SEMIREFLEXIVE, LEFT AND RIGHT PRODUCTS OF TWO SUBCATEGORIES

PRODUSE SEMIREFLEXIVE DE STÂNGA ȘI DE DREAPTA A DOUA SUBCATEGORII

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Abstract. *We propose a synthesis of the author's results concerning the semireflexive product, the left product and the right product of two subcategories. The possibility of describing the R–semireflexive subcategories using the indicated products is shown.*

Keywords: *reflexives subcategories, coreflexive subcategories, semireflexive subcategories, semireflexive products, left and right products.*

Rezumat. *Propunem o sinteză a rezultatelor autorilor privind produsul semireflexiv, produsul de stânga și produsul de dreapta a două subcategorii. Este prezentată posibilitatea descrierii subcategoriilor R- semireflexive folosind produsele indicate.*

Cuvinte-cheie: *subcategorii reflexive, subcategorii coreflexive, subcategorii semireflexive, produse semireflexive, produse de stânga și de dreapta.*

INTRODUCTION

The main results are formulated for the category C_2V of vector topological locally convex Hausdorff spaces.

In the category C_2V we consider the following bicategory structures:

- (Epi, M_f) = (the class of epimorphisms, the class of strict monomorphisms);
- (E_u, M_p) = (the class of universal epimorphisms, the class of precise monomorphisms) = (the class of surjective applications, the class of topological embedding);
- (E_p, M_u) = (the class of precise epimorphisms, the class of universal monomorphisms) [5];

– $(E_f, Mono)$ = (the class of strict epimorphisms, the class of monomorphisms).

We will consider the following subcategories:

- Π, the subcategory of complete spaces with weak topology [9];
- S, the subcategory of spaces with weak topology [9];
- sN, the subcategory of strict nuclear spaces [7];
- N, the subcategory of nuclear spaces [11];
- Sc, the subcategory of Schwartz spaces [12];
- Γ_0, the subcategory of complete spaces [13];
- $q\Gamma_0$, the subcategory of quasicomplete spaces [14];
- sR, the subcategory of semireflexive spaces [9];
- iR, the subcategory of inductive semireflexive spaces [1];
- $\widetilde{M}$, the subcategory of spaces with Mackey topology [13].

The last category is coreflective and the others are reflective.

First of all, we will formulate several well-known notions and results.

Definition 1. *A full subcategory R of the category C is called reflective in the inclusion functor $i : R \to C$ possesses a left adjunct $r : C \to R$.*

Theorem 1. *The full subcategory R is reflective if for any object X of the category C there is an object $rX \in |R|$ and a morphism $r^X : X \to rX$ with the next universal property:*

for any object $A \in |R|$ and any morphism $f : X \to A$ there is a unique morphism $g : rX \to A$ so that $f = g \cdot r^X$.

In this case the pair rX, r^X, or the object rX, or the morphism r^X is called R–replique of the object X.

Definition 2. *Let A be a class of morphisms of the category C. The reflective subcategory R of category C is called A–reflective if for any object X of the category C it's R–replique $r^X \in A$.*

Dual notions: coreflective subcategory K, coreflector functor k:

$C \to K$, K–coreplique of the object $X : k^X : kX \to X$, *A–coreflective subcategory*.

Theorem 2. *Let (P,I) be a bicategory structure in the category C that possesses products and is P – co-locally small. The full subcategory R of the category C is P–reflective if it is closed under products and I –subobjects.*

We mention that in category $C_2 V$ any non-null reflective subcate- gory is mono-reflecive, thus it is $(Epi \cap M_u)$ – reflective, and any non-null coreflective subcategory is $(Mono \cap E_u)$ – coreflective.

For a reflective subcategory R of category C with the reflector functor $r : C \to R$ we denote

$$\varepsilon R = \{e \in Epi | r(e) \in Iso\}.$$

Dual: For a coreflective subcategory K or category C with core- flector functor $k : C \to K$ we denote

$$\mu K = \{m \in Mono | k(m) \in Iso\}.$$

Thus, if K (respectively: R) is a non-null coreflective (respectively: reflective) subcategory, then $\mu K \subset E_u \cap Mono$ (respectively: $\varepsilon R \subset E pi \cap M_u$).

Let E (respectively: M) be a class of *epi* (respectively: *mono*) and let A be a subcategory $QE(A)$ *(respectively:* $SM(A)$*).* Therefore $Q_E(A)$ (respectively: $S_M(A)$) is the full subcategory of category C_2V consisting of all E – factorobjects (respectively: M - subobjects) of objects from A.

SEMIREFLEXIVE PRODUCT OF TWO SUBCATEGORIES

We will examine the main properties of the semireflexive product and some cases when the reflector functors keeps the class of injections of bicategory structures.

Definition 3. *Let R be a reflective subcategory and A a subcategory of the category C. The object X of the category C is called* (R, A)*– semireflexive if its R–replique belongs to a subcategory A. The full subcategory of all* (R, A)*–semireflexive objects is called the semireflexive product of subcategories R and A , and is denoted by*

$$L = R \times_{sr} A.$$

The monograph [2] also introduces and studies the dual concept: the semicoreflexive product.

We mention the following properties of the semireflexive product.

Theorem 3. *The semireflexive product has the following properties:*

1. $R \times_{sr} A = R \times_{sr} (R \cap A)$.

2. Let $\{A_i | i \in I\}$ *be a class of subcategories of the category* C_2V*. Then* $R \times_{sr} (\cap A_i) = \cap (R \times_{sr} A_i)$.

3. If $A_i \subset A_2$*, then* $R \times_{sr} A_1 \subset R \times_{sr} A_2$.

4. If $R \subset A$*, then* $R \times_{sr} A = C$.

5. If $A \subset R$*, then* $A \subset R \times_{sr} A$.

Definition 4 [2]**.** *The reflective subcategory R of category* C_2V *is called c–reflective if it contains the subcategory I of the spaces with weak topology and the reflector functor* $r : C_2V \to R$ *is left exact.*

Remark. The lattice R_c of *c*–reflective subcategories does not contain all the elements between I and C_2V. For example, the subcategory of nuclear spaces N is not *c*–reflective. The subcategory of strict nuclear spaces sN is *c*–reflective.

Definition 5 [2]**.** *The pair of subcategories (K,R) of the category* C_2V *that verifies the conditions* $\mu K = \varepsilon R$ *is called a "pair of conjugated subcategories" and K is called "conjugated coreflective subcategory" of the reflective subcategory R and conversely.*

The pair $(\widetilde{M}, S)$ is the smallest pair of conjugated subcategories and (C_2V, C_2V) is the biggest. Conjugated coreflective subcategories of the reflective subcategories Sc and sN were described in article [4].

In known cases, the semireflexive product of two subcategories is a reflective subcategory, closed under products and under closed subspaces. We will present some sufficient conditions for the semireflexive product of two subcategories to exhibit these same properties.

Theorem 4. *Let (P, I) be a bicategory structure in the category* C_2V*, R and A two reflective subcategories so that A is P–reflective and reflector functor* $r : C_2V \to R$ *has the property* $r(I) \subset I$ *. Then the semireflexive product*

$$L = R \times_{sr} A$$

of R and A is a reflective subcategory of the category C_2V.

From Theorem 4 it follows another case which allows us to construct a lot of examples when semireflexive product of two subcategories is a reflective subcategory.

Theorem 5. *Let R be a c – reflective subcategory and A a reflective subcategory of the category C_2V. Then the semireflexive product $R \times_{sr} A$ is a reflective subcategory.*

SEMIREFLEXIVE SUBCATEGORIES

Let L be a reflective subcategory, and A – a subcategory of the category C_2V. The subcategory A is called the L –semireflexive subcategory if it is closed in correlation to (εL)–subobjects and (εL)– factorobjects. The class of L –semireflexive subcategories is noted $R_f^s(\varepsilon L)$. The dual concept is also introduced.

Let R be a reflective subcategory and A a subcategory. We denoted by $\lambda_R(A)$ the full subcategory of all T objects with properties: for any object $X \in |C_2V|$ all morphism $f: X \to T$ is extended by the morphism $r^X: X \to rX$: for a morphism g, $f = g \cdot r^X$.

Theorem 6 ([2], Theorem 12.2.6)**.** *Let R and L be two reflective subcategories. The following statements are equivalent:*

1. $R = L \times_{sr} R$.
2. $R \in R_f^s(\varepsilon L)$.
3. $R = L \times_{sr} H$ *for any H with property* $R \subset H \subset \lambda_R(L)$.

Exemple (See [2], chapter 16). **1.** *S s*emireflexive subcategories:

–the *sR* subcategory of semireflexive spaces;

- the $l\Gamma_0$ subcategory of complete locally subspaces [12];

-the Γ subcategory of complete spaces with weak topology;

-the *B-iR* subcategory of *B* inductive semireflexive spaces [3];

-the *p-sR* subcategory of *p* – semireflexive space [10].

2.The subcategory *iR* of inductive-semireflexive spaces [1] is a (*Sh*)–semireflexive subcategory.

Remark. Let R and L be two reflective subcategories and $R \subset L$. Then $\varepsilon L \subset \varepsilon R$ and any R – semireflexive subcategory is also L – semireflexive.

THE LEFT AND RIGHT PRODUCTS OF TWO SUBCATEGORIES

Definition 7 ([12], Definition 11.1.4)**.** *Let R be a reflective subcategory and K a coreflective subcategory of category C_2V.*

The subcategory $V = S_{\mu K}(R)$ is called right product of subcategories R and K and is noted $V = K \times_d R$.

The subcategory $W = Q_{\varepsilon R}(K)$ is called left product of subcategories K and R and is noted $W = K \times_s R$.

It is not known whether the left (right) product is a coreflective (i.e. reflective) subcategory except in the cases (see [2], problems 11.1.23).

Theorem 7 [6]**.** *Let K be a coreflective subcategory, R – a reflective subcategory and $\widetilde{M} \subset K$ or $S \subset R$. Then $K \times_s R$ is a coreflective subcategory and $R \times_d Q$ is a reflective subcategory*.

Let K be a coreflective subcategory and R a reflective subcategory of the category C_2V with these respective functors $k : C_2V \to K$ and $r : C_2V \to R$. For any object X of the category C_2V let $k^X : kX \to X$, $r^X : X \to rX$ be the K–coreplique and R–replique and let $k^{rX} : krX \to rX$ be the K–coreplique of the object rX.

Then

$$r^X \cdot k^X = k^{rX} \cdot k(r^X). \tag{1}$$

On the morphisms k^X and $k(r^X)$ construct the pushout

$$n^X \cdot k^X = g \cdot k(r^X) \tag{2}$$

From equality (1) it follows that there exists a morphism u^X such that

$$u^X \cdot n^X = r^X, \tag{3}$$

$$u^X \cdot g = k^{rX}. \tag{4}$$

Since $k^X \in E_u$, and (2) is a pushout, it follow that $g \in E_u$ and from equality (4) it results that $g \in Mono$. Therefore g is a bijective map. Since k^{rX} is also a bijective map, we conclude that u^X is bijective too. Then from the fact that u^X is *mono* and from equality (4) it follows that g is the K–coreplique of the object nX. We denote by $K \times_d R$ the full subcategory of the category C_2V consisting of all objects which are isomorphic to objects of the form nX *(see Figure.1)*.

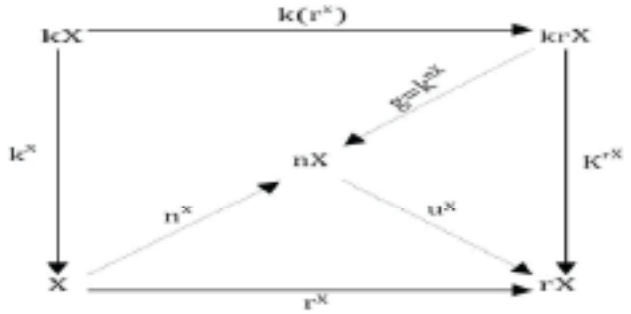


Figure 1. Commutative diagram of the Pushout and Morphisms in the Category C_2V.

Theorem 8 ([6], Theorem 2.5)**.** *Let $V = K \times_d R$. Then the next statements are equivalent:*

1. *V is a reflective subcategory of the category C_2V.*
2. *For any object X of the category C_2V the morphism n^X is the V–replique of the object X.*
3. *For any object X of the category C_2V the morphism nX is an epi.*
4. *For any object X of the category C_2V the morphism u^X is the R–replique of the object nX.*

5. $X \in |V| \Leftrightarrow r^X \cdot k^X$ *is the K–replique of the object rX.*

Theorem 9. *Let* (K,T) *be a pair of conjugated subcategories and* Γ *a* M_p*–reflective subcategory* $(\Gamma_0 \subset \Gamma)$ *of category* C_2V*. Then*

$$T\times_{sr}\Gamma = K \times_{d}(T\cap\Gamma).$$

Examples. 1. Since $(\widetilde{M}, S)$ is a pair of conjugated subcategories in the category C_2V and $\Pi = S \cap \Gamma_0$, we put

$$S \times_{sr}\Gamma_0 = \widetilde{M}\times_{d}\Pi = \Pi.$$

2. Let $q\Gamma_0$ be a subcategory of quasicomplete spaces, and sR the subcategory of semireflexive spaces [9]. Then

$$S \times_{sr}(q\Gamma_0) = \widetilde{M}\times_{d}(S \cap q\Gamma_0) = sR.$$

3. Subcategory Sc of Schwartz spaces is c – reflective. Let K be the coreflective subcategory of the category C_2V for which $(K\ ,\ Sc)$ is a pair of conjugated subcategories. Then

$$Sc \times_{sr}\Gamma_0 = iR = K \times_{d}(Sc \cap \Gamma_0),$$

iR is a subcategory of inductive semireflexive spaces (see [1], Theorem 1.5).

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