Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

On a Method of Prolongation of Quasigroups

Elena Cuznețov

Abstract

We consider a method of prolongation of finite quasigroups using two transversals, which intersect exactly in one cell, and study the recursive 1-differentiability of such prolongations.

Keywords: quasigroup, transversal, prolongation, recursive derivative, recursively differentiable quasigroup.

The construction of prolongations was first studied by Bruck (1944) who considered finite idempotent quasigroups for this purpose [1]. Note that the notion "prolongation" was introduced by Belousov in 1967 [2].

Constructions of prolongations of finite quasigroups has been given by Osborn (1961), Yamamoto (1961), Denes and Pasztor (1963), Belousov and Belyavskaya (1968), Belyavskaya (1969), Deriyenko and Dudek (2008, 2013) and others.

The recursive derivative of order k of a quasigroup $(Q, \cdot)$, denoted by $(\overset{k}{\cdot})$, where $k \in N$, is defined as follows:

$$x \overset{0}{\cdot} y = x \cdot y;$$
$$x \overset{1}{\cdot} y = y \cdot (x \cdot y);$$
$$x \overset{k}{\cdot} y = (x \overset{k-2}{\cdot} y) \cdot (x \overset{k-1}{\cdot} y), for\ all\ k \geq 2.$$

A binary quasigroup $(Q, \cdot)$ is called recursively r-differentiable if its recursive derivatives $(\overset{k}{\cdot})$, where $k = 1, ..., r$, are quasigroup operations.

It is an open problem to find the maximum order of recursive differentiability of a finite binary (or n-ary) quasigroup. Some estimations are given in [3]–[6], where, in particular, it is shown that:

© 2024 by Elena Cuznețov

1. The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q-2$;
2. There exist finite binary recursively $(q-2)$ - differentiable quasigroups of any primary order $q \geq 3$.

The recursive differentiability of some known prolongations is studied in [7]. We give a new method of prolongation and investigate the possibility to extend the recursive differentiability of a finite binary quasigroup to its prolongations in the present work.

In the given method we use 2 transversals having exactly 1 cell in their intersection. The idea of the proposed method of prolongations is shown in the following example, where we give the prolongation of a quasigroup of order 5, with the transversals marked by colored and, respectively, uncolored squares, which intersect in the cell $(1,1)$, with 2 new elements denoted by ξ_1 and ξ_2.

$\cdot$	1	2	3	4	5
1	2	4	1	3	5
2	1	3	5	2	4
3	5	2	4	1	3
4	4	1	3	5	2
5	3	5	2	4	1

$\longrightarrow$

$\circ$	1	2	3	4	5	ξ_1	ξ_2
1	ξ_1	4	1	3	5	2	ξ_2
2	1	ξ_1	ξ_2	2	4	3	5
3	5	2	ξ_1	1	ξ_2	4	3
4	4	ξ_2	3	ξ_1	2	5	1
5	3	5	2	ξ_2	ξ_1	1	4
ξ_1	ξ_2	1	5	4	3	ξ_1	2
ξ_2	2	3	4	5	1	ξ_2	ξ_1

Proposition 1. *For 2 fixed transversals of a finite quasigroup $(Q,\cdot)$, that intersect exactly in one cell, there exist 12 possible prolongations $(Q',\circ)$, with two new elements ξ_1 and ξ_2, given bellow in I-XII. Moreover, the prolongations I, IV, VI, VII, X and XI are not recursively 1-differentiable, as it is shown after their Cayley tables.*

I.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	a	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_2	ξ_1
ξ_2	...	...	ξ_2	...	...	ξ_1	a

$\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_1$
$\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_1$

II.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	a	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	ξ_1	a
ξ_2	...	...	a	...	...	ξ_2	ξ_1

III.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	ξ_2	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	a	ξ_1
ξ_2	...	...	a	...	...	ξ_1	ξ_2

IV.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	ξ_2	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_1	ξ_2
ξ_2	...	...	ξ_2	...	...	a	ξ_1

$\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_1$
$\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_1$

V.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	a	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_1	ξ_2
ξ_2	...	...	ξ_1	...	...	ξ_2	a

VI.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	a	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	ξ_2	a
ξ_2	...	...	a	...	...	ξ_1	ξ_2

$\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_2$
$\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_2$

VII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	ξ_1	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	a	ξ_2
ξ_2	...	...	a	...	...	ξ_2	ξ_1

$\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_2$
$\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_2$

VIII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	ξ_1	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_2	ξ_1
ξ_2	...	...	ξ_1	...	...	a	ξ_2

IX.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	ξ_2	a
ξ_2	...	...	ξ_2	...	...	a	ξ_1

X.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	a	ξ_1
ξ_2	...	...	ξ_1	...	...	ξ_2	a

$x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_1 = a$
$x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_2 = a$

XI.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_2	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	ξ_1	a
ξ_2	...	...	ξ_1	...	...	a	ξ_2

$x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_2 = a$
$x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_1 = a$

XII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_2	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	a	ξ_2
ξ_2	...	...	ξ_2	...	...	ξ_1	a

We study if the prolongations of a finite recursively 1-differentiable quasigroup, using 2 transversals which intersect exactly in one cell is recursively 1-differentiable. According to Proposition 1, it remains to consider a total of 6 possible cases.

Proposition 2. *There exist exactly 240 latin squares of order 5, having 2 transversals which intersect exactly in one cell, one of which is on the main diagonal (fixed).*

Proposition 3. *There does not exist recursively 1-differentiable prolongations of quasigroups of order 5, obtained using the main diagonal with the fixed order* $\{2, 3, 4, 5, 1\}$ *and an arbitrary second transversal which intersects the main diagonal exactly in one cell.*

Acknowledgments. This work was partially supported by the Institutional Research Program 011302 "MANSDP" for 2024-2027, Moldova State University.

References

[1] R. H. Bruck. *Some results in the theory of quasigroups.* Trans. Amer. Math. Soc., vol. 55, 1944, pp. 19–52.

[2] V. Belousov. *Foundations of the Theory of Quasigroups and Loops (in Russian).* Nauka, Moscow, 1967.

[3] S. Gonsales, E. Couselo, V. T. Markov, and A. A. Nechaev. *Recursive MDS codes and recursively differentiable quasigroups.* Diskret. Mat., vol. 10(2), 1998, pp. 3–29.

[4] V. T. Markov, A. A. Nechaev., S. S. Skazhenik, and E. O. Tveritinov. *Pseudogeometries with clusters and an example of a recursive* $[4, 2, 3]_{42}$*-code.* J. Math. Sci., vol. 163, no. 5, 2009, pp. 563–571.

[5] P. Syrbu. *On the order of recursive differentiability of finite binary quasigroups.* Bul. Acad. Ştiinţe Repub. Mold., Mat., no. 3(103), 2023, pp. 103–106.

[6] P. Syrbu and E. Cuzneţov. *On recursively differentiable k-quasigroups.* Bul. Acad. Ştiinţe Repub. Mold., Mat., no. 2(99), 2022, pp. 68–75.

[7] P. Syrbu and E. Cuzneţov. *On recursive 1-differentiability of quasigroup prolongations.* Bul. Acad. Ştiinţe Repub. Mold., Mat., no. 2(101), 2023, pp. 102–109.

Elena Cuzneţov

Moldova State University
E–mail: lenkacuznetova95@gmail.com
ORCID: https://orcid.org/0009-0004-3092-3786