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# Generalized Berezin transform and Schatten class operators on weighted Bergman spaces

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**Abstract.** In this paper we derive certain elementary properties of the generalized Berezin transform $B_\alpha,\ \alpha > -1$ and show that if $T$ is a bounded linear operator from the weighted Bergman space $L^2_a(dA_\alpha)$ into itself, and $G(w) = \left\|TK_w^{(\alpha)}\right\|^p,\ w \in \mathbb{D},\ 0 < p < \infty$, then $B_0^n G$ is subharmonic for all $n \in \mathbb{N}$ and $B_0^n \phi \to \psi$, the least harmonic majorant of $G$. Further, we describe the Schatten class characterization of a bounded linear operator $T$ in terms of the generalized Berezin transform of $T$. We have also shown that if $T \geq 0$ or $T$ is in trace class $S_1$, then $tr(T) = (\alpha+1)\int\limits_{\mathbb{D}} (B_\alpha T)(z) d\lambda(z)$ where $d\lambda(z) = \frac{dA(z)}{\left(1-|z|^2\right)^2}$. Further, if $A$ is compact and positive and $0 < p \leq 1$, then we show that $\int\limits_{\mathbb{D}} B_\alpha(A^p)(z) d\lambda(z)$ is finite if and only if $\sum\limits_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)}\right\rangle\right|^p$ is finite for some orthonormal basis $\left\{\xi_n^{(\alpha)}\right\}_{n=1}^{\infty}$ in $L^2_a(dA_\alpha)$.

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## 1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $dA(z) = \frac{1}{\pi} dxdy$ denote the normalized area measure defined on $\mathbb{D}$. Let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),\ \alpha > -1,\ z \in \mathbb{D}$. Let $L^2(\mathbb{D}, dA_\alpha)$ be the Hilbert space of all Lebesgue measurable functions on $\mathbb{D}$ that are absolutely square-integrable with respect to the measure $dA_\alpha(z)$. The weighted Bergman space $L^2_a(dA_\alpha)$ is the subspace of all analytic functions of $L^2(\mathbb{D}, dA_\alpha)$. The spaces $L^2_a(dA_\alpha)$ are closed subspaces of the corresponding spaces $L^2(\mathbb{D}, dA_\alpha),\ \alpha > -1$ and these are all reproducing kernel Hilbert spaces. For $\alpha = 0$, we shall denote $L^2_a(dA_0) = L^2_a(\mathbb{D})$ as the unweighted Bergman space of $\mathbb{D}$ whose reproducing kernel is given by $K(z,w) = \frac{1}{(1-z\overline{w})^2},\ z, w \in \mathbb{D}$ and the normalized reproducing kernel of $L^2_a(\mathbb{D})$ is given by $k_z(w) = \frac{1-|z|^2}{(1-w\overline{z})^2}$. Assume $K_z(w) = \overline{K(z,w)}$. The reproducing kernel of $L^2_a(dA_\alpha)$ is given by $K^{(\alpha)}(z,w) = [K(z,w)]^{1+\frac{\alpha}{2}} = \frac{1}{(1-z\overline{w})^{\alpha+2}}$ for $z, w \in \mathbb{D}$.

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Let $K_z^{(\alpha)}(w) = [K_z(w)]^{1+\frac{\alpha}{2}} = \overline{K^{(\alpha)}(z,w)}$. If $\langle \cdot, \cdot \rangle_\alpha$ denotes the inner product in $L^2(dA_\alpha) = L^2(\mathbb{D}, dA_\alpha)$, then $\langle h, K_z^{(\alpha)} \rangle_\alpha = h(z)$, for every $h \in L_a^2(dA_\alpha)$ and $z \in \mathbb{D}$. The orthogonal projection $P_\alpha$ from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto the closed subspace $L_a^2(dA_\alpha)$ is given by $(P_\alpha f)(z) = \left\langle f, K_z^{(\alpha)} \right\rangle_\alpha = \displaystyle\int_{\mathbb{D}} f(w) \frac{1}{(1-\overline{z}w)^{\alpha+2}} dA_\alpha(z)$ for $f \in L^2(\mathbb{D}, dA_\alpha)$ and $z \in \mathbb{D}$. The normalized reproducing kernels of $L_a^2(dA_\alpha)$ are the functions $k_z^{1+\frac{\alpha}{2}}(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$. The sequence of functions $\{e_n^{(\alpha)}\} = \left\{ \frac{z^n}{\gamma_{n,\alpha}} \right\}$ form an orthonormal basis [12] for $L_a^2(dA_\alpha)$ where

$$\gamma_{n,\alpha}^2 = \|z^n\|^2 = (\alpha+1) \int_{\mathbb{D}} |z|^{2n} (1-|z|^2)^\alpha dA(z) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}.$$

Let $L^\infty(\mathbb{D}, dA)$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, define $\|\phi\|_\infty = ess\ sup\{|\phi(z)| : z \in \mathbb{D}\} < \infty$. The space $L^\infty(\mathbb{D})$ is a Banach space with respect to the $\|\cdot\|_\infty$. Let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$ and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$.

If $\phi \in L^1(\mathbb{D}, dA_\alpha)$, the generalized Berezin transform of $\phi$ is defined by

$$(B_\alpha \phi)(z) = \left\langle \phi k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = \int_{\mathbb{D}} (\phi \circ \phi_z)(w) dA_\alpha(w),\ z \in \mathbb{D}.$$

Notice that $k_z^{1+\frac{\alpha}{2}} \in L^\infty(\mathbb{D})$ for all $z \in \mathbb{D}$, so the definition makes sense. For $\alpha = 0$, the transformation $B_0$ will be referred to as Berezin transform. It turns out [6] that the Berezin transform behaves well with respect to the invariant measure $\frac{dA(z)}{(1-|z|^2)^2}$.

The Berezin transform plays an important role in the theory of Toeplitz operators on the Bergman space and the Fock space. In this paper we discuss some of the ergodicity properties of the Berezin transform.

We now define the Schatten class operators defined on a Hilbert space $H$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators defined from the Hilbert space $H$ into itself. Let $\mathcal{LC}(H)$ be the set of all compact operators in the space $\mathcal{L}(H)$. For any non-negative integer $n$, the $n^{th}$ singular value of an operator $T \in \mathcal{LC}(H)$ is defined by

$$s_n(T) = \inf \{\|T - M\| : M \in \mathcal{LC}(H), rank\ M \leq n\},$$

where $\|\cdot\|$ is the operator norm. Also, $s_0(T) = \|T\|$ and $s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$. We can conclude $s_n(T)$ is the distance with respect to the operator norm of $T$ from the set of operators of rank at most $n$ in $\mathcal{L}(H)$. According to the spectral theorem the singular values of the compact operator $T$ are actually the square roots of the eigenvalues of $T^*T$ in case that the Hilbert space $H$ is separable and infinite dimensional. In other words, $s_n(T)$ can be defined for any $T \in \mathcal{L}(H)$ but clearly $s_n(T) \to 0$ if and only if $T$ is compact. If $T$ is a compact operator on a separable Hilbert space $H$, then there exist orthonormal sets $\{e_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ in $H$ such that $Tx = \displaystyle\sum_{n=0}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n,\ x \in H$, where $\lambda_n$ is the $n^{th}$ singular value of $T$. For

$0 < p < \infty$, the Schatten $p$-class of $H$, denoted as $S_p$ is the space [12] of all compact operators $T$ on $H$ with its singular value sequence $\{\lambda_n\}_{n=0}^{\infty}$ belonging to the $p$-summable sequence space $l^p$. The Schatten class $S_p$ is a Banach space with the norm $\|T\|_p = \left[\sum_{n=0}^{\infty} |\lambda_n|^p\right]^{\frac{1}{p}}$. The Schatten class $S_1$ is called the trace class of $H$ and $S_2$ is called the Hilbert-Schmidt class.

The plan layout of this paper is as follows: In Section 2, some elementary properties of the Berezin transform $B_\alpha,\ \alpha > -1$ are derived. We show that the Berezin transform $B_\alpha$ defined on $L^p(\mathbb{D}, d\eta_\alpha(z)),\ 1 \le p \le \infty$ is a contraction where $d\eta_\alpha(z) := \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}}$. Further, we show that $B_\alpha$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta_\alpha)$ and also $B_\alpha$ is a self-adjoint operator from $L^2(\mathbb{D}, d\eta_\alpha)$ into itself. If $f \in L^1(\mathbb{D}, dA_\alpha)$, then $(B_\alpha f)(z) = \int_{\mathbb{D}} (f \circ \phi_z)(w) dA_\alpha(w),\ z \in \mathbb{D}$. A function $\phi \in L^1(\mathbb{D}, dA_\alpha)$ is harmonic if and only if $B_\alpha \phi = \phi$. In Section 3, we consider a bounded real-valued subharmonic function $\phi$ on $\mathbb{D}$. We show that if $T$ is a bounded linear operator from the weighted Bergman space $L_a^2(dA_\alpha)$ into itself, and $G(w) = \left\|TK_w^{(\alpha)}\right\|^p,\ w \in \mathbb{D},\ 0 < p < \infty$ then $B_0^n G$ is subharmonic for all $n \in \mathbb{N}$ and $B_0^n \phi \to \psi$, the least harmonic majorant of $G$. In Section 4, we describe the Schatten class characterization of a bounded linear operator $T$ from $L_a^2(dA_\alpha)$ into itself in terms of the generalized Berezin transform of $T$. We have shown that if $T \ge 0$ or $T$ is in trace class $S_1$, then $tr(T) = (\alpha+1)\int_{\mathbb{D}} (B_\alpha T)(z) d\lambda(z)$ where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$. Further, we show that if $A$ is compact and positive and $0 < p \le 1$, then $\int_{\mathbb{D}} B_\alpha(A^p)(z) d\lambda(z)$ is finite if and only if $\sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)}\right\rangle\right|^p$ is finite for some orthonormal basis $\left\{\xi_n^{(\alpha)}\right\}_{n=1}^{\infty}$ in $L_a^2(dA_\alpha)$.

## 2 Generalized Berezin transform of integrable functions

In this section, we derive some elementary properties of the Berezin transform $B_\alpha,\ \alpha > -1$. We establish that the Berezin transform $B_\alpha$ defined on $L^p(\mathbb{D}, d\eta_\alpha(z)),\ 1 \le p \le \infty$ is a contraction where $d\eta_\alpha(z) := \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}}$. Further, we establish that $B_\alpha$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta_\alpha)$ and also $B_\alpha$ is a self-adjoint operator from $L^2(\mathbb{D}, d\eta_\alpha)$ into itself. If $f \in L^1(\mathbb{D}, dA_\alpha)$, then $(B_\alpha f)(z) = \int_{\mathbb{D}} (f \circ \phi_z)(w) dA_\alpha(w),\ z \in \mathbb{D}$. We have shown that the function $f \in L^1(\mathbb{D}, dA_\alpha)$ is harmonic if and only if $f$ is a fixed point of $B_\alpha$. If $f \in L^1(\mathbb{D}, dA_\alpha),\ \alpha > -1$, the generalised Berezin transform of $f$ is defined by

$$(B_\alpha f)(z) = \left\langle f k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle = \int_{\mathbb{D}} f(w) \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w),\ z \in \mathbb{D}. \tag{1}$$

The functions $k_z \in L^\infty(\mathbb{D})$ for all $z \in \mathbb{D}$. Hence the definition makes sense. For $\alpha = 0$, we shall write $(B_0 f)(z) = \widetilde{f}(z)$. The function $\widetilde{f}$ is an infinitely differentiable function on $\mathbb{D}$. It can be verified by differentiating under the integral sign. If $f \in L^\infty(\mathbb{D})$, then $B_\alpha f$ is also bounded and $\|B_\alpha f\| \leq \|f\|_\infty$. The operator $B_\alpha$ is a contraction in $L^\infty(\mathbb{D})$. Since by Cauchy-Schwarz inequality, we have

$$|(B_\alpha f)(z)| = \left|\left\langle f k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle\right| \leq \|f\|_\infty \left\|k_z^{1+\frac{\alpha}{2}}\right\|_2^2 = \|f\|_\infty.$$

If $f$ is a constant function then $f = B_\alpha f$ for all $\alpha > -1$ and so the norm of $B_\alpha$ on $L^\infty(\mathbb{D})$ is equal to 1. Further if $f \geq 0$ then $B_\alpha f \geq 0$ and if $f \geq g$ then $B_\alpha f \geq B_\alpha g$ as $B_\alpha$ is an integral operator with positive kernel.

**Lemma 1.** *The operator $B_\alpha$ defined on $L^p(\mathbb{D}, d\eta_\alpha(z))$, $1 \leq p \leq \infty$ is a contraction where*

$$d\eta_\alpha(z) := \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}} = (\alpha+1)\frac{dA(z)}{(1-|z|^2)^2}.$$

*Further, $B_\alpha$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta_\alpha)$. Again, $B_\alpha$ is a self-adjoint operator from $L^2(\mathbb{D}, d\eta_\alpha)$ into itself.*

*Proof.* Since $L^1(\mathbb{D}, d\eta_\alpha) \subset L^1(\mathbb{D}, dA_\alpha)$, hence the generalized Berezin transform $B_\alpha$ is defined on the space $L^1(\mathbb{D}, d\eta_\alpha)$. Further

$$|(B_\alpha f)(z)| = \left|\int_{\mathbb{D}} f(w) \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w)\right| \leq \int_{\mathbb{D}} |f(w)| \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w) = B_\alpha(|f|)(z).$$

$$\begin{aligned}
\text{Thus } \int_{\mathbb{D}} |(B_\alpha f)(z)| \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}} &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(w)| \left|k_z^{1+\frac{\alpha}{2}}(w)\right|^2 dA_\alpha(w)\right) \frac{dA_\alpha(z)}{(1-|z|^2)^{2+\alpha}} \\
&= \int_{\mathbb{D}} |f(w)| \int_{\mathbb{D}} \frac{dA_\alpha(z)}{|1-\overline{z}w|^{4+2\alpha}} dA_\alpha(w) \\
&= \int_{\mathbb{D}} |f(w)| \left\langle K_w^{(\alpha)}, K_w^{(\alpha)} \right\rangle dA_\alpha(w) \\
&= \int_{\mathbb{D}} |f(w)| \frac{dA_\alpha(w)}{(1-|w|^2)^{2+\alpha}},
\end{aligned}$$

the change of the order of integration being justified by the positivity of the integrand. Thus it follows that $B_\alpha$ is a contraction on $L^1(\mathbb{D}, d\eta_\alpha)$. The same is true for $L^\infty(\mathbb{D})$. The result follows from the Marcinkiewicz interpolation theorem [9]. We now verify that $B_\alpha$ is a self-adjoint operator on $L^2(\mathbb{D}, d\eta_\alpha)$. For $\phi \in L^2(\mathbb{D}, d\eta_\alpha)$,

$$\begin{aligned}
\langle B_\alpha \phi, \phi \rangle &= \int_{\mathbb{D}} (B_\alpha \phi)(z) \overline{\phi(z)} d\eta_\alpha(z) \\
&= \int_{\mathbb{D}} (B_\alpha \phi)(z) \overline{\phi(z)} K^{(\alpha)}(z,z) dA_\alpha(z) \\
&= \int_{\mathbb{D}} \left\langle \phi k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle \overline{\phi(z)} K^{(\alpha)}(z,z) dA_\alpha(z)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \phi(w) k_z^{1+\frac{\alpha}{2}}(w) \overline{k_z^{1+\frac{\alpha}{2}}(w)} dA_\alpha(w) \right) \overline{\phi(z)} K^{(\alpha)}(z,z) dA_\alpha(z) \\
&= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \phi(w) \left| k_z^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w) \right) \overline{\phi(z)} K^{(\alpha)}(z,z) dA_\alpha(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \phi(w) \left| K^{(\alpha)}(z,w) \right|^2 dA_\alpha(w) \overline{\phi(z)} dA_\alpha(z) \\
&= \int_{\mathbb{D}} \phi(w) K^{(\alpha)}(w,w) dA_\alpha(w) \int_{\mathbb{D}} \overline{\phi(z)} \frac{\left| K^{(\alpha)}(z,w) \right|^2}{K^{(\alpha)}(w,w)} dA_\alpha(z) \\
&= \int_{\mathbb{D}} \phi(w) d\eta_\alpha(w) \int_{\mathbb{D}} \overline{\phi(z)} \left| k_w^{1+\frac{\alpha}{2}}(z) \right|^2 dA_\alpha(z) \\
&= \int_{\mathbb{D}} \phi(w) \overline{\left( \int_{\mathbb{D}} \phi(z) \left| k_w^{1+\frac{\alpha}{2}}(z) \right|^2 dA_\alpha(z) \right)} d\eta_\alpha(w) \\
&= \int_{\mathbb{D}} \phi(w) \overline{(B_\alpha \phi)(w)} d\eta_\alpha(w) \\
&= \langle \phi, B_\alpha \phi \rangle_{L^2(\mathbb{D}, d\eta_\alpha)} .
\end{aligned}
$$

□

**Lemma 2.** *For $f \in L^1(\mathbb{D}, dA_\alpha)$, $\alpha > -1$*
*(i) $(B_\alpha f)(z) = \int_{\mathbb{D}} (f \circ \phi_z)(w) dA_\alpha(w)$, $z \in \mathbb{D}$.*
*(ii)$B_\alpha C_a^{(\alpha)} f = C_a^{(\alpha)} B_\alpha f$, where $a \in \mathbb{D}$ and $C_a^{(\alpha)} f = f \circ \phi_a$.*

*Proof.* Notice that $\phi_a'(z) = -k_a(z)$ and $J_{\phi_a}(z) = |k_a(z)|^2$ for all $a, z \in \mathbb{D}$. Since $(k_z \circ \phi_z) k_z = 1$ for all $z \in \mathbb{D}$, hence

$$
\begin{aligned}
(B_\alpha f)(z) &= \int_{\mathbb{D}} f(w) \left| k_z^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w) \\
&= \int_{\mathbb{D}} (f \circ \phi_z)(w) \left| (k_z \circ \phi_z)(w) \right|^{2+\alpha} \left| k_z^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w) \\
&= \int_{\mathbb{D}} (f \circ \phi_z)(w) \left| (k_z \circ \phi_z)(w) \right|^{2+\alpha} |k_z(w)|^{2+\alpha} dA_\alpha(w) \\
&= \int_{\mathbb{D}} (f \circ \phi_z)(w) dA_\alpha(w).
\end{aligned}
$$

This proves $(i)$. To prove $(ii)$, let $f \in L^1(\mathbb{D}, dA_\alpha)$, $a \in \mathbb{D}$, $z \in \mathbb{D}$. Then

$$
\begin{aligned}
B_\alpha(f \circ \phi_a)(z) &= \int_{\mathbb{D}} (f \circ \phi_a \circ \phi_z)(w) dA_\alpha(w) \\
&= \int_{\mathbb{D}} (f \circ \phi_a)(w) \left| k_z^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w) \\
&= \int_{\mathbb{D}} f(w) \left| (k_z \circ \phi_a)(w) \right|^{2+\alpha} |k_a(w)|^{2+\alpha} dA_\alpha(w)
\end{aligned}
$$

$$= \int_{\mathbb{D}} f(w) \left| k_{\phi_a(z)}^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w)$$
$$= (B_\alpha f)(\phi_a(z)) = (C_a^{(\alpha)} B_\alpha f)(z).$$

Thus $B_\alpha C_a^{(\alpha)} = C_a^{(\alpha)} B_\alpha$ for all $a \in \mathbb{D}$. This proves $(ii)$. □

**Lemma 3.** *A function $\phi \in L^1(\mathbb{D}, dA_\alpha)$ is harmonic if and only if $B_\alpha \phi = \phi$.*

*Proof.* If $\phi \in L^1(\mathbb{D}, dA_\alpha)$ is harmonic, then so is $\phi \circ \phi_a$ for any $a \in \mathbb{D}$; by the mean value property,

$$(B_\alpha \phi)(z) == \int_{\mathbb{D}} \phi(\phi_z(w)) dA_\alpha(w) = (\phi \circ \phi_z)(0) = \phi(z).$$

Conversely, if $B_\alpha \phi = \phi$ then $\phi$ is harmonic. This follows from [1]. □

From Lemma 3 it is not difficult to see that if $\phi \in L^1(\mathbb{D}, d\eta_\alpha)$ then $B_\alpha \phi = \phi$ if and only if $\phi$ is harmonic. This is so as $L^1(\mathbb{D}, d\eta_\alpha) \subset L^1(\mathbb{D}, dA_\alpha)$.

## 3 An ergodicity property of the Berezin transform

In this section, we consider a bounded real-valued subharmonic function $\phi$ on $\mathbb{D}$. We show that if $T$ is a bounded linear operator from $L_a^2(dA_\alpha)$ into itself, and $G(w) = \left\| T K_w^{(\alpha)} \right\|^p$, $w \in \mathbb{D}$, $0 < p < \infty$ then $B_0^n G$ is subharmonic for all $n \in \mathbb{N}$ and $B_0^n \phi \to \psi$, the least harmonic majorant of $G$.

**Definition 1.** Let $\phi \in L^1(\mathbb{D}, dA)$ is a real-valued subharmonic function on $\mathbb{D}$. If there exists a real-valued function $g \in L^1(\mathbb{D}, dA)$ harmonic on $\mathbb{D}$ and such that $g(z) \geq \phi(z)$ for all $z \in \mathbb{D}$, then we say the function $\phi$ admits an integrable harmonic majorant. The harmonic function $\psi \in L^1(\mathbb{D}, dA)$ is called the least harmonic majorant of $\phi$ if $\psi(z) \leq g(z)$ for all $z \in \mathbb{D}$ if $g \in L^1(\mathbb{D}, dA)$ is any other harmonic majorant of $\phi$.

Given a bounded real-valued subharmonic function on $\mathbb{D}$, the boundary values of its least harmonic majorant can be obtained as given in Lemma 4.

**Lemma 4.** *Suppose $\phi$ is a bounded real-valued subharmonic function on $\mathbb{D}$. Define $\phi$ on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ by $\phi(e^{it}) = \limsup_{r \to 1} \phi(re^{it})$. Let $\psi$ be the Poisson extension of $\phi \mid_{\mathbb{T}}$ into $\mathbb{D}$. Then $\psi$ is the least harmonic majorant of $\phi$ and $\psi \in L^\infty(\mathbb{D})$.*

*Proof.* Let $g$ be the least harmonic majorant of $\phi$. Except for $e^{it}$ in a set of zero measure (arc-length measure), we get

$$\lim_{r \to 1} g(re^{it}) \geq \limsup_{r \to 1} \phi(re^{it}) = \phi(e^{it}) = \lim_{r \to 1} \psi(re^{it}).$$

The bounded harmonic function $g - \psi$ has nonnegative radial limits a.e. on $\mathbb{T}$; hence $g \geq \psi$ on $\mathbb{D}$. We shall show that $g \leq \psi$. Since harmonicity and subharmonicity is

invariant under Möbius transforms, we need to show only that $\psi(0) \geq g(0)$. Without loss of generality, we may assume $\phi \leq 0$; hence also $g \leq 0,\ \psi \leq 0$. Using Fatou's Lemma [9] to the function $t \mapsto \phi(re^{it})$, we obtain

$$\begin{aligned}\psi(0) &= \frac{1}{2\pi}\int_0^{2\pi} \limsup_{r\to 1} \phi(re^{it})dt \geq \limsup_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \phi(re^{it})dt \\ &= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \phi(re^{it})dt = g(0).\end{aligned}$$

The function $\psi \in L^\infty(\mathbb{D})$. For proof see [4]. This proves the lemma. □

**Theorem 1.** *If $\phi \in L^\infty(\mathbb{D}, dA)$ is a real-valued subharmonic function on $\mathbb{D}$ then (i) $B_0^n\phi \to \psi$, the least harmonic majorant of $\phi$. (ii) $B_0^n\phi$ are subharmonic for all $n \in \mathbb{N}$.*

*Proof.* Let $\phi \in L^\infty(\mathbb{D}, dA)$ be a real-valued subharmonic function on $\mathbb{D}$. Define $\phi$ on $\mathbb{T}$ by $\phi(e^{it}) = \limsup_{r\to 1} \phi(re^{it})$. Let $\psi$ be the Poisson extension of $\phi\mid_{\mathbb{T}}$ into $\mathbb{D}$. Then $\psi$ is the least harmonic majorant of $\phi$ and $\psi \in L^\infty(\mathbb{D}) \subset L^1(\mathbb{D}, dA)$. From [5], it follows that for all $u \in \mathbb{D}$,

$$\phi(u) = \psi(u) + \frac{1}{4}\int_{\mathbb{D}} ln|\phi_u(w)|^2 d\gamma(w)$$

for some positive Borel measure $\gamma$ on $\mathbb{D}$. In case $\phi$ is twice continuously differentiable, $d\gamma(w) = (\Delta\phi)(w)dA(w)$ where $\Delta := 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplace operator. The measure $\pi d\gamma$ is called the Riesz measure of $\phi$. Let $h(w) = ln|w|^2$. Notice that $|\phi_u(w)| = |\phi_w(u)|$. Hence $\phi(u) = \psi(u) + \frac{1}{4}\int_{\mathbb{D}} (h \circ \phi_w)(u)d\gamma(w)$. Thus

$$B_0\phi(z) = B_0\psi(z) + \int_{\mathbb{D}} |k_u(z)|^2 \frac{1}{4}\int_{\mathbb{D}} (h \circ \phi_w)(u)d\gamma(w)dA(u). \tag{2}$$

Further since $\psi \in L^1(\mathbb{D}, dA)$ is harmonic, hence it follows from [7] that $\psi \circ \phi_u, u \in \mathbb{D}$ is also harmonic. Using the mean value property of harmonic functions [3], it follows that for all $z \in \mathbb{D},\ (B_0\psi)(z) = \int_{\mathbb{D}} (\psi \circ \phi_z)(w)dA(w) = (\psi \circ \phi_z)(0) = \psi(z)$. Now $\phi \leq \psi,\ \psi \in L^\infty(\mathbb{D}) \subset L^1(\mathbb{D}, dA)$ and $B_0\psi = \psi$. It follows from (2) that

$$(B_0\phi)(z) = \psi(z) + \int_{\mathbb{D}} |k_u(z)|^2 \frac{1}{4}\int_{\mathbb{D}} (h \circ \phi_w)(u)d\gamma(w)dA(u).$$

Further as the integrand above is nonpositive, we may interchange the order of integration to obtain

$$(B_0\phi)(z) = \psi(z) + \frac{1}{4}\int_{\mathbb{D}} B_0(h \circ \phi_w)(z)d\gamma(w).$$

Thus by repeating this process, $n$ times we get

$$(B_0^n\phi)(z) = \psi(z) + \frac{1}{4}\int_{\mathbb{D}} B_0^n(h\circ\phi_w)(z)d\gamma(w). \tag{3}$$

Now we shall show that $B_0^n(h\circ\phi_w)(z) \to 0$ as $n\to\infty$ for all $z, w \in \mathbb{D}$. It follows from Lemma 2 that $B_0^n(h\circ\phi_w) = (B_0^n h)\circ\phi_w$. So we need only to show that $B_0^n h \to 0$. Recall that $h(w) = ln|w|^2$ and a straightforward calculation shows that $B_0h(w) = |w|^2 - 1$. Thus $h \le B_0h$. Repeating this process, one can show that $h \le B_0h \le B_0^2h \le B_0^3h \le \cdots \le 0$. Thus the limit $\Theta(w) := \lim_{n\to\infty}(B_0^n h)(w)$ must exist and $\Theta \le 0$, and from the Lebesgue monotone convergence theorem, $B_0\Theta = \Theta$. We shall show that $\Theta \equiv 0$. Suppose $\Theta \neq 0$. Since $0 \ge \Theta(w) \ge (B_0h)(w) = |w|^2 - 1$, we get $\lim_{|w|\to 1}\Theta(w) = 0$ and therefore $\Theta$ must attain its infimum at some point $z_0 \in \mathbb{D}$. Suppose $z_0 = 0$ (replacing $\Theta$ by $\Theta_0\phi_{z_0}$ otherwise). Then

$$\Theta(0) = (B_0\Theta)(w) = \int_{\mathbb{D}}\Theta(w)dA(w) > \inf_{\mathbb{D}}\Theta(w)\int_{\mathbb{D}}dA(w) = \Theta(0),$$

which is a contradiction. Thus $B_0^n(h\circ\phi_w)(z) \to 0$. Now because $\gamma$ is a positive measure, using Lebesgue monotone convergence theorem [9] we have $B_0^n\phi \to \psi$ as $n\to\infty$. This proves $(i)$. To prove $(ii)$, let $0 < r < 1$. Since $B_0\phi(z) = \psi(z) + \frac{1}{4}\int_{\mathbb{D}} B_0(h\circ\phi_w)(z)d\gamma(w)$, we obtain $\frac{1}{2\pi}\int_0^{2\pi}(B_0\phi)(re^{it})dt = \frac{1}{2\pi}\int_0^{2\pi}\psi(re^{it})dt + \frac{1}{2\pi}\int_0^{2\pi} B_0(h\circ\phi_w)(re^{it})d\gamma(w)dt$. Since $B_0(h\circ\phi_w)(z) \le 0$ for all $z\in\mathbb{D}$, we may interchange the order of integration. Hence we obtain $\frac{1}{2\pi}\int_0^{2\pi}(B_0\phi)(re^{it})dt = \psi(0) + \frac{1}{4}\int_{\mathbb{D}}\left(\frac{1}{2\pi}\int_0^{2\pi} B_0(h\circ\phi_w)(re^{it})dt\right)d\gamma(w)$. The function $B_0(h\circ\phi_w)(z) = |\phi_w(z)|^2 - 1$ is a subharmonic function of $z$. Hence we obtain $\frac{1}{2\pi}\int_0^{2\pi} B_0(h\circ\phi_w)(re^{it})dt \ge B_0(h\circ\phi_w)(0)$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi}(B_0\phi)(re^{it})dt \ge \psi(0) + \frac{1}{4}\int_{\mathbb{D}} B_0(h\circ\phi_w)(0)d\gamma(w) = (B_0\phi)(0),$$

for $0 < r < 1$. Repeating the same procedure for $\phi\circ\phi_a$, $a\in\mathbb{D}$, one can show that $B_0\phi$ satisfies the sub-mean value property and that $B_0\phi$ is subharmonic on $\mathbb{D}$. Since $\phi \le \psi$ we obtain $B_0\phi \le B_0\psi = \psi$. Thus $B_0\phi$ has an integrable harmonic majorant. We may proceed by induction to show that $B_0^n\phi$ is subharmonic for all $n\in\mathbb{N}$. This proves $(ii)$. □

**Theorem 2.** *Suppose $T \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ and $0 < p < \infty$. Let $G(w) = \left\|TK_w^{(\alpha)}\right\|^p$, $w\in\mathbb{D}$. Then $B_0^nG$ is subharmonic function for all $n\in\mathbb{N}$ and $B_0^nG \to \psi$, the least harmonic majorant of $G$.*

*Proof.* Without loss of generality, we shall assume that $T \neq 0$. We shall first show that $\left\|TK_w^{(\alpha)}\right\|^p$ is a real-valued subharmonic function on $\mathbb{D}$ and the result will follow from Theorem 1. Notice that the function $G$ has isolated zeros in $\mathbb{D}$. Further, away from the zeros of $G$, it follows that since $G(w) = \left\langle TK_w^{(\alpha)}, TK_w^{(\alpha)} \right\rangle^{\frac{p}{2}}$, we have

$$\frac{\partial G}{\partial \overline{w}} = \frac{p}{2} \left\langle TK_w^{(\alpha)}, TK_w^{(\alpha)} \right\rangle^{\frac{p}{2}-1} \left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle$$

$$\text{where } K_w^{(\alpha)^*} = \frac{\partial}{\partial \overline{w}} \frac{1}{(1-z\overline{w})^{\alpha+2}} = (-)\frac{(\alpha+2)(1-z\overline{w})^{\alpha+1}(-z)}{(1-z\overline{w})^{2\alpha+4}} = \frac{(\alpha+2)z}{(1-z\overline{w})^{\alpha+3}}\,.$$

Differentiating one more time, we get using Cauchy-Schwarz inequality that

$$\begin{aligned}
\frac{\partial^2 G}{\partial w \partial \overline{w}} &= \frac{p}{2}\left(\frac{p}{2}-1\right)\left\langle TK_w^{(\alpha)}, TK_w^{(\alpha)} \right\rangle^{\frac{p}{2}-2} \left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle \left\langle TK_w^{(\alpha)}, TK_w^{(\alpha)^*} \right\rangle \\
&\quad + \frac{p}{2}\left\langle TK_w^{(\alpha)}, TK_w^{(\alpha)} \right\rangle^{\frac{p}{2}-1} \left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)^*} \right\rangle \\
&= \frac{p}{2}\left(\frac{p}{2}-1\right)\left\|TK_w^{(\alpha)}\right\|^{p-4} \left|\left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle\right|^2 + \frac{p}{2}\left\|TK_w^{(\alpha)}\right\|^{p-2} \left\|TK_w^{(\alpha)^*}\right\|^2 \\
&= \frac{p}{2}\left\|TK_w^{(\alpha)}\right\|^{p-4} \left[\left(\frac{p}{2}-1\right)\left|\left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle\right|^2 + \left\|TK_w^{(\alpha)}\right\|^2 \left\|TK_w^{(\alpha)^*}\right\|^2\right] \\
&\geq \frac{p}{2}\left\|TK_w^{(\alpha)}\right\|^{p-4} \left[\left\|TK_w^{(\alpha)}\right\|^2 \left\|TK_w^{(\alpha)^*}\right\|^2 - \left|\left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle\right|^2\right] \\
&= \frac{p}{2}\left\|TK_w^{(\alpha)}\right\|^{p-4} \left[\left\|TK_w^{(\alpha)}\right\| \left\|TK_w^{(\alpha)^*}\right\| + \left|\left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle\right|\right] \\
&\qquad\qquad \left[\left\|TK_w^{(\alpha)}\right\| \left\|TK_w^{(\alpha)^*}\right\| - \left|\left\langle TK_w^{(\alpha)^*}, TK_w^{(\alpha)} \right\rangle\right|\right] \\
&\geq 0.
\end{aligned}$$

Hence $G$ is a subharmonic function [7] on $\mathbb{D}$. The result follows from Theorem 1. □

## 4 Schatten class operators in $\mathcal{L}\left(L_a^2(dA_\alpha)\right)$

In this section, we describe the Schatten class characterization of a bounded linear operator $T$ from $L_a^2(dA_\alpha)$ into itself in terms of the generalized Berezin transform of $T$. Further we show that if $T \geq 0$ or $T \in S_1$, then $tr(T) = (\alpha+1)\int\limits_{\mathbb{D}} (B_\alpha T)(z)d\lambda(z)$. Further, we have shown that if $A$ is compact and positive and $0 < p \leq 1$, then $\int\limits_{\mathbb{D}} B_\alpha(A^p)(z)d\lambda(z)$ is finite if and only if $\sum\limits_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p$ is finite for some orthonormal basis $\{\xi_n^{(\alpha)}\}$ in $L_a^2(dA_\alpha)$.

**Lemma 5.** *Suppose $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ and $T \geq 0$. Let $f \in L^2_a(\mathbb{D})$ with $\|f\| = 1$. Then*

*(i)* $\langle T^p f, f\rangle \geq \langle Tf, f\rangle^p$ *for all* $p \geq 1$ ;

*(ii)* $\langle T^p f, f\rangle \leq \langle Tf, f\rangle^p$ *for all* $0 < p \leq 1$.

For proof see [12].

**Theorem 3.** *Suppose $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ and assume that either $T \geq 0$ or $T \in S_1$. Then*

$$tr(T) = (\alpha+1)\int_{\mathbb{D}} (B_\alpha T)(z)d\lambda(z), \quad \text{where} \quad d\lambda(z) = \frac{dA(z)}{\left(1-|z|^2\right)^2}.$$

*Proof.* Consider the orthonormal basis $\left\{e_n^{(\alpha)}\right\}_{n=0}^{\infty}$ of $L^2_a(dA_\alpha)$. Then

$$\begin{aligned}
tr(T) = \sum_{n=1}^{\infty}\left\langle Te_n^{(\alpha)}, e_n^{(\alpha)}\right\rangle &= \sum_{n=1}^{\infty}\int_{\mathbb{D}}\left(Te_n^{(\alpha)}\right)(z)\overline{e_n^{(\alpha)}(z)}dA_\alpha(z)\\
&= \sum_{n=1}^{\infty}\int_{\mathbb{D}}\left\langle Te_n^{(\alpha)}, K_z^{(\alpha)}\right\rangle\overline{e_n^{(\alpha)}(z)}dA_\alpha(z)\\
&= \int_{\mathbb{D}}\left\langle T\left(\sum_{n=1}^{\infty}e_n^{(\alpha)}(z)\overline{e_n^{(\alpha)}(z)}\right), K_z^{(\alpha)}\right\rangle dA_\alpha(z)\\
&= \int_{\mathbb{D}}\left\langle TK_z^{(\alpha)}, K_z^{(\alpha)}\right\rangle dA_\alpha(z)\\
&= \int_{\mathbb{D}}\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle K^{(\alpha)}(z,z)dA_\alpha(z)\\
&= \int_{\mathbb{D}}\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle \frac{(\alpha+1)\left(1-|z|^2\right)^\alpha dA(z)}{\left(1-|z|^2\right)^{\alpha+2}}\\
&= (\alpha+1)\int_{\mathbb{D}}\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle \frac{dA(z)}{\left(1-|z|^2\right)^2}\\
&= (\alpha+1)\int_{\mathbb{D}}\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle d\lambda(z)\\
&= (\alpha+1)\int_{\mathbb{D}}(B_\alpha T)(z)d\lambda(z).
\end{aligned}$$

□

**Theorem 4.** *Suppose $T \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ and $d\lambda(w) = \frac{dA(w)}{\left(1-|w|^2\right)^2}$ is the Möbius invariant area measure on $\mathbb{D}$. Then the following hold:*
*(i) If $0 < p \leq 2$ and $\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^p d\lambda(w) < \infty$ then $T \in S_p$.*
*(ii) If $2 \leq p < \infty$ and $T \in S_p$ then $\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^p d\lambda(w) < \infty$.*
*(iii) The operator $T$ is Hilbert-Schmidt if and only if*

$$\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^2 d\lambda(w) = \frac{1}{(\alpha+1)} \int\limits_{\mathbb{D}} \left\|TK_w^{(\alpha)}\right\|^2 dA_\alpha(w) < \infty.$$

*Proof.* Let $0 < p \leq 2$ and assume $\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^p d\lambda(w) < \infty$. This implies

$$\int\limits_{\mathbb{D}} \left\langle Tk_w^{1+\frac{\alpha}{2}}, Tk_w^{1+\frac{\alpha}{2}} \right\rangle^{\frac{p}{2}} d\lambda(w) < \infty.$$

Now $0 < p \leq 2$ if and only if $0 < \frac{p}{2} \leq 1$. Hence

$$\begin{aligned}\int\limits_{\mathbb{D}} \left\langle |T|^p k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle d\lambda(w) &= \int\limits_{\mathbb{D}} \left\langle (T^*T)^{\frac{p}{2}} k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle d\lambda(w) \\ &\leq \int\limits_{\mathbb{D}} \left\langle (T^*T) k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle^{\frac{p}{2}} d\lambda(w) < \infty.\end{aligned}$$

Therefore $|T|^p \in S_1$. This implies $T \in S_p$. For more details see [12]. The proof of (i) follows. For the proof of (ii), assume $2 \leq p < \infty$ and let $T \in S_p$. Then it follows from [12] that $|T|^p = (T^*T)^{\frac{p}{2}} \in S_1$. By Lemma 5 and Theorem 3

$$\begin{aligned}\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^p d\lambda(w) &= \int\limits_{\mathbb{D}} \left\langle Tk_w^{1+\frac{\alpha}{2}}, Tk_w^{1+\frac{\alpha}{2}} \right\rangle^{\frac{p}{2}} d\lambda(w) \\ &= \int\limits_{\mathbb{D}} \left\langle (T^*T) k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle^{\frac{p}{2}} d\lambda(w) \\ &\leq \int\limits_{\mathbb{D}} \left\langle (T^*T)^{\frac{p}{2}} k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle d\lambda(w) \\ &= \int\limits_{\mathbb{D}} \left\langle |T|^P k_w^{1+\frac{\alpha}{2}}, k_w^{1+\frac{\alpha}{2}} \right\rangle d\lambda(w) < \infty.\end{aligned}$$

When $T \in S_2$, then it follows from (i) and (ii) that $\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^2 d\lambda(w) < \infty$. But

$$\int\limits_{\mathbb{D}} \left\|Tk_w^{1+\frac{\alpha}{2}}\right\|^2 d\lambda(w) = \int\limits_{\mathbb{D}} \left\|TK_w^{(\alpha)}\right\|^2 \frac{dA(w)}{\left(1-|w|^2\right)^2} \left(1-|w|^2\right)^{\alpha+2}$$

$$= \frac{1}{(\alpha+1)} \int_{\mathbb{D}} \left\| TK_w^{(\alpha)} \right\|^2 (\alpha+1) \left(1-|w|^2\right)^{\alpha} dA(w)$$
$$= \frac{1}{(\alpha+1)} \int_{\mathbb{D}} \left\| TK_w^{(\alpha)} \right\|^2 dA_\alpha(w).$$

The result follows. □

**Corollary 1.** *If $\phi$ is a non-negative function on $\mathbb{D}$, then* $Tr(T_\phi^{(\alpha)}) = (\alpha+1)\int_{\mathbb{D}} \phi(z) d\lambda(z)$.

*Proof.* From Theorem 3 and Fubini's theorem [9], it follows that

$$\begin{aligned}
Tr(T_\phi^{(\alpha)}) &= \int_{\mathbb{D}} \left\langle T_\phi k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle K^{(\alpha)}(z,z) dA_\alpha(z) \\
&= \int_{\mathbb{D}} K^{(\alpha)}(z,z) dA_\alpha(z) \int_{\mathbb{D}} \phi(w) \left| k_z^{1+\frac{\alpha}{2}}(w) \right|^2 dA_\alpha(w) \\
&= \int_{\mathbb{D}} dA_\alpha(z) \int_{\mathbb{D}} \phi(w) \left| K^{(\alpha)}(z,w) \right|^2 dA_\alpha(w) \\
&= \int_{\mathbb{D}} \phi(w) dA_\alpha(w) \int_{\mathbb{D}} \left| K^{(\alpha)}(z,w) \right|^2 dA_\alpha(z) \\
&= \int_{\mathbb{D}} \phi(w) K^{(\alpha)}(w,w) dA_\alpha(w) \\
&= \int_{\mathbb{D}} \phi(w) \frac{(\alpha+1)(1-|w|^2)^\alpha dA(w)}{(1-|w|^2)^{\alpha+2}} \\
&= (\alpha+1) \int_{\mathbb{D}} \phi(w) \frac{dA(w)}{(1-|w|^2)^2} = (\alpha+1) \int_{\mathbb{D}} \phi(w) d\lambda(w).
\end{aligned}$$

□

**Corollary 2.** *If $p \geq 1$ and $\phi \in L^p(\mathbb{D}, d\lambda)$, then $T_\phi^{(\alpha)} \in S_p$.*

*Proof.* We only need to prove the result for $p = 1$. The case $p = +\infty$ is trivial. The result will follow by Marcinkiewicz interpolation theorem [9].

Suppose $\phi \in L^1(\mathbb{D}, d\lambda)$ and let $\{e_n^{(\alpha)}\}$ be an orthonormal basis in $L_a^2(dA_\alpha)$. For any $n \geq 1$,

$$\left\langle T_\phi^{(\alpha)} e_n^{(\alpha)}, e_n^{(\alpha)} \right\rangle = \int_{\mathbb{D}} \left| e_n^{(\alpha)}(z) \right|^2 \phi(z) dA_\alpha(z).$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \left\langle T_\phi^{(\alpha)} e_n^{(\alpha)}, e_n^{(\alpha)} \right\rangle \right| &\leq \int_{\mathbb{D}} \sum_{n=1}^{\infty} \left| e_n^{(\alpha)}(z) \right|^2 |\phi(z)| dA_\alpha(z) \\
&\leq \int_{\mathbb{D}} K^{(\alpha)}(z,z) |\phi(z)| dA_\alpha(z)
\end{aligned}$$

$$= \int_{\mathbb{D}} |\phi(z)| \frac{(\alpha+1)(1-|z|^2)^\alpha dA(z)}{(1-|z|^2)^{\alpha+2}}$$
$$= (\alpha+1) \int_{\mathbb{D}} |\phi(z)| d\lambda(z) < \infty.$$

From [12], it follows that $T_\phi^{(\alpha)} \in S_1$ and $\left\|T_\phi^{(\alpha)}\right\|_{S_1} \leq (\alpha+1) \int_{\mathbb{D}} |\phi(z)| d\lambda(z)$. □

**Definition 2.** A sequence $\{w_n\}$ in $\mathbb{D}$ is called a sampling sequence for the weighted Bergman space $L_a^2(dA_\alpha)$ if there exists a positive constant $C$ such that

$$C^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty} \left(1-|w_n|^2\right)^{\alpha+2} |f(w_n)|^2 \leq C\|f\|^2 \tag{4}$$

for all $f \in L_a^2(dA_\alpha)$. This condition can be written as

$$C^{-1}\|f\|^2 \leq \sum_{n=1}^{\infty} \left|\left\langle f, k_{w_n}^{1+\frac{\alpha}{2}} \right\rangle\right|^2 \leq C\|f\|^2.$$

A sequence $\{h_n^{(\alpha)}\}$ in $L_a^2(dA_\alpha)$ is called a frame for $L_a^2(dA_\alpha)$ if there exist positive constants $M_1$ and $M_2$ such that

$$M_1\|f\|^2 \leq \sum_{n=1}^{\infty} \left|\left\langle f, h_n^{(\alpha)} \right\rangle\right|^2 \leq M_2\|f\|^2$$

for all $f \in L_a^2(dA_\alpha)$. The numbers $M_1$ and $M_2$ are certainly not unique. The optimal upper constant, $\inf M_2$ will be called the upper frame bound for $\{h_n^{(\alpha)}\}$. Similarly the optimal lower constant, $\sup M_1$ will be called the lower frame bound for $\{h_n^{(\alpha)}\}$. A frame is called tight if its lower and upper frame bounds are the same. Also a frame is called Parseval or normalised tight if its lower and upper frame bounds are both 1. For more details about frame and sampling sequence refer [2].

It thus follows from [6],[10] and [11] that $\{w_n\}$ is a sampling sequence for the weighted Bergman space $L_a^2(dA_\alpha)$ if and only if the sequence $\left\{k_{w_n}^{1+\frac{\alpha}{2}}\right\}$ is a frame in $L_a^2(dA_\alpha)$.

**Lemma 6.** *Let $\left\{\xi_n^{(\alpha)}\right\}$ be an orthonormal basis for $L_a^2(dA_\alpha)$ and $\left\{h_n^{(\alpha)}\right\}$ be a frame for $L_a^2(dA_\alpha)$ for fixed $\alpha$ with upper frame bound equal to $L_2$ and lower frame frame bound equal to $L_1$. Define $E : L_a^2(dA_\alpha) \to L_a^2(dA_\alpha)$ by $E\left(\sum_{k=1}^{\infty} b_k \xi_k^{(\alpha)}\right) = \sum_{k=1}^{\infty} b_k h_k^{(\alpha)}$. Then $E \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ and $\sqrt{L_1} \leq \|E\| \leq \sqrt{L_2}$ and $EE^*$ is invertible on $L_a^2(dA_\alpha)$.*

*Proof.* Let $g \in L_a^2(dA_\alpha)$. Then it follows from Cauchy-Schwarz inequality [9] that

$$\left|\left\langle E\left(\sum_{k=1}^{N} b_k \xi_k^{(\alpha)}\right), g \right\rangle\right|^2 = \left|\sum_{k=1}^{N} b_k \left\langle h_k^{(\alpha)}, g \right\rangle\right|^2 \leq \sum_{k=1}^{N} |b_k|^2 \sum_{k=1}^{N} \left|\left\langle h_k^{(\alpha)}, g \right\rangle\right|^2$$

$$\leq L_2\|g\|^2 \left\|\sum_{k=1}^{N} b_k \xi_k^{(\alpha)}\right\|^2$$

where $L_2$ is the upper frame bound for $\left\{h_n^{(\alpha)}\right\}$. By the Hahn-Banach Theorem [8], $E$ extends to an operator in $\mathcal{L}\left(L_a^2(dA_\alpha)\right)$ with $\|E\|^2 \leq L_2$ and $\sum_{k=1}^{\infty} |b_k|^2 < \infty$. If $\left\langle g, h_k^{(\alpha)} \right\rangle = 0$ for all $k$, then it follows from the Definition 2 that $g = 0$. Therefore, $\overline{RangeE} = L_a^2(dA_\alpha)$. For any vector $g \in L_a^2(dA_\alpha)$, we obtain

$$E^*g = \sum_{n=1}^{\infty} \left\langle E^*g, \xi_n^{(\alpha)} \right\rangle \xi_n^{(\alpha)} = \sum_{n=1}^{\infty} \left\langle g, E\xi_n^{(\alpha)} \right\rangle \xi_n^{(\alpha)} = \sum_{n=1}^{\infty} \left\langle g, h_n^{(\alpha)} \right\rangle \xi_n^{(\alpha)}.$$

Hence $\|E^*g\|^2 = \sum_{n=1}^{\infty} \left|\left\langle g, h_n^{(\alpha)} \right\rangle\right|^2 \geq L_1\|g\|^2$, where $L_1$ is the lower frame bound for $\left\{h_n^{(\alpha)}\right\}$. This implies $\|E\|^2 = \|E^*\|^2 \geq L_1$ and $E^*$ is one-to-one and $RangeE^*$ is closed. Again, for any $g \in L_a^2(dA_\alpha)$, we obtain $L_1\|g\|^2 \leq \|E^*g\|^2 = \langle EE^*g, g\rangle \leq \|EE^*g\|\|g\|$. Therefore, $\|EE^*g\| \geq L_1\|g\|$ for all $g \in L_a^2(dA_\alpha)$ and $EE^*$ is one-to-one. Further, $RangeEE^*$ is closed. As $(RangeEE^*)^\perp = ker(EE^*) = \{0\}$, the operator $EE^*$ is onto and thus $EE^*$ is invertible. □

**Lemma 7.** *Let $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ and $2 \leq p < \infty$. Then the following are equivalent. (i) $\int_{\mathbb{D}} B_\alpha(|A|^p)(z)d\lambda(z) < \infty$.*

*(ii) $\sum_{n=1}^{\infty} \|A\xi_n^{(\alpha)}\|^p < \infty$ for every orthonormal basis $\{\xi_n^{(\alpha)}\}$ in $L_a^2(dA_\alpha)$.*

*(iii) $\sum_{n=1}^{\infty} \|Ah_n^{(\alpha)}\|^p < \infty$ for every frame $\{h_n^{(\alpha)}\}$ in $L_a^2(dA_\alpha)$.*

*Proof.* From [12], it follows that $A \in S_p$ if and only if $|A|^p \in S_1,\ p \geq 1$. Further from Theorem 3, we obtain $|A|^p \in S_1,\ p \geq 1$ if and only if $\int_{\mathbb{D}} \left\langle |A|^p k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}} \right\rangle d\lambda(z) < \infty$. Thus $(i)$ is equivalent to say that $A \in S_p$. Thus it follows from [12] that $(i)$ is equivalent to $(ii)$. Further, $(iii)$ implies $(ii)$ as every orthonormal basis is a frame. To show $(i)$ implies $(iii)$, consider an orthonormal basis $\left\{\xi_n^{(\alpha)}\right\}$ and frame $\left\{h_n^{(\alpha)}\right\}$ for $L_a^2(dA_\alpha)$ and consider the operator $E$ defined in Lemma 6. If $(i)$ holds, then $S = AE \in S_p$. Further, applying condition $(ii)$ to $S$ we obtain

$$\sum_{n=1}^{\infty} \|Ah_n^{(\alpha)}\|^p = \sum_{n=1}^{\infty} \|AE\xi_n^{(\alpha)}\|^p = \sum_{n=1}^{\infty} \|S\xi_n^{(\alpha)}\|^p < \infty.$$

Thus conditions $(i), (ii), (iii)$ are equivalent. □

*Remark* 1. It follows from [12] that $\|A\|_p^p = \sup \sum_{n=1}^{\infty} \|A\xi_n^{(\alpha)}\|^p$ where the supremum is taken over all orthonormal bases $\{\xi_n^{(\alpha)}\}$. Since every orthonormal basis is a frame with frame bounds equal to 1, we obtain

$$\|A\|_p^p = \sup \sum_{n=1}^{\infty} \|A\xi_n^{(\alpha)}\|^p \leq \sup \sum_{n=1}^{\infty} \|Ah_n^{(\alpha)}\|^p. \tag{5}$$

Thus

$$\sum_{n=1}^{\infty} \|Ah_n^{(\alpha)}\|^p = \sum_{n=1}^{\infty} \|AE\xi_n^{(\alpha)}\|^p \leq \|AE\|_p^p.$$

Further, it follows from [12] that $\|AE\|_p \leq \|A\|_p \|E\|$. Hence we obtain using Lemma 6 that $\|AE\|_p \leq \|A\|_p$ wherever the frame $\{h_n^{(\alpha)}\}$ has upper bound $\leq 1$.

Thus $\sup \sum_{n=1}^{\infty} \|Ah_n^{(\alpha)}\|_p^p \leq \|A\|_p^p$ where the supremum is taken over all frames $\{h_n^{(\alpha)}\}$ with upper frame bound less than or equal to 1.

**Lemma 8.** *Let $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ and $1 \leq p < \infty$. Then the following are equivalent.*

*(i)* $\int_{\mathbb{D}} B_\alpha(A^p)(z)d\lambda(z) < \infty$.

*(ii)* $\sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p < \infty$ *for every orthonormal basis* $\{\xi_n^{(\alpha)}\}$ *in* $L_a^2(dA_\alpha)$.

*(iii)* $\sum_{n=1}^{\infty} \left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle\right|^p < \infty$ *for every frame* $\{h_n^{(\alpha)}\}$ *in* $L_a^2(dA_\alpha)$.

*Further, if $A = A^*$ then*

$$\|A\|_p^p = \sup \sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p = \sup \sum_{n=1}^{\infty} \left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle\right|^p,$$

*where the first supremum is taken over all orthonormal bases and the second supremum is taken over all frames with upper frame bound $\leq 1$.*

*Proof.* Assume $1 \leq p < \infty$. The equivalence of $(i)$ and $(ii)$ follows from Theorem 3 and [12]. Further $(iii)$ implies $(ii)$ as every orthonormal basis is a frame in $L_a^2(dA_\alpha)$. To prove $(i)$ implies $(iii)$, we shall first assume $A \geq 0$. Then $A = S^*S$ where $S$ is the square root of $A$. Further $\left\|Sh_n^{(\alpha)}\right\|^2 = \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle$ and hence $(i)$ implies $(iii)$ follows from Lemma 7 and the fact [12] that $A \in S_p$ if and only if $S \in S_{2p}$. If $A$ is not positive we can write [12] as $A = (A_1 - A_2) + i(A_3 - A_4)$ where each $A_k$ is positive and each $A_k \in S_p$. From the first part, since each $A_k$ is positive it follows

that $\sum_{n=1}^{\infty}\left|\left\langle A_k h_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p < \infty$ for $1 \le k \le 4$. Therefore it follows from Minkowski inequality that $\sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p < \infty$. From the canonical decomposition for self-adjoint compact operators and since every orthonormal basis is a frame with the frame bounds 1, we obtain

$$\|A\|_p^p \le \sup \sum_{n=1}^{\infty}\left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)}\right\rangle\right|^p = \sup \sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p.$$

If $\{h_n^{(\alpha)}\}$ is a frame with upper frame bound 1, then by the norm estimate for $E$ in Lemma 6, we obtain $\|h_n^{(\alpha)}\| \le 1$ for all $n$. Thus from Lemma 7, it follows that

$$\sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p \le \sum_{n=1}^{\infty}\|Ah_n^{(\alpha)}\|_p^p \le \|A\|_p^p.$$

This shows that $\sup \sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p \le \|A\|_p^p$. This completes the proof of the lemma. □

**Theorem 5.** *Let $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ be positive and assume $0 < p \le 1$. Then the following are equivalent:*

*(i)* $\int_{\mathbb{D}} B_\alpha(A^p)(z)d\lambda(z) < \infty$.

*(ii)* $\sum_{n=1}^{\infty}\left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)}\right\rangle\right|^p < \infty$ *for some orthonormal basis* $\{\xi_n^{(\alpha)}\}$ *in* $L_a^2(dA_\alpha)$.

*(iii)* $\sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p < \infty$ *for some frame* $\left\{h_n^{(\alpha)}\right\}$ *in* $L_a^2(dA_\alpha)$.

*In this case* $\|A\|_p^p = \inf \sum_{n=1}^{\infty}\left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p$, *where the infimum is taken over all Parseval frames* $\left\{h_n^{(\alpha)}\right\}$ *in* $L_a^2(dA_\alpha)$.

*Proof.* It follows from [12] that if $A \ge 0$ and $p > 0$ then $A \in S_p$ if and only if $A^p \in S_1$. From Theorem 3, it follows that if $A \ge 0$ then $A^p \in S_1$ if and only if $\int_{\mathbb{D}}\left\langle A^p k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle d\lambda(z) < \infty$. Thus $(i)$ holds if and only if $A \in S_p$. Assume $0 < p \le 1,\ A \ge 0$.

To prove $(i)$ implies $(ii)$, let $Af = \sum_{n=1}^{\infty}\lambda_n^{(\alpha)}\langle f, u_n^{(\alpha)}\rangle u_n^{(\alpha)}$ be the canonical decomposition of $A$ where $\lambda_n^{(\alpha)} \in \mathbb{C}$ for all $n$ and $\sum_{n=1}^{\infty}\left|\lambda_n^{(\alpha)}\right|^p < \infty$ and $\left\{\lambda_n^{(\alpha)}\right\}$ is

the singular value sequences of $A$ and $\left\{u_n^{(\alpha)}\right\}$ is an orthonormal set in $L_a^2(dA_\alpha)$. Since $A$ is positive, hence $\lambda_n^{(\alpha)} \geq 0$ for all $n$ and we obtain $Af = 0$ if and only if $\left\langle f, u_n^{(\alpha)} \right\rangle = 0$ for all $n$. Thus if $\left\{u_n^{(\alpha)'}\right\}$ is an orthonormal basis for $kerA$, then $\left\{\xi_n^{(\alpha)}\right\} = \left\{u_n^{(\alpha)}\right\} \bigcup \left\{u_n^{(\alpha)'}\right\}$ is an orthonormal basis for $L_a^2(dA_\alpha)$. Now since $Au_n^{(\alpha)} = \lambda_n^{(\alpha)} u_n^{(\alpha)}$, for all $n$ and $\left\{\lambda_n^{(\alpha)}\right\} \in l^p$, we obtain

$$\sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p = \sum_{n=1}^{\infty} \left|\left\langle Au_n^{(\alpha)}, u_n^{(\alpha)} \right\rangle\right|^p = \sum_{n=1}^{\infty} \left(\lambda_n^{(\alpha)}\right)^p = \|A\|_p^p < \infty.$$

For the proof of (ii) implies (i), refer to [12].
Notice also that whenever $\left\{\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right\} \in l^p$ where $0 < p \leq 1$, then

$$\|A\|_p^p = \|A^p\|_1^p = \sum_{n=1}^{\infty} \left|\left\langle A^p \xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right| \leq \sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p < \infty$$

and $\|A\|_p^p = \inf \sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p$ where the infimum is taken over all orthonormal bases $\left\{\xi_n^{(\alpha)}\right\}$. The condition (ii) implies (iii) follows since every orthonormal basis is a frame with both upper and lower frame bounds equal to 1. Further,

$$\inf \sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^{2(1-p)} \left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle\right|^p \leq \inf \sum_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right|^p,$$

where the first infimum is taken over all frames $\left\{h_n^{(\alpha)}\right\}$ with lower frame bound at least 1 and the second infimum is taken over all orthonormal bases $\left\{\xi_n^{(\alpha)}\right\}$. To prove (iii) implies (i), let $\left\{\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle\right\} \in l^p$ for some frame $\left\{h_n^{(\alpha)}\right\}$. Now let $D = E^*AE$. Then $D \geq 0$ and

$$\left\langle D\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle = \left\langle E^*AE\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle = \left\langle AE\xi_n^{(\alpha)}, E\xi_n^{(\alpha)} \right\rangle = \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle.$$

Hence $\left\{\left\langle D\xi_n^{(\alpha)}, \xi_n^{(\alpha)} \right\rangle\right\} \in l^p$. Thus from the equivalence of (i) and (ii), it follows that $D \in S_p$. As the space $S_p$ is a two-sided ideal in $\mathcal{L}\left(L_a^2(dA_\alpha)\right)$, the operator

$$(EE^*)A(EE^*) = EDE^* \in S_p. \tag{6}$$

Multiplying both sides of (6) by $(EE^*)^{-1}$, we obtain $A \in S_p$. Hence (iii) implies (i). Now let $\left\{h_n^{(\alpha)}\right\}$ be a frame with lower frame bound $\geq 1$. We shall show now that

$$\|A\|_p^p \leq \sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^{2(1-p)} \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle. \tag{7}$$

Suppose the right hand side in (7) is finite. Otherwise the inequality is trivial to check.
Assume $h_n^{(\alpha)} \neq 0$ for all $n$. From [12], we obtain

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^{2(1-p)} \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle^p \\ &= \sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^2 \left\langle A\frac{h_n^{(\alpha)}}{\left\|h_n^{(\alpha)}\right\|}, \frac{h_n^{(\alpha)}}{\left\|h_n^{(\alpha)}\right\|}\right\rangle^p \\ &\geq \sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^2 \left\langle A^p\frac{h_n^{(\alpha)}}{\left\|h_n^{(\alpha)}\right\|}, \frac{h_n^{(\alpha)}}{\left\|h_n^{(\alpha)}\right\|}\right\rangle = \sum_{n=1}^{\infty} \left\langle A^p h_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle.\end{aligned}$$

Since (i) and (iii) are equivalent, it follows that $A^p \in S_1$.
If $A^p f = \sum\limits_{k=1}^{\infty} \mu_k^{(\alpha)} \left\langle f, u_k^{(\alpha)}\right\rangle u_k^{(\alpha)}$ is the canonical decomposition for $A^p$, then $\left\langle A^p h_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle = \sum\limits_{k=1}^{\infty} \mu_k^{(\alpha)} \left|\left\langle h_n^{(\alpha)}, u_k^{(\alpha)}\right\rangle\right|^2$ for each $n$. Since the lower frame bound of $\left\{h_n^{(\alpha)}\right\}$ is $\geq 1$, it follows from Fubini's theorem [9] and [12] that

$$\begin{aligned}\sum_{n=1}^{\infty} \left\langle A^p h_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle &= \sum_{k=1}^{\infty} \mu_k^{(\alpha)} \sum_{n=1}^{\infty} \left|\left\langle h_n^{(\alpha)}, u_k^{(\alpha)}\right\rangle\right|^2 \\ &\geq \sum_{k=1}^{\infty} \mu_k^{(\alpha)} \left\|u_k^{(\alpha)}\right\|^2 = \sum_{k=1}^{\infty} \mu_k^{(\alpha)} = \|A^p\|_1 = \|A\|_p^p.\end{aligned}$$

Thus $\|A\|_p^p = \inf \sum\limits_{n=1}^{\infty} \left|\left\langle A\xi_n^{(\alpha)}, \xi_n^{(\alpha)}\right\rangle\right|^p = \inf \sum\limits_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^{2(1-p)} \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle^p$, where the first infimum is taken over all orthonormal bases and the second infimum is taken over all frames with lower bound $\geq 1$.

If $h_n^{(\alpha)}$ is a Parseval frame, then we obtain $\left\|h_n^{(\alpha)}\right\| \leq 1$ for all $n$. Thus it follows that

$$\sum_{n=1}^{\infty} \left\|h_n^{(\alpha)}\right\|^{2(1-p)} \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle^p \leq \sum_{n=1}^{\infty} \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle^p$$

and therefore $\|A\|_p^p = \inf \sum\limits_{n=1}^{\infty} \left|\left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)}\right\rangle\right|^p$, where the infimum is taken over all Parseval frames $\left\{h_n^{(\alpha)}\right\}$ as every orthonormal basis is a Parseval frame. □

**Corollary 3.** *Let $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ and $0 < p \leq 2$. Then the following are equivalent.*
*(i)* $\int\limits_{\mathbb{D}} B_\alpha(|A|^p)(z)d\lambda(z) < \infty$.

*(ii)* $\sum_{n=1}^{\infty} \left\| A\xi_n^{(\alpha)} \right\|^p < \infty$ *for some orthonormal basis* $\left\{ \xi_n^{(\alpha)} \right\}$ *in* $L_a^2(dA_\alpha)$.
*(iii)* $\sum_{n=1}^{\infty} \left\| Ah_n^{(\alpha)} \right\|^p < \infty$ *for some frame* $\left\{ h_n^{(\alpha)} \right\}$ *in* $L_a^2(dA_\alpha)$.

*Proof.* Let $H = A^*A$ and apply Theorem 5 to $H$. The result follows from the fact that an operator $A \in S_p$ if and only if $H \in S_{p/2}$ and $\left\langle Hh_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle = \left\| Ah_n^{(\alpha)} \right\|^2$. □

**Corollary 4.** *Let* $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$. *Then* $A$ *belongs to the Hilbert Schmidt class* $S_2$ *if and only if* $\sum_{n=1}^{\infty} \left\| Ak_{w_n}^{1+\frac{\alpha}{2}} \right\|^2 < \infty$ *where* $\{w_n\}$ *is a sampling sequence for the weighted Bergman space* $L_a^2(dA_\alpha)$.

*Proof.* Notice that $\{w_n\}$ is a sampling sequence for $L_a^2(dA_\alpha)$ if and only if the sequence $\left\{ k_{w_n}^{1+\frac{\alpha}{2}} \right\}$ is a frame in $L_a^2(dA_\alpha)$. For proof see [6] and [10]. To establish the corollary, observe that the following are equivalent:
(i) $A \in S_2$.
(ii) $\sum_{n=1}^{\infty} \left\| Ah_n^{(\alpha)} \right\|^2 < \infty$ for every frame $\left\{ h_n^{(\alpha)} \right\}$.
(iii) $\sum_{n=1}^{\infty} \left\| Ah_n^{(\alpha)} \right\|^2 < \infty$ for some frame $\left\{ h_n^{(\alpha)} \right\}$.
This follows from Lemma 7 and Corollary 3 and from the fact that $A \in S_2$ if and only if $A^*A \in S_1$ and $\left\langle A^*Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle = \left\| Ah_n^{(\alpha)} \right\|^2$. Thus $A \in S_2$ if and only if $\sum_{n=1}^{\infty} \left\| Ak_{w_n}^{1+\frac{\alpha}{2}} \right\|^2 < \infty$ where $\{w_n\}$ is a sampling sequence for $L_a^2(dA_\alpha)$. □

**Corollary 5.** *Let* $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ *and assume* $A \geq 0$. *Then* $A \in S_1$ *if and only if* $\sum_{n=1}^{\infty} \left\langle Ak_{w_n}^{1+\frac{\alpha}{2}}, k_{w_n}^{1+\frac{\alpha}{2}} \right\rangle < \infty$ *where* $\{w_n\}$ *is a sampling sequence for* $L_a^2(dA_\alpha)$.

*Proof.* Notice that if $A \in \mathcal{LC}\left(L_a^2(dA_\alpha)\right)$ and $A \geq 0$, then the following are equivalent:
(i) $A \in S_1$.
(ii) $\sum_{n=1}^{\infty} \left| \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle \right| < \infty$ for every frame $\left\{ h_n^{(\alpha)} \right\}$ in $L_a^2(dA_\alpha)$.
(iii) $\sum_{n=1}^{\infty} \left| \left\langle Ah_n^{(\alpha)}, h_n^{(\alpha)} \right\rangle \right| < \infty$ for some frame $\left\{ h_n^{(\alpha)} \right\}$ in $L_a^2(dA_\alpha)$.
This follows from Lemma 8 and Theorem 5. Now since $\{w_n\}$ is a sampling sequence for $L_n^{2,\alpha}(\mathbb{D})$, hence $\left\{ k_{w_n}^{1+\frac{\alpha}{2}} \right\}$ is a frame in $L_a^2(dA_\alpha)$ and therefore $\sum_{n=1}^{\infty} \left\langle Ak_{w_n}^{1+\frac{\alpha}{2}}, k_{w_n}^{1+\frac{\alpha}{2}} \right\rangle < \infty$ as $A \in S_1$ is positive. □

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