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Some Types of LCA Groups with Local Ring of Continuous Endomorphisms

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Abstract

We describe the structure of some locally compact abelian (LCA) groups whose ring of continuous endomorphisms is local.

Keywords: LCA groups, rings of continuous endomorphisms of LCA groups, local rings, topologically indecomposable groups, purely topologically indecomposable groups, co-purely topologically indecomposable groups, LCA groups with finite exchange property.

1 Introduction

Let $\mathcal{L}$ be the class of LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of all continuous endomorphisms of X. One may ask:

For which groups $X \in \mathcal{L}$, the ring $E(X)$ is local,

i.e., its non-invertible elements form an ideal.

A. Orsatti in [1], P.A. Krylov in [2], and R.B. Warfield in [3] have characterized various types of discrete groups in $\mathcal{L}$ with this property.

In the following, we will answer the corresponding question for different non-discrete types of groups in $\mathcal{L}$.

2 The results

First we introduce some notations. If $X \in \mathcal{L}$, we let $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote respectively the connected component of

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X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X, and the character group of X with its usual topology. We will use the group of rationals $\mathbb{Q}$ taken discrete, and the group of reals $\mathbb{R}$ with its usual topology. If $p \in \mathbb{P}$ and $n \in \mathbb{N}$, we denote by $\mathbb{Z}(p^n)$ the cyclic group of order p^n, by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group, and by $\mathbb{J}_p$ the group of p-adic integers, all taken discrete. Further, we denote by $\mathbb{Z}_p$ the group of p-adic integers with its unique compact topology, and by $\mathbb{Q}_p$ the group of all p-adic numbers with its usual locally compact topology.

Theorem 1. *Let $X \in \mathcal{L}$. If $E(X)$ is local, then either X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$, or else X is residual.*

Corollary 1. *Let X be a non-residual group in $\mathcal{L}$. The following conditions are equivalent:*

(i) *$E(X)$ is a field.*

(ii) *$E(X)$ is local.*

(iii) *X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$.*

Definition 1.

(i) *The pseudo-socle of an abelian group X, denoted by $soc(X)$, is the pure subgroup of X generated by all minimal pure fully invariant subgroups of X.*

(ii) *A group $X \in \mathcal{L}$ is said to be purely topologically indecomposable if every closed, pure subgroup of X is topologically indecomposable.*

Theorem 2. *Let X be a residual, purely topologically indecomposable group in $\mathcal{L}$ with the following properties:*

1) *$t(X)$ is closed in X.*

2) *If $k(X) \neq \{0\}$ and $d(X) = \{0\}$, then $t(X) \neq \{0\}$.*

The following conditions are equivalent:

(i) $E(X)$ *is a discrete valuation domain.*

(ii) $E(X)$ *is local.*

(iii) X *is topologically isomorphic either with a pure subgroup* G *of* $\mathbb{J}_p$ *such that* $soc(G) = G$, *or with one of the groups* $\mathbb{Q}_p$, $\mathbb{Z}(p^\infty)$, *or* $\mathbb{Z}(p^n)$, *where* $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$.

Definition 2.

(i) *The pseudo-radical,* $rad(X)$, *of a connected group* $X \in \mathcal{L}$ *is the connected component of the intersection of all maximal connected topologically fully invariant closed subgroups of* X. *If* X *has no maximal connected topologically fully invariant closed subgroups, then* $rad(X)$ *is taken to be all of* X.

(ii) *A group* $X \in \mathcal{L}$ *is said to be co-purely topologically indecomposable if for any closed, pure subgroup* C *of* X, *the quotient group* X/C *is topologically indecomposable.*

Corollary 2. *Let* X *be a residual, co-purely topologically indecomposable group in* $\mathcal{L}$ *with the following properties:*

1) $\overline{rX}/\cap_{n\in\mathbb{N}_0}\overline{nX}$ *is compact for some* $r \in \mathbb{N}_0$.

2) *If* $c(X) \neq X$ *and* $m(X) = X$, *then* $\cap_{n\in\mathbb{N}_0}\overline{nX} \neq X$.

The following conditions are equivalent:

(i) $E(X)$ *is a discrete valuation domain.*

(ii) $E(X)$ *is local.*

(iii) X *is topologically isomorphic either with a quotient group of* $\mathbb{J}_p^*$ *by a closed pure subgroup* A *with* $rad(A) = \{0\}$, *or with one of the groups* $\mathbb{Q}_p$, $\mathbb{Z}_p$, *or* $\mathbb{Z}(p^n)$, *where* $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$.

Definition 3. *Let $\mathcal{S}$ be a subclass of $\mathcal{L}$ closed under taking finitary topological direct products, closed subgroups, and topological isomorphisms. A group X in $\mathcal{S}$ is said to have the finite exchange property in $\mathcal{S}$ if X satisfies the following condition:*

For any group Y in $\mathcal{S}$ and any its pair of topological direct sum decompositions

$$Y = X' \oplus G = Y_1 \oplus \cdots \oplus Y_n,$$

where $G, Y_1, \ldots, Y_n \in \mathcal{S}$, $n \in \mathbb{N}_+$, and $X' \cong X$, there exist closed subgroups Y_i' and Z_i of Y_i such that $Y_i = Y_i' \oplus Z_i$ for all $i \in \{1, \ldots, n\}$, and $Y = X' \oplus Y_1' \oplus \cdots \oplus Y_n'$.

Theorem 3. *Let X be a non-zero topologically indecomposable group in $\mathcal{L}$. The ring $E(X)$ is local if and only if X has the finite exchange property in $\mathcal{L}$.*

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