-1}) = \lambda_g(e_1^{j-1},g,e_{j+n}^{2n-1}) = \{y\}$. As a result, we have $\bigcup_{b\in\lambda_g(e_i^{i+n-1})}\lambda_g(e_1^{i-1},b,e_{i+n}^{2n-1}) = \bigcup_{b\in\lambda_g(e_j^{j+n-1})}\lambda_g(e_1^{j-1},b,e_{j+n}^{2n-1})$. In all other cases, $\bigcup_{b\in\lambda_g(a_i^{i+n-1})}\lambda_g(a_1^{i-1},b,a_{i+n}^{2n-1}) = \{y\} = \bigcup_{b\in\lambda_g(a_j^{j+n-1})}\lambda_g(a_1^{j-1},b,a_{j+n}^{2n-1})$. Thus, we conclude that the n-ary operation λ_g is (i,j)-associative.

We next prove that $g\mapsto\lambda_g$ is a homomorphism and an injection. To the injectivity, assume that $\lambda_{g_1}=\lambda_{g_2}$ for $g_1,g_2\in G$. It can be seen that for each $g\in G$ an element x comes from the domain of an inner left hypertranslation λ_g. It follows that $\lambda_{g_1}(\overset{n}{x}) = \lambda_{g_2}(\overset{n}{x})$, and thus, $\{g_1\}=\{g_2\}$. So, $g_1=g_2$.

Now let $g,g_1,\dots,g_n\in G\cup\{x,y\}$. If $g,g_1,\dots,g_n\in G$, by the superassociativity of a hyperoperation $\diamond$, we have

$$\begin{aligned}\lambda_{\diamond(g,g_1,\dots,g_n)}(a_1^n) &= \diamond(\diamond(g,g_1,\dots,g_n),a_1^n) = \diamond(g,\diamond(g_1,a_1^n),\dots,\diamond(g_n,a_1^n))\\ &= \diamond(g,\lambda_{g_1}(a_1^n),\dots,\lambda_{g_n}(a_1^n)) = \bigcup_{\substack{c_i\in\lambda_{g_i}(a_1^n)\\ i\in\{1,\dots,n\}}}\diamond(g,c_1,\dots,c_n)\\ &= \bigcup_{\substack{c_i\in\lambda_{g_i}(a_1^n)\\ i\in\{1,\dots,n\}}}\lambda_g(c_1,\dots,c_n) = \bullet(\lambda_g,\lambda_{g_1},\dots,\lambda_{g_n})(a_1^n).\end{aligned}$$

Now let $a_1=\cdots=a_n=x$, then according to the definition of an extension λ_A, we obtain

$$\begin{aligned}\lambda_{\diamond(g,g_1,\dots,g_n)}(\overset{n}{x}) &= \diamond(g,g_1,\dots,g_n) = \lambda_g(g_1,\dots,g_n) = \bigcup_{\substack{c_i\in\{g_i\}\\ i\in\{1,\dots,n\}}}\lambda_g(c_1,\dots,c_n)\\ &= \bigcup_{\substack{c_i\in\lambda_{g_i}(\overset{n}{x})\\ i\in\{1,\dots,n\}}}\lambda_g(c_1,\dots,c_n) = \bullet(\lambda_g,\lambda_{g_1},\dots,\lambda_{g_n})(\overset{n}{x}),\end{aligned}$$

which means $\lambda_{\diamond(g,g_1,\dots,g_n)}(\overset{n}{x}) = \bullet(\lambda_g,\lambda_{g_1},\dots,\lambda_{g_n})(\overset{n}{x})$.

In the case when $(a_1,\dots,a_n)\in(G\cup\{x,y\})^n\setminus(G^n\cup\{(\overset{n}{x})\})$, according to the definition, we have $\lambda_{\diamond(g,g_1,\dots,g_n)}(a_1,\dots,a_n)=\{y\}$ and

$$\begin{aligned}\bullet(\lambda_g,\lambda_{g_1},\dots,\lambda_{g_n})(a_1,\dots,a_n) &= \bigcup_{\substack{x_i\in\lambda_{g_i}(x_1,\dots,x_n)\\ i\in\{1,\dots,n\}}}\lambda_g(x_1,\dots,x_n)\\ &= \bigcup_{\substack{c_i\in\{y\}\\ i\in\{1,\dots,n\}}}\lambda_g(c_1,\dots,c_n) = \lambda_g(y,\dots,y) = \{y\}.\end{aligned}$$

From these, consequently, $\lambda_{\diamond(g,g_1,\dots,g_n)} = \bullet(\lambda_g,\lambda_{g_1},\dots,\lambda_{g_n})$. □

Characterizations of the embeddability of any Menger hyperalgebra into the algebra of associative n-ary hyperoperations is now proposed in the following result.

Theorem 3.4. *A Menger hyperalgebra $(G, \diamond)$ of rank n is isomorphically represented by associative n-ary hyperoperations if and only if it is quasi-associative.*

Proof. The proof follows similarly to that of Theorem 3.3, using an analogous construction of the hyperoperation λ_g. □

The following corollary follows directly from Theorems 2.1 and 3.4.

Corollary 3.5. *Any Menger hyperalgebra $(G, \diamond)$ of rank n satisfying both identities* (4) *and* (6) *is isomorphically represented by n-ary hyperoperations satisfying both* (1) *and* (3) *defined on some set.*

This section completes with a discussion of a Menger hyperalgebra represented by autodistributive n-ary hyperoperations.

Lemma 3.6. *Let $A \neq \emptyset$ and $i = 1, \ldots, n$. Then for autodistributive n-ary hyperoperations f, g_i, h_i on A, the identity*

$$\bullet(f, g_1^{i-1}, \bullet(f, h_1^n), g_{i+1}^n) = \bullet(f, \bullet(f, \bullet(f, g_1^{i-1}, h_1, g_{i+1}^n)), \ldots, \bullet(f, g_1^{i-1}, h_n, g_{i+1}^n)),$$

holds.

Proof. Let $a_1, \ldots, a_n$ be elements in A. We have

$$\begin{aligned}
&\bullet(f, g_1^{i-1}, \bullet(f, h_1^n), g_{i+1}^n)(a_1^n) \\
&= \bigcup_{b \in \bullet(f, h_1^n)(a_1^n)} f(g_1(a_1^n), \ldots, g_{i-1}(a_1^n), b, g_{i+1}(a_1^n), \ldots, g_n(a_1^n)) \\
&= \bigcup_{b \in f(h_1(a_1^n), \ldots, h_n(a_1^n))} f(g_1(a_1^n), \ldots, g_{i-1}(a_1^n), b, g_{i+1}(a_1^n), \ldots, g_n(a_1^n)) \\
&= \bigcup_{\substack{c_k \in f(g_1(a_1^n), \ldots, g_{i-1}(a_1^n), h_k(a_1^n), g_{i+1}(a_1^n), \ldots, g_n(a_1^n)) \\ k=1,\ldots,n}} f(c_1^n) \\
&= \bigcup_{\substack{c_k \in \bullet(f, \bullet(f, g_1^{i-1}, h_k, g_{i+1}^n))(a_1^n) \\ k=1,\ldots,n}} f(c_1^n) \\
&= \bullet(f, \bullet(f, \bullet(f, g_1^{i-1}, h_1, g_{i+1}^n)), \ldots, \bullet(f, g_1^{i-1}, h_n, g_{i+1}^n))(a_1^n).
\end{aligned}$$

□

Having this lemma at hand, we prove the following theorem.

Theorem 3.7. *Let $(G,\diamond)$ be a Menger hyperalgebra of rank n. Then $(G,\diamond)$ is homomorphically represented by autodistributive n-ary hyperoperations if and only if $(G,\diamond)$ is quasi-autodistributive.*

Proof. The proof of necessity follows from Lemma 3.6. Thus, it is left give a proof only another direction. Suppose now that $(G,\diamond)$ is a quasi-autodistributive Menger hyperalgebra of rank n. For every $g \in G$, we define the n-ary hyperoperation ζ on G by the following $\zeta_g(a_1^n) = \diamond(g, a_1^n)$. Each $\emptyset \neq A \subseteq G$ induces the hyperoperation ζ_A defined by $\zeta_A(a_1^n) = \diamond(A, a_1^n)$. Then for all $a_i, b_i \in G$ and $i = 1, \ldots, n$ by the satisfaction of quasi-autodistributivity of the hyperoperation $\diamond$ on G, we have

$$\zeta_g(a_1^{i-1}, \zeta_g(b_1^n), a_{i+1}^n) = \diamond(g, a_1^{i-1}, \diamond(g, b_1^n), a_{i+1}^n) = \bigcup_{c \in \diamond(g, b_1^n)} \diamond(g, a_1^{i-1}, c, a_{i+1}^n)$$

$$= \bigcup_{\substack{c_k \in \diamond(g, a_1^{i-1}, b_k, a_{i+1}^n) \\ k=1,\ldots,n-1}} \diamond(g, c_1^n) = \zeta_g(\zeta_g(a_1^{i-1}, b_1, a_{i+1}^n), \ldots, \zeta_g(a_1^{i-1}, b_n, a_{i+1}^n)),$$

which shows that ζ_g is autodistributive. It is not difficult to prove that $g \mapsto \zeta_g$ is a homomorphism by using the same argument given in Theorem 2.1. This completes the proof. □

Corollary 3.8. *Any quasi-(i,j)-associative (quasi-associative, quasi-superassociative) Menger hyperalgebra $(G,\diamond)$ of rank n satisfying the identity of quasi-autodistributive in the same time can be homomorphically represented by autodistributive and (i,j)-associative (associative, superassociative) n-ary hyperoperations defined on some set.*

References

[1] **C. Borlido, B. McLean**, *Difference-restriction algebras of partial functions: Axiomatisations and representations*, Algebra Universalis, **83**(3) (2022) 24.

[2] **K. Denecke**, *Partial Menger algebras and their weakly isomorphic representation*, Mathematics Open, **1** (2022) 2250003.

[3] **K. Denecke, H. Hounnon**, *Partial Menger algebras of terms*, Asian-Eur. J. Math., **14(6)** (2021) 2150092.

[4] **W.A. Dudek**, *In memoriam: Valentin S. Trokhimenko*, Quasigroups Related Systems, **28** (2020), 171-176.

[5] **W.A. Dudek, V.S. Trokhimenko**, *Algebras of multiplace functions*, De Gruyter, Berlin, (2012), 389 pp.

[6] **W.A. Dudek, V.S. Trokhimenko**, *Menger algebras of associative and self-distributive n-ary operations*, Quasigroups Related Systems, **26** (2018) 45-52.

[7] **W.A. Dudek, V.S. Trokhimenko**, *Menger algebras of idempotent n-ary operations*, *Stud. Sci. Math. Hung.*, **5(2)** (2019) 260-269.

[8] **W.A. Dudek, V.S. Trokhimenko**, *Menger algebras of k-commutative n-place functions*, Georgian Math. J., **28(3)** (2021) 355-361.

[9] **W.A. Dudek, V.S. Trokhimenko**, *On (i, j)-commutativity in Menger algebras of n-place functions*, Quasigroups Related Systems, **24** (2016), 219-230.

[10] **P. Junlouchai, T. Kumduang, C. Siwapornanan**, *The structures of full terms preserving a partition under different operations*, Quasigroups Related Systems, **32**(2) (2024), 261-276.

[11] **T. Kumduang**, *Menger hypercompositional algebras represented by medial n-ary hyperoperations*, Quasigroups Related Systems, **30**(2) (2022), 263-274.

[12] **T. Kumduang**, *Representations of Menger hypercompositional algebras by some types of commutative hyperoperations*, Acta Math. Univ. Comen., New Ser., **92**(2) (2023), 101-111.

[13] **T. Kumduang**, *Weak embeddability of the partial Menger algebra of formulas*, Quasigroups Related Systems, **31**(2) (2023), 269-284.

[14] **T. Kumduang, S. Leeratanavalee**, *Menger hyperalgebras and their representations*, Commun. Algebra, **49** (2021) 1513-1533.

[15] **T. Kumduang, S. Sriwongsa**, *Superassociative structures of terms and formulas defined by transformations preserving a partition*, Commun. Algebra **51**(8) (2023), 3203-3220.

[16] **T. Kumduang, K. Wattanatripop, S. Leeratanavalee**, *Semiabelian hyperoperations with applications to Menger hyperalgebras*, Thai J. Math. **20**(Spec. Iss.: Annual Meeting in Mathematics 2021) (2022), 15-24.

[17] **B. McLean**, *Complete representation by partial functions for signatures containing antidomain restriction*, Int. J. Algebra Comput., **34**(8) (2024), 1131-1158.

[18] **A. Nongmanee, S. Leeratanavalee**, *Algebraic connections between Menger algebras and Menger hyperalgebras via regularity*, Algebra Discrete Math., **36**(1) (2023), 61-73.

[19] **A. Nongmanee, S. Leeratanavalee**, *Some results in Menger hyperalgebras*, Palest. J. Math., **14**(1) (2025), 124-132.

Received June 06, 2025

Department of Mathematics, Faculty of Science and Technology,
Rajamangala University of Technology Rattanakosin, Nakhon Pathom 73170, Thailand
E-mail: thodsaporn.kum@rmutr.ac.th