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Groupoids and Quasigroups Obeying Certain Laws

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Abstract

In this paper we study under which conditions the binary topological groupoid with some algebraic properties can be "transformed" into a topological quasigroup with the same algebraic properties.

Keywords: *AG*-groupoid, *AD*-groupoid, Ward quasigroup, Schröder quasigroup, Stein's quasigroup, topological quasigroup.

1 Introduction

The results established are related to the results of M. Choban and L. Chiriac in [2] and to the research papers [1,3,4,5,6]. Our main goal is to prove a new connection between groupoids and quasigroups. It is shown that if P is an open compact neighborhood of an $(2,1)$-identity e of an $Ward$ quasigroups G, then P contains an open compact $Ward$ subquasigroup Q with $(2,1)$-identity e. We study properties of *AG*- and *AD*-topological quasigroups with multiple identities.

2 Basic Notions

A non-empty set G is said to be a *groupoid* relatively to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a,b) of elements of G there is a unique element $ab \in G$.

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A groupoid $(G, \cdot)$ is called a *quasigroup* if for every $a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions.

A groupoid $(G, \cdot)$ is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid $(G, \cdot)$ is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is said to be *subtractive*, if the following conditions hold: $b \cdot (b \cdot a) = a$ and $a \cdot bc = c \cdot ba$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *AD-groupoid* if it satisfies the law $a \cdot bc = c \cdot ba$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

We define a *Ward groupoid* as any groupoid $(G, \cdot)$ containing an element $e \in G$ such that $a^2 = a \cdot a = e$ and $(a \cdot b) \cdot c = a \cdot (c \cdot (e \cdot b))$, for all $a, b, c \in G$. A groupoid $(G, \cdot)$ is called a *Schröder Second Law groupoid* if it satisfies the law $(ab) \cdot (ba) = a$ for all $a, b \in G$ [6]. The identity $(ab) \cdot (ba) = b$ for all $a, b \in G$ is known as *Stein's Third Law* [6]. The concept of (n, m)-identities was introduced by M.M. Choban and L.L. Chiriac in [3].

3 Main Results

We study the problem formulated below from several perspectives.

Problem. Under which conditions the binary topological groupoid with the algebraic properties $P_1, P_2, ..., P_k$ can be "transformed" into a topological quasigroup with the algebraic properties $P_1, P_2, ..., P_k$?

Theorem 1. *If $(G, \cdot)$ is an AG and AD-multiplicative topological groupoids and the following conditions hold:*

1. $x^2 = x \cdot x = x$ for every $x \in G$,

2. $(xy) \cdot (yx) = x$ for all $x, y \in G$.

3. if $xa = ya$ then $x = y$ for all $x, y, a \in G$,

then $(G, \cdot)$ is a Schröder, medial, AG and AD-topological quasigroups.

Example 1. *Let $G = \{1, 2, 3, 4\}$. We define the binary operation "$(\circ)$":*

$(\circ)$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Then $(G, \circ)$ is a Schröder, AG and AD-non-associative quasigroup.

Theorem 2. *If $(G, \cdot)$ is an AD-multiplicative topological groupoids and the following conditions hold:*

1. *$x^2 = x \cdot x = x$ for every $x \in G$,*
2. *$(xy) \cdot (yx) = y$ for all $x, y \in G$.*
3. *if $xa = ya$ then $x = y$ for all $x, y, a \in G$,*

then $(G, \cdot)$ is a Stein and AD-topological quasigroup.

Example 2. *Let $G = \{1, 2, 3, 4, 5\}$. We define the binary operation "$(\circ)$":*

$(\circ)$	1	2	3	4	5
1	1	4	5	2	3
2	3	2	1	5	4
3	4	5	3	1	2
4	5	3	2	4	1
5	2	1	4	3	5

Then $(G, \circ)$ is a Stein and AD-non-associative quasigroup.

Theorem 3. *If $(G, \cdot)$ is an AD-multiplicative topological groupoid, $e \in G$ and the following conditions hold:*

1. *$xe = x$ for every $x \in G$,*
2. *$x^2 = x \cdot x = e$ for every $x \in G$,*
3. *if $xa = ya$ then $x = y$ for all $x, y, a \in G$,*

then $(G, \cdot)$ is a $Ward$, subtractive and AD-topological quasigroup with a $(2, 1)$-identity e.

Theorem 4. *Let $(G, \cdot)$ be a $Ward$ topological quasigroup with an $(2, 1)$-identity e and $x^2 = e$ for every $x \in G$. If P is an open compact neighborhood such that $e \in P$, then P contains an open compact $Ward$ subquasigroup $(Q, \cdot)$ with an $(2, 1)$-identity of $(G, \cdot)$.*

References

[1] V. D. Belousov. *Foundation of the theory of quasigroups and loops.* Moscow, Nauka, 1967.

[2] M. M. Choban and L. L. Kiriyak. *The topological quasigroups with multiple identities.* Quasigroups and Related Systems, vol. 9, 2002, pp. 19–31.

[3] Natalia Bobeica and Liubomir Chiriac. *On Topological AG-groupoids and Paramedial Quasigroups with Multiple Identities.* ROMAI Journal, vol. 6, no. 1, 2010, pp. 1–14.

[4] L. L. Chiriac, L. Chiriac Jr, and N. Bobeica. *On topological groupoids and multiple identities.* Buletinul Academiei de Ştiinţe a RM, Matematica, no. 1(59), 2009, pp. 67–78.

[5] Liubomir Chiriac and Natalia Bobeica. *On topological quasigroups and multiple identities.* Topology and its Applications, vol. 340, 1 December 2023, 108759, https://doi.org/10.1016/j.topol.2023.108759.

[6] A. Sade. *Quasigroupes obeissant a certaines lois.* Rev. Fas. Sci. Univ. Istambul, vol. 22, 1957, pp. 151–184.

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