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Optimality Criterion and Algorithm for Solving the Traveling Salesman Problem

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Abstract

Changing the direction of research from exact methods to methods such as genetics, with heuristic approaches, is promising for creating the desired algorithm. To this end, the paper presents a method for solving the traveling salesman problem, considering it as a transportation-type problem that can be easily solved by a modified distribution method. The main results are related to the expansion of the application of the latest technology. The criteria for optimality of the solution are presented. A new algorithm has been developed, as well as a new, fairly general, test problem generator with a given optimal solution, which is consistent with the hypothesis about the need to conduct mass computational experiments to build an effective way to solve the traveling salesman problem.

Keywords: signs of optimality of a solution, three-element replacement operation, modified distribution method, traveling salesman problem.

1 Introduction

The study relates to the traveling salesman problem, which consists of finding the optimal route that passes through each of the given points once and then returns to the starting point. The goal of the traversal is to minimize a function related to the route cost, distance traveled, or time spent [1]. According to its mathematical formulation, the traveling salesman problem is related to transport-type problems and can

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be used in other situations: to optimize the data transmission route between different nodes, in creating cost-effective production, drone mission planning, to develop more advanced software systems useful for technologies such as driverless cars, and in developing software for processing large amounts of information [2],[3],[4]. In continuation of the research [5], the goal here is to develop a new computational algorithm for solving the traveling salesman problem, based on the use of the three-element replacement operation and the technology of the modified distribution method.

2 Optimality criteria

Let us present two criteria for the optimality of the solution to the traveling salesman problem. One criterion is presented as a necessary feature of optimality, and the other as a sufficient condition for the optimality of the solution. Definition: The operation of three-element replacement is the replacement of three elements (i_1, j_1), (i_2, j_2), (i_3, j_3) of a given route with a cyclic sequence of other three elements. Let c be the matrix of transport costs, $x_{a_1b_1}, x_{a_2b_2}, ..., x_{a_nb_n}$ – arbitrary feasible route (solution) to the traveling salesman problem. We form combinations of three unit elements $x_{i_1j_1} = 1$, $x_{i_2j_2} = 1$, and $x_{i_3j_3} = 1$ in this solution and calculate

$$C(i_1, j_1, i_2, j_2, i_3, j_3) = c(i_1j_1 + c(i_2j_2) + c(i_3j_3).$$

Then for the optimal plan and any of its 3-element combinations, the following inequality must be satisfied

$$C(i_1, j_1, i_2, j_2, i_3, j_3 \leq \min(C(i_2, j_1, i_3, j_2, i_1, j_3, C(i_3, j_1, i_1, j_2, i_2, j_3).$$

If the inequality is not satisfied for a given solution, then it is not optimal, since after performing a three-element replacement operation $x_{i_2j_1} = 1$, $x_{i_3j_2} = 1$,$x_{i_1j_3} = 1$ or $x_{i_3j_1} = 1$, $x_{i_1j_2} = 1$,$x_{i_2j_3} = 1$ (depending on the order of the term on the right-hand side of the inequality at which the minimum is achieved), and putting $x_{i_1j_1} = 0$,

$x_{i_2j_2} = 0$,$x_{i_3j_3} = 0$, we will get the best solution. This three-element replacement operation is used to obtain a new feasible cyclic solution with a lower cost value, and the solution is again checked for finality. A necessary criterion for the optimality of a solution is that it is impossible to perform a three-element replacement operation for an optimal cyclic solution. Based on the optimality criterion, it is possible to construct schemes for solving the traveling salesman problem within the framework of the technology of the modified distribution method for solving the transport problem. At the same time, its calculation stages change significantly in connection with the requirement for the cyclicality of the route and the elimination of the degeneracy of the solution in the corresponding transport problem. Let us consider another criterion that is sufficient for the optimality of the cyclic set of $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ in the traveling salesman problem.

Theorem 1. *Let there be some vector $v = (v-1, v_2, ..., v_n)$ and the set of $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ satisfies the v-optimality condition: $c_{i_rj_r} - v_{j_r} \leq c_{i_rk} - v_k$ for all k. Then this set is a solution to the traveling salesman problem.*

A sufficient criterion can be used to test a given solution for optimality. To do this, a system of inequalities $c_{i_rj_r} - v_{j_r} \leq c_{i_rk} - v_k$ is constructed for each element of the $c_{i_rj_r}$. If the vector v can be determined from these inequalities, then the set $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ is optimal, which follows from the cyclicity of the set and the equality

$$sum_{r=1}^{n} c_{i_rj_r} = sum_{r=1}^{n} v_{j_r} + sum_{r=1}^{n} (c_{i_rj_r} - v_{j_r}),$$

in which on the right-hand side one sum over v is constant, and the other is minimal due to the v-minimality of the elements $c_{i_rj_r}$, $r = \overline{1,n}$.

3 Solution of the traveling salesman problem as a special problem of the transport type

At the initial stage of Algorithm 1, two methods were used to obtain a non-degenerate cyclic solution: using the cost matrix ms (mc method)

and the evaluation matrix mo (mo method). When using the matrix mc (mo), its elements were first ordered in ascending (descending) order, and then, based on the first element selected from the ordered sequence, the next element was selected, representing a cyclical continuation of the previous one. Such iterations are carried out until a final cyclic solution is obtained. At the stage of calculating potentials, the constructed solution must be supplemented with fictitious zeros to eliminate degeneracy and in such a way that it is possible to complete the process of determining all $2n$ potentials.

Algorithm 1.

1. Definition of the matrix $c_{i_r j_r}$ of order n.
2. Formation of the initial route (xv, yv), $f = sum_{i=1}^{n} c(xv_i, yv_i)$.
3. Start of iteration of searching for the optimal solution.
4. Generating a non-degenerate solution x corresponding to the transport problem and consisting of n unit elements $x(xv_i, yv_i) = 1$ and $n-1$ dummy zeros $x(xve_j, yve_j)$, $j = \overline{1,n}$. Calculation of potentials u_i and v_j, $i, j = \overline{1,n}$.
5. Definition of the evaluation matrix $muv = muv(u_i, v_j)$.
6. For each zero position (i, j) of the matrix x, $x_{ij} = 0$, and with a positive value of the evaluation matrix muv, $muv(u_i, v_j) > 0$, calculation of two cycles (1) and (2) is performed.
7. One cycle is built starting from cell zero to cell one in a row i_1,

$$(i_1^1, j_1^1), (i_2^1, j_2^1), ..., (i_k^1, j_k^1) \tag{1}$$

and another cycle starting from the zero cell in the direction of the fictitious zero in the row i_1,

$$(i_1^\epsilon, j_1^\epsilon), (i_2^\epsilon, j_1^\epsilon), ..., (i_k^\epsilon, j_k^\epsilon). \tag{2}$$

8. Recalculation of the initial solution (xv, yv) for each of the cycles.
9. Formation of new solutions (xm_1^1, ym_1^1) and $(xm_1^\epsilon, ym_1^\epsilon)$.
10. Checking the cyclicality of new solutions.
11. Selecting the best cyclical solution.

12. Finding the best solution based on (xm_1^1, ym_1^1) and $(xm_1^\epsilon, ym_1^\epsilon)$ solutions using 3-element replacement operations.
13. Based on the best cyclic solution found, we form a new initial solution (xv, yv).
14. Go to step 3, to the next iteration.
15. Steps 3–14 are repeated a specified number of times if we do not know in advance the optimal solution to the problem being solved; when solving test problems such that the optimal solution s known in advance, the iterations continue until the final solution is obtained.
The presented Algorithm 1 is a sequential application of 3-element substitution operations to solve the traveling salesman problem. The new cyclic solution is calculated by the cycle within the framework of the modified distributive method.

4 Construction of the traveling salesman problem with a predetermined optimal solution

Test problems are necessary to check the correctness of the calculation schemes and evaluate their efficiency. Algorithm 2 is based on the representation of the cost matrix as a decomposition into components. The cost matrix c is represented as the sum of three matrices: co is the optimality matrix, cb is the base matrix, and cr is the reduction matrix.

Algorithm 2.
1. Generate a cyclic substitution (xo, yo) that corresponds to the optimal route

$$(i_1, j_1), (i_2, j_2), ..., (i_n, j_n).$$

2. Formation of the optimality matrix co, in which the elements in positions $(xo(k), yo(k))$ are non-zero, as well as the diagonal elements,

$$co(i,j) = \begin{cases} -a, & \text{for } i = i_k, j = j_k, k = \overline{1,n} \\ Inf & \text{if } i = j \\ 0, & \text{in other cases} \end{cases}$$

where a is a positive integer constant.
3. Generate the base matrix cb with elements

$$cb(i,j) = \begin{cases} 0, & \text{for } i = i_k, j = j_k, \text{or } i = j, \\ b_{ij}, & \text{in other cases} \end{cases}$$

where b_{ij} are integer non-negative random numbers not less than a.
4. We define two random positive integer vectors p and q and a reduction matrix cr,

$$cr(i,j) = \begin{cases} p_i + q_j, & i \neq j, \\ 0, & i = j \end{cases} .$$

5. We construct the cost matrix c of the traveling salesman problem with a given optimal solution (xo, yo),

$$c = co + cb + cr.$$

Theorem 2. *Traveling salesman problem with cost matrix $c = co + cb + cr$, constructed according to Algorithm 2, has an optimal solution (xo, yo).*

Proof. From the construction of the matrix c it follows that, after reduction by rows and columns, in the matrix $co + cb = c - p - q$, in positions $(xo(i)), yo(i))$, $i = \overline{1,n}$, there are zeros, and in the remaining positions there are non-negative numbers. By construction, the sequence $(xo(i), yo(i))$ forms a cyclic route. Therefore, $(xo(i), yo(i))$, $i = \overline{1,n}$, is the optimal route of the traveling salesman problem with matrix c.

5 Conclusion

By successively applying three-element replacement operations in the current cycle of the modified distribution method, a good approximation to an acceptable solution occurs without large requirements for RAM. Computational experiments were carried out in various variants of generating the initial reference plan: using the above-described mc

method, the mo method, or with the selection of a random route, which was completed to the reference plan. If the new solution is better than the current one, it becomes the initial solution. This adaptation of the exact modified distributive method proves useful and is recommended for obtaining an acceptable solution to the traveling salesman problem of sufficiently large size in the allotted time.

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