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The stability analysis of approximate schemes for the plane fractional order diffusion equation

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Abstract

The evolution equation of fractional order by the space variables is considered in this paper. This equation models the diffusion factor in the process to transport any substance in some medium. An approximate scheme to solve numerically this problem is constructed, using the decomposition principle.

Keywords: diffusion equation, equation of fractional order, approximate scheme.

1 Introduction

The evolution equation simulates a lot of the problems that appear in physique, ecologies, hydrogeology, finance etc. For example, in the mathematical modeling of the problem to transport any substance in atmosphere the main factors are the diffusion process, absorbtion of substance and advection convection process. The classical model of this evolution problem with one space variable uses the usual partial derivatives of first and second order. In recent years many authors use the fractional space derivative to modeling such process. In this article it is considered the same problem with two space variables of the form

$$\frac{\partial\varphi}{\partial t} - d_+(x)\frac{\partial^\alpha\varphi}{\partial_+x^\alpha} - d_-(x)\frac{\partial^\alpha\varphi}{\partial_-x^\alpha} - d_+(y)\frac{\partial^\alpha\varphi}{\partial_+y^\alpha} - d_-(y)\frac{\partial^\alpha\varphi}{\partial_-y^\alpha} = f(x,y,t),$$

$$\varphi(x,y,0) = s(x,y), \qquad \varphi(x,y,t) = 0 \qquad on \qquad the \qquad \partial D, \qquad (1)$$

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in the domain $D = [0, a] \times [0, b]$ with the boundary ∂D and the time interval $[0, T]$, where $1 < \alpha \leq 2, 0 < x < a, 0 < y < b, 0 \leq t \leq T, d_+(x) \geq 0, d_-(x) \geq 0$. The left-hand (+) and the right-hand (-) fractional derivatives of order α in (1) are defined by Riemann-Liouville formulas

$$\frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_0^x \frac{\varphi(\xi, y, t) d\xi}{(x-\xi)^{\alpha+1-m}},$$

$$\frac{\partial^\alpha \varphi}{\partial_- x^\alpha} = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial x^m} \int_x^a \frac{\varphi(\xi, y, t) d\xi}{(\xi-x)^{\alpha+1-m}}, \tag{2}$$

where m is a less integer such that $m - 1 < \alpha \leq m$. The analogous formula holds for the fractional derivative by variable y.

2 The decomposition principle in construction of the weighted approximate schemes

To construct an approximate scheme for (1) let τ be the time step, $t_n = n\tau$ and h is a space step of grid, $x_i = ih, y_k = kh \;\; i, k = 0, 1, 2, \ldots, M$. Let $\varphi_{i,k}^n$ be the numerical approximate value of $\varphi(x_i, y_k, t_n)$. Using the notations

$$A_x = -d_+(x) \frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} - d_-(x) \frac{\partial^\alpha \varphi}{\partial_- x^\alpha}, \quad A_y = -d_+(y) \frac{\partial^\alpha \varphi}{\partial_+ y^\alpha} - d_-(y) \frac{\partial^\alpha \varphi}{\partial_- y^\alpha} \tag{3}$$

the equation (1) can be written in the form

$$\frac{\partial \varphi}{\partial t} + A_x \varphi + A_y \varphi = f. \tag{4}$$

The operators A_x, A_y are defined in the space of functions $\varphi(x, y, t)$ that satisfies the initial and boundary conditions of the problem (1) and for any fixed value of t this function belongs to $L^2(D)$. Also we consider that the operators A_x, A_y are positive defined and are discretized using the following Grunwald formula for the left-hand and

right-hand fractional derivatives:

$$\frac{\partial^\alpha \varphi}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{m=0}^{i+1} g_m \varphi(x_i - (m-1)h, y_k, t_n) + o(h)$$

$$\frac{\partial^\alpha \varphi}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{m=0}^{M-i+1} g_m \varphi(x_i + (m-1)h, y_k, t_n) + o(h), \qquad (5)$$

and the analogous formulas for the variable y, where

$$g_0 = 1, \qquad g_m = (-1)^m \frac{\alpha(\alpha-1)...(\alpha-m+1)}{m!}, \qquad m = 1, 2, 3, \ldots$$

We denote the corresponding operators defined by the right sides of the equality (5) through Λ_x^+ and Λ_x^-. For the variable y the analogous operators are denoted by Λ_y^+ and Λ_y^- using corresponding equality by the variable y. Thus the operator A_x is approximated by $\Lambda_x = \Lambda_x^+ + \Lambda_x^-$ and the operator A_y is approximated by $\Lambda_y = \Lambda_y^+ + \Lambda_y^-$.

The approximation of the operators Λ_x, Λ_y by the variable t on the interval $[t_{n-1}, t_{n+1}]$ is defined as the weighted average with the parameter r of each operator Λ_x^+ and Λ_x^- at the time points t_{n-1} and t_{n+1}. Thus at the point t_n the value of the operator $\Lambda_x \varphi$ is replaced by the formula

$$\Lambda_x^n \varphi = \frac{d_i^+}{h^\alpha}[r\Lambda_x^+ \varphi(t_{n-1}) + (1-r)\Lambda_x^+ \varphi(t_{n+1})]$$

$$+ \frac{d_i^-}{h^\alpha}[r\Lambda_x^- \varphi(t_{n-1}) + (1-r)\Lambda_x^- \varphi(t_{n+1})]. \qquad (6)$$

The operator $\Lambda_y \varphi$ is replaced by the analogous formula. The first order time derivative is discretized using the central finite difference formula. In the next formulas we use the notation $\varphi(x_i, y_k, t_n) = \varphi_{i,k}^n$ and the inferior index will be omitted. According to the decomposition principle, equation (4) with two space variables is approximated on the interval $t_{n-1} \le t \le t_{n+1}$ by a system of four equations. All equations

of this system has the similar form. We consider, for example, the first equation on time interval $[t_{n-1/2}, t_{n-1}]$

$$\frac{\varphi^{n-\frac{1}{2}} - \varphi^{n-1}}{\tau} + \Lambda_x^n \varphi^{n-\frac{3}{4}} = 0. \tag{7}$$

In detailed form this equation becomes

$$\frac{\varphi_{i,j}^{n-1/2} - \varphi_{i,j}^{n-1}}{\tau} = \frac{d_i^+}{h^\alpha}[r\sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1/2} + (1-r)\sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1}]$$

$$+\frac{d_i^-}{h^\alpha}[r\sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1/2} + (1-r)\sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1}].$$

For the second and third equations of the system (7) the function $f \neq 0$ and is approximated by addition of the the sum $r f_{i,j}^{n-1/2} + (1-r) f_{i,j}^{n-1}$ at the right side of the precedent equality. Through simple transformations we obtain

$$\varphi_{i,j}^{n-1/2} - (1-r)[\xi_i \sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1/2} - \eta_i \sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1/2}] =$$

$$\varphi_{i,j}^{n-1} + r[\xi_i \sum_{k=0}^{i+1} g_k \varphi_{i-k+1,j}^{n-1} + \eta_i \sum_{k=0}^{M-i+1} g_k \varphi_{i+k-1,j}^{n-1}] \tag{8}$$

for $i = 1, 2, \ldots, M-1, \qquad n = 1, 2, \ldots, N-1$, where

$$\xi_i = \frac{\tau d_i^+}{h^\alpha}, \qquad \eta_i = \frac{\tau d_i^-}{h^\alpha}.$$

Using the boundary conditions it is obtained the matrix form of (8)

$$[I - (1-r)A]\Phi^{n-1/2} = (I + rA)\Phi^{n-1}, \tag{9}$$

$$\Phi^{n-1/2} = [\varphi_{1,j}^{n-1/2}, \varphi_{2,j}^{n-1/2}, \ldots, \varphi_{M-1,j}^{n-1/2}]' \quad \text{and}$$

$$\Phi^{n-1} = [\varphi_{1,j}^{n-1}, \varphi_{2,j}^{n-1}, \ldots, \varphi_{M-1,j}^{n-1}]'$$

and the elements of the matrix $A = (a_{i,j}), i, j = 1, 2, \ldots, M-1$ are defined by following formulas:

$$a_{i,j} = \left\{ \begin{array}{cc} (\xi_i + \eta_i)g_1, & j = i, \\ \xi_i g_2 + \eta_i g_0, & j = i-1, \\ \xi_i g_0 + \eta_i g_2, & j = i+1, \\ \xi_i g_{i-j+1}. & j < i-1, \\ \eta_i g_{j-i+1}, & j > i+1. \end{array} \right\}. \qquad (10)$$

For approximate scheme (7) the following theorem is true:

Theorem 1. *The order of local error approximation of equation (7) by the system (9) is* $O(\tau + h)$ *for* $0 \le r < \frac{1}{2}$ *and this order is* $O(\tau^2 + h)$ *for the value of* $r = \frac{1}{2}$.

3 Stability of the approximate scheme

In order to prove the stability of scheme (7) (in its matrix form (9)) we first remark that the coefficients ξ_i, η_i are non-negative. Also the Grunwald coefficients g_k satisfy the following properties

$$g_1 = -\alpha, g_k \ge 0 (k > 1), \quad \sum_{k=0}^{\infty} g_k = 0, \sum_{k=0, k\neq 1}^{N} g_k \le \alpha, \quad N = 1, 2, \ldots \qquad (11)$$

The stability of scheme (7) is considered in sense of recurrence. In matrix form first equation of this scheme can be written in the form

$$\Phi^{n-1/2} = P\Phi^{n-1} : with\ P = [I - (1-r)A]^{-1}(I + rA),$$

and Φ^0 is obtained from the initial condition of the problem.

Theorem 2. *The approximate scheme (11) of the initial problem is unconditionally stable for* $0 \le r \le 1/2$. *If* $1/2 < r \le 1$, *then the scheme (7) is conditionally stable when*

$$(\xi + \eta)\alpha \le \frac{1}{2r-1}\frac{h^\alpha}{\tau},$$

where $\xi = max(\xi_x, \xi_y)$ *and* $\xi_x = max(d_+(x))$, $\eta = max(d_-(x))$ *on the interval* $[0, a]$, $\xi_y = max(d_+(y))$, $\eta = max(d_-(y))$ *on* $[0, b]$.

Proof. According to (10) we have for the diagonal elements of the matrix A $a_{ii} = (\xi_i + \eta_i)g_1 = -(\xi_i + \eta_i)\alpha$. The sum of non-diagonal elements of line i , denoted by R_i can be evaluated as follows

$$R_i = \sum_{k=1,k\neq i}^{M-1} A_{ik} = \sum_{k=0,k\neq 1}^{i} \xi_i g_k + \sum_{k=0,k\neq 1}^{M-i} \eta_i g_k \leq (\xi_i + \eta_i)\alpha.$$

Using Gherschgorin theorem for every eigenvalue λ there exist A_{ii} such that

$$\mid \lambda - A_{ii} \mid \leq R_i, \qquad |\lambda + (\xi_i + \eta_i)\alpha| \leq (\xi_i + \eta_i)\alpha.$$

From the last inequality the real parts of the eigenvalue of matrix A are non-positive. The eigenvalue of matrix P is

$$\lambda_P = \frac{1 + r\lambda}{1 - (1 - r)\lambda}.$$

If $0 \leq r \leq 1/2$ the inequality $\mid \lambda_P \mid \leq 1$ holds for any r, then the spectral radius of the matrix P is not greater than 1, therefore $\parallel P \parallel \leq 1$ and the algorithm is stable. If $1/2 \leq r \leq 1$, the inequality $\mid \lambda_P \mid \leq 1$ is established under the condition $(\xi_i + \eta_i)\alpha \leq \frac{1}{2r-1}$. Hence, the scheme (9) is conditionally stable under the restriction

$$(\xi + \eta)\alpha \leq \frac{1}{2r - 1}\frac{h^\alpha}{\tau}.$$

The proof of stability for equations (2)-(4) of (7) is analogous.

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