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Invariant Transformations of Loop Transversals in Loops. 1. The Case of Isomorphism

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Abstract

A special class of invariant transformations of loop transversals in loops is investigated. The considered transformations correspond to arbitrary isomorphisms of transversal operations in loops. Isomorphisms of loop transversal operations with the same unit 1 are investigated.

Keywords: quasigroups, loops, transversals in loops, invariant transformations, isomorphism.

1 Introduction

The notion of left (right) transversal in a group to its subgroup is well known in group theory, group representation theory, and in quasigroup theory [1, 3]. This notion has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some concrete groups to their subgroups are of special interest.

In the work [6], the author gives a natural generalization of the notion of transversal for a suitable class of loops. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloop (see [8] and the **Condition A** below).

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Let a loop L and its proper subloop R be set, and some loop transversal $T_0 = \{t_i\}_{i \in E}$ in L to R is given and fixed. How to describe all other loop transversals in L to R? In other words, what kind of transformations are admissible over loop transversal T_0 to get that the obtained sets are loop transversals too? And how to describe the set of all such admissible transformations?

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, they are transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies, and isostrophies are well studied.

In the present work, the author investigates what transformations of loop transversals correspond to the first well-known transformation of transversal operations – to an isomorphism. A similar investigation for loop transversals in groups can be found in [7].

2 Necessary definitions and statements

2.1 Quasigroups, loops, and transversals in loops

Definition 1 . *A system $\langle E, \cdot \rangle$ is called [2] a **right (left) quasigroup** if, for arbitrary $a, b \in E$, the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If a system $\langle E, \cdot \rangle$ is both a right and left quasigroup, then it is called a **quasigroup**. If, in a right (left) quasigroup $\langle E, \cdot \rangle$, there exists an element $e \in E$ such that $x \cdot e = e \cdot x = x$, for every $x \in E$, then the system $\langle E, \cdot \rangle$ is called a **right (left) loop** (the element e is called a **unit** or **identity element**). If a system $\langle E, \cdot \rangle$ is both a right and left loop, then it is called a **loop**.*

Definition 2 . *([8]) Let $\langle L, \cdot \rangle$ be a loop and $\langle R, \cdot \rangle$ be its proper subloop. Then a **left coset** of R is a set of the form*

$$xR = \{xr \mid r \in R\},$$

and a ***right coset*** *has the form*

$$Rx = \{rx \,|\, r \in R\}.$$

Definition 3. *([8]) A loop L has a* ***left (right) coset decomposition by its proper subloop*** *R, if the left (right) cosets form a partition of the loop L, i.e., for some set of indexes E,*

1. $\bigcup\limits_{i\in E}(a_iR) = L$;
2. *for every $i, j \in E$, $i \neq j$*

$$(a_iR) \cap (a_jR) = \varnothing.$$

Definition 4. *(**Left Condition A**) A product at the left of an arbitrary element a of a loop L and an arbitrary left coset R_i of a loop L by its proper subloop R is a left coset of the loop L by its proper subloop R too, i.e., for every $a, b \in L$, there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{1}$$

2.2 The concept of a transversal in a loop to its subloop. Elementary properties.

Definition 5. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and the left (right)* ***Condition A*** *is fulfilled. Let $\{R_x\}_{x\in E}$ be the set of all left (right) cosets in L to R that form a left (right) coset decomposition of the loop L. The set $T = \{t_x\}_{x\in E} \subset L$ is called a* ***left (right) transversal*** *in L to R if T is a complete set of representatives of the left (right) cosets R_x in L to R, i.e., there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.*

Definition 6. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and both left and right* ***Conditions A*** *are fulfilled. The set $T = \{t_x\}_{x\in E}$ is called a* ***two-sided transversal*** *in L to R, if it is a left and a right transversal in L to R.*

Remark 7. *In Definitions 5, we assume that $t_1 = e$ (for simplicity and analogously to the case of transversals in groups).*

On the set E, it is possible to define correctly the following operation:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \quad (2)$$

if T is the left (non-reduced) transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \quad (3)$$

if T is the right (non-reduced) transversal in L to R.

Further we will study only the left transversals in L to R (the study of the right transversals is similar).

Definition 8. *If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then the left transversal $T = \{t_x\}_{x \in E}$ is called a **loop transversal**.*

The following statement is known (see [6]):

Lemma 9. *A system $< E, \overset{(T)}{\cdot}, 1 >$ is a left loop with the two-sided unit 1.*

2.3 Representation of loops by cosets

Let $\langle L, \cdot, e \rangle$ be a loop, $\langle R, \cdot, e \rangle$ be its subloop, and the left **Condition A** is fulfilled in $\langle L, \cdot, e \rangle$. Let $T = \{t_x\}_{x \in E}$ be a left transversal in L to R. Define the following mapping:

$$\begin{aligned} f &: L \times E \to E, \\ f &: (g, x) \to y = \hat{g}(x), \\ \hat{g}(x) &= y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot R) = t_y \cdot R. \end{aligned} \quad (4)$$

By virtue of the left **Condition A**, this definition (a *left action* of the loop L on the set E) is correct.

Lemma 10. *The mapping $\hat{g}$ is a permutation on the set E, for every element $g \in L$.*

From the previous Lemma, we obtain a permutation representation of the loop $\langle L, \cdot, e\rangle$: the mapping φ

$$\begin{aligned} \varphi &: g \to \hat{g}, \\ \varphi &: L \to \hat{L} \subset S_E, \end{aligned}$$

(where S_E is the symmetric group on E), and the multiplication of permutations from $\hat{L}$ is defined as follows:

$$\hat{g}_1 * \hat{g}_2 = \hat{g}_3 \quad \overset{def}{\Leftrightarrow} \quad g_1 \cdot g_2 = g_3 \quad \text{in a loop } \langle L, \cdot, e\rangle. \tag{5}$$

It is easy to see that

$$\varphi(g_1) * \varphi(g_2) = \hat{g}_1 * \hat{g}_2 = \hat{g}_3 = \widehat{g_1 \cdot g_2} = \varphi(g_1 \cdot g_2),$$

i.e., φ is a homomorphism of loops, from $\langle L, \cdot, e\rangle$ to $\left\langle \hat{L}, *, \mathbf{id} \right\rangle$.

Lemma 11. *For an arbitrary left transversal $T = \{t_x\}_{x\in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$, the following statements are true:*

1. *$\forall r \in R$: $\hat{r}(1) = 1$;*
2. *$\forall x, y \in E$:*

$$\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_x^{-1}(y) = x \backslash y,$$

where $\hat{t}_x^{-1}$ is the inverse of a permutation $\hat{t}_x$ in S_E, and "$\backslash$" is the left division in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\begin{aligned} \hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \\ \hat{t}_x^{-1}(1) = x\backslash 1; \quad \hat{t}_x^{-1}(x) = 1. \end{aligned}$$

Let us show how any two left transversals T and P in a loop L to its subloop R are connected.

Lemma 12 (see also [5]). *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$- be left transversals in L to R. Then there exists a set of elements $\{r_{(x)}\}_{x\in E}$ from R such that:*

1. $p_x = t_x r_{(x)}\ \forall x \in E$;
2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{r}^*_{(x)}(y)$, *where $\hat{r}^*_{(x)}$ is some suitable permutation from $\hat{R}$.*

This set $\{r_{(x)}\}_{x\in E}$ may be called a **derivation set** for the transversal T (and for the transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

3 The transformations which correspond to isomorphisms of the transversal operations

Let us remind the definitions of a left multiplicative group and of a left inner mappings group of a loop.

Definition 13. *Let $< E, \cdot, e >$ be a loop. Then the group*

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

*generated by all left translations L_a of the loop $< E, \cdot, e >$, is called the **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup*

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a\cdot b} L_a L_b, : a, b \in E >$$

*generated by all permutations $l_{a,b}$, is called the **left inner mappings group** of the loop $< E, \cdot, e >$.*

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a loop L to its subloop R, $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ be its

transversal operations. We will fix one of transversals, for example, $T = \{t_x\}_{x\in E}$ and will consider the following group:

$$M_L(T) \stackrel{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{LM(L)}\alpha^{-1} >,$$

generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \overset{(T)}{\cdot}, 1 >) \subseteq \alpha\widehat{LM(L)}\alpha^{-1}.$$

Lemma 14. *The following statement is true:*

$$N_{St_1(S_E)}(\widehat{LM(L)}) \subseteq M_L(T) \subseteq St_1(S_E).$$

Lemma 15. *Let the loops $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ be isomorphic, and let $\varphi : E \to E$ be an isomorphism (note that $\varphi(1) = 1$). Then*

1. *$\widehat{P} = r_0^{-1}\widehat{T}r_0$ for some $r_0 \in R^* = M_L(T)$;*
2. *$\varphi \equiv r_0$.*

Lemma 16. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in L to R, $r_0 \in R^* = M_L(T)$ and let*

$$\hat{p}_{x'} \stackrel{def}{=} r_0^{-1}\hat{t}_x r_0 \quad \forall x \in E.$$

The following statements are true:

1. *$P = \{p_{x'}\}_{x'\in E}$ is a loop transversal in G to H;*
2. *The transversal operations $< E, \overset{(P)}{\cdot}, 1 >$ and $< E, \overset{(T)}{\cdot}, 1 >$ are isomorphic, and the isomorphism is set up by the mapping $\varphi(x) = r_0(x)$.*

It means that "conjugated" loop transversals in L to R correspond to isomorphic loop transversal operations and vice versa.

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