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Endomorphisms of abelian group isotope semisymmetrizations

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Abstract. Semisymmetrization and Mendelsohnization allow quasigroup homotopies to be reduced to homomorphisms. We demonstrate the semisymmetrization of an abelian group isotope is isomorphic to the direct product of its Mendelsohnization and another specific isotope if and only if its order is not divisible by 3. We also provide an example of a quasigroup such that the idempotent replica of its semisymmetrization is trivial.

1. Introduction

Quasigroup homotopies can be preferable to homomorphisms in some contexts, such as incidence geometry [5]. However, they lack certain convenient algebraic properties - for example, equational identities often fail to be preserved under homotopic images [1]. The semisymmetrization functor Δ, described by Smith [5], as well as the similar Mendelsohnization functor Γ, described by Krapež and Petrić [2], allow homotopies between arbitrary quasigroups to be converted into homomorphisms between semisymmetric quasigroups. In fact, the category of quasigroups with homotopies can be shown to embed as a subcategory of the category of semisymmetric quasigroups with homomorphisms.

In the specific case where a given quasigroup Q is isotopic to some abelian group A, Smith [6] demonstrated (using slightly different notation) that the following sequence is exact:

$$0 \to \bar{A} \to Q^{\Delta} \to Q^{\Gamma} \to 0$$

where $(\bar{A}, \circ)$ is the totally symmetric isotope [7] [8] of A defined by $a \circ b = -a - b$ for $a, b \in A$. Further, Q^{Γ} is isomorphic to the *idempotent replica* $V(Q^{\Delta})$ of Q^{Δ}, the largest idempotent quotient of Q^{Δ}.

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The main result of this paper is to prove:

$$0 \to Q^{\Gamma} \to Q^{\Delta} \to \bar{A} \to 0$$

is also exact, and $\bar{A}$ is isomorphic to the largest commutative quotient of Q^{Δ}. Stronger, if and only if $3 \nmid |Q|$ then $Q^{\Delta} \cong Q^{\Gamma} \times \bar{A}$. We also show Q^{Γ} is distributive, and give an idempotent endomorphism λ of Q^{Δ} fixing the Q^{Γ} subquasigroup when $3 \nmid |Q|$.

More, we provide an answer in the negative for problem 5.3 of [6] regarding whether $Q^{\Gamma} \cong V(Q^{\Delta})$ in the general case; indeed, we give an example of a quasigroup for which $V(Q^{\Delta})$ is trivial.

2. Preliminaries

A *quasigroup* $(Q, \cdot, /, \backslash)$ is a set Q with binary operations $(\cdot, /, \backslash)$ such that

$$\begin{aligned} a\backslash(a \cdot b) &= b \\ a \cdot (a\backslash b) &= b \\ (a \cdot b)/b &= a \\ (a/b) \cdot b &= a \end{aligned}$$

For brevity, we may denote $a \cdot b$ by juxtaposition ab, and $ab \cdot c$ is to be interpreted as $(a \cdot b) \cdot c$. Throughout this paper we assume all quasigroups to be finite. A given quasigroup is

- *semisymmetric* if $ab \cdot a = b$ or equivalently $a \cdot ba = b$
- *idempotent* if $aa = a$
- *commutative* if $ab = ba$
- *distributive* if $a \cdot bc = ab \cdot ac$ and $ab \cdot c = ac \cdot bc$
- *Mendelsohn* if semisymmetric and idempotent
- *totally symmetric* if semisymmetric and commutative

A *homotopy* between quasigroups $\phi : Q_1 \to Q_2$ is a triple of functions (ϕ_1, ϕ_2, ϕ_3) such that $\phi_1(a) \cdot \phi_2(b) = \phi_3(c)$; if ϕ_1, ϕ_2 and ϕ_3 are permutations then ϕ is an *isotopy* and Q_1 is said to be an *isotope* of Q_2 and vice versa. If $\phi_1 = \phi_2 = \phi_3$, then ϕ is a *homomorphism* or an *isomorphism* if bijective.

Given a quasigroup Q, we can construct the *semisymmetrization* $(Q^\Delta, *)$ defined on the direct cube Q^3 of the underlying set of Q as:

$$[a, b, c] * [d, e, f] = [f/b, d\backslash c, ae]$$

Q^Δ is always a semisymmetric quasigroup. Given a homotopy $\phi : Q_1 \to Q_2$ then then

$$\phi^\Delta : Q_1^\Delta \to Q_2^\Delta; [a, b, c] \mapsto [\phi_1(a), \phi_2(b), \phi_3(c)]$$

is a homomorphism [5].

Likewise, the *Mendelsohnization* $(Q^\Gamma, *)$ is defined on the direct square Q^2 of the underlying set as:

$$[a, b] * [c, d] = [(ad)/b, c\backslash(ad)]$$

Q^Γ is always a Mendelsohn quasigroup and

$$\phi^\Gamma : Q_1^\Gamma \to Q_2^\Gamma; [a, b] \mapsto [\phi_1(a), \phi_2(b)]$$

is a homomorphism [2], [6].

In particular, if Q_1 is isotopic to Q_2 then $Q_1^\Delta \cong Q_2^\Delta$ and $Q_1^\Gamma \cong Q_2^\Gamma$. In proofs up to isomorphism, if Q is a quasigroup isotopic to abelian group A, without loss of generality we will perform calculations in Q^Δ, Q^Γ in their isomorphic copies A^Δ, A^Γ using additive notation for the sake of conciseness and clarity.

In [6], Smith identifies the subquasigroup of elements $[a, a, -a] \in Q^\Delta$ as the kernel of a homomorphism $p\colon Q^\Delta \to Q^\Gamma$ sending $[a, b, c] \mapsto [a + c, b + c]$. However, he describes the exact sequence (1) in terms of Mendelsohn extensions, whereas for the purposes of this paper we describe the kernel of p as the isotope $\bar{A}$ of A via isomorphism $a \in \bar{A} \mapsto [a, a, -a] \in Q^\Delta$.

3. Results

Proposition 3.1. *Up to isotopy, there exists at least one quasigroup Q of order greater than* 1 *such that $V(Q^\Delta)$ is trivial.*

Proof. Suppose some surjective homomorphism $h : Q^\Delta \to V(Q^\Delta)$. Every element $a \in V(Q^\Delta)$ is idempotent and thus a subquasigroup of order 1, so the preimage of each a must be a normal subquasigroup of Q^Δ i.e. there exists some partition of Q^Δ into disjoint normal subquasigroups of order $|Q|^3/|V(Q^\Delta)|$. Now suppose the order of Q is prime; then if there exists an element of $b \in Q^{\Delta}$ such that the subquasigroup generated by b is of order greater than $|Q|^2$, then the normal closure of b must be the entirety of Q and $V(Q^{\Delta})$ is trival [4].

Let Q be the semisymmetric quasigroup of order 5 with Cayley table:

	0	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

The subquasigroup of Q^{Δ} generated by the element $[0, 1, 2]$ is of order greater than 25.

$$\begin{aligned}
[0,1,2] * [0,1,2] &= [3,2,1] \\
[3,2,1] * [3,2,1] &= [4,4,4] \\
[4,4,4] * [0,1,2] &= [1,4,3] \\
[0,1,2] * [4,4,4] &= [2,3,4] \\
[4,4,4] * [4,4,4] &= [0,0,0] \\
[4,4,4] * [1,4,3] &= [2,3,0] \\
[2,3,4] * [4,4,4] &= [1,0,3] \\
[4,4,4] * [2,3,0] &= [4,1,2] \\
[1,4,3] * [0,1,2] &= [1,3,0] \\
[1,4,3] * [3,2,1] &= [3,0,3] \\
[3,2,1] * [1,4,3] &= [1,0,1] \\
[1,4,3] * [1,4,3] &= [2,2,2] \\
[1,4,3] * [2,3,4] &= [0,4,4] \\
[2,3,4] * [1,4,3] &= [0,3,3] \\
[1,4,3] * [0,0,0] &= [4,3,1] \\
[0,0,0] * [1,4,3] &= [3,1,4] \\
[2,3,0] * [1,4,3] &= [0,1,3] \\
[1,4,3] * [1,0,3] &= [2,2,1] \\
[1,0,3] * [1,4,3] &= [3,2,2] \\
[1,4,3] * [4,1,2] &= [1,1,0]
\end{aligned}$$

$$[4,1,2]*[1,4,3]=[4,4,0]$$
$$[1,4,3]*[1,3,0]=[4,2,4]$$
$$[1,4,3]*[3,0,3]=[2,0,1]$$
$$[1,0,1]*[1,4,3]=[3,0,2]$$
$$[1,4,3]*[0,4,4]=[0,3,2]$$

The proof is complete. □

Lemma 3.2. *Given a quasigroup Q, an element of Q^Δ is idempotent if and only if it is of the form $[a,b,ab]$ for some $a,b\in Q$, and the idempotent elements of Q^Δ form a subquasigroup iff Q is an abelian group isotope.*

Proof. If $[a,b,c]\in Q^\Delta$ is idempotent $[a,b,c]*[a,b,c]=[c/b,a\backslash c,ab]=[a,b,c]$ meaning $c/b=a, a\backslash c=b, ab=c$, all of which are equivalent to $ab=c$. Then because $[a,b,ab]*[c,d,cd]=[(cd)/b,c\backslash(ab),ad]$, the idempotent elements form a subquasigroup iff for all $a,b,c,d\in Q^\Delta$ we have $((cd)/b)\cdot(c\backslash(ab))=ad$; this is equivalent to identity 8.2.122 of [1], which defines the variety of abelian group isotopes. □

For any abelian group isotope Q, define Q^I to be the subquasigroup of idempotent elements of Q^Δ.

Lemma 3.3. *Given an abelian group isotope Q, then Q^I is isomorphic to Q^Γ.*

Proof. Let $h: Q^\Gamma\to Q^\Delta$ send $[a,b]\mapsto[a-b,b,a]$; clearly h is bijective onto Q^I. More,

$$h([a,b]*[c,d])=h([a-b+d,a-c+d])=[-b+c,a-c+d,a-b+d]$$
$$h([a,b])*h([c,d])=[a-b,b,a]*[c-d,d,c]=[-b+c,a-c+d,a-b+d]$$

so h is an isomorphism. □

For any abelian group isotope Q, define a relation $\sim$ on Q^Δ where

$$[x,-x+y+z,y]\sim[v,-v+w+z,w]$$

Lemma 3.4. *Given an abelian group isotope Q, then Q^I is normal in Q^Δ by congruence $\sim$.*

Proof. Define a relation $\prec$ on Q^Δ where

$$[a,b,a+b]*[c,d,e] \prec [f,g,f+g]*[c,d,e]$$

or equivalently

$$[-b+e,a+b-c,a+d] \prec [e-g,-c+f+g,d+f]$$

that is to say for $\alpha,\beta \in Q^\Delta$ then $\alpha \prec \beta$ if α and β are in the same right coset of Q^I.

If we let $a=y, b=-x, c=-z, d=0, e=0, f=w, g=-v$ it becomes clear $\alpha \prec \beta \Rightarrow \alpha \sim \beta$. Conversely, letting $x=-b+e, y=a+d, z=-c-d+e, v=e-g, w=d+f$ demonstrates $\alpha \sim \beta \Rightarrow \alpha \prec \beta$.

Likewise, $\succ$ where

$$[a,b,c,]*[d,e,d+e] \succ [a,b,c]*[g,f,g+f]$$

or

$$[-b+d+e,c-d,a+e] \succ [-b+f+g,c-g,a+f]$$

defines the left cosets of Q^I. Then letting $a=y, b=-y-z, c=0, d=x-y-z, e=0, f=w-y, g=v-w-z$ shows $\alpha \succ \beta \Rightarrow \alpha \sim \beta$ and letting $x=-b+d+e, y=a+e, z=-a-b+c, v=-b+f+g, w=a+f$ demonstrates $\alpha \sim \beta \Rightarrow \alpha \succ \beta$.

Now

$$[x,y,z]*[a,-a+b+c,b] = [b-y,-a+z,-a+b+c+x]$$
$$= [b-y,-(b-y)+(-a+b+c+x)+(-c-x-y+z),-a+b+c+x]$$

and

$$[x,y,z]*[d,c-d+e,e] = [e-y,-d+z,c-d+e+x]$$
$$= [e-y,-(e-y)+(c-d+e+x)+(-c-x-y+z),c-d+e+x]$$

so $\sim$ is a left congruence. More,

$$[a,-a+b+c,b]*[x,y,z] = [a-b-c+z,b-x,a+y]$$
$$= [a-b-c+z,-(a-b-c+z)+(a+y)+(-c-x-y+z),a+y]$$

and

$$[d,-d+e+c,e]*[x,y,z] = [d-e-c+z,e-x,d+y]$$
$$= [d-e-c+z,-(d-e-c+z)+(d+y)+(-c-x-y+z),d+y]$$

so $\sim$ is a right congruence. Because Q^Δ is semisymmetric, it follows that $\sim$ also respects left and right division, so $\sim$ is a congruence. $\square$

Proposition 3.5. *Given a quasigroup Q isotopic to abelian group A, the quotient quasigroup Q^{Δ}/Q^{I} is isomorphic to $\bar{A}$.*

Proof. Let $h : Q^{\Delta} \to \bar{A}$ send $[a, b, c] \mapsto a+b-c$. Then $h([a, -a+b+c, b]) = c$ and $h([d, -d + e + c, e]) = c$ so $\alpha \sim \beta$ implies $h(\alpha) = h(\beta)$. Likewise $h([a, b, c]) = h([d, e, f])$ means $a+b-c = d+e-f$ so $[a, b, c] = [a, -a+c+(a+b-c), c] \sim [d, e, f] = [d, -d+f+(d+e-f), f]$ meaning $h(\alpha) = h(\beta)$ implies $\alpha \sim \beta$.

More,

$$h([a, b, c] * [d, e, f]) = h([-b + f, c - d, a + e]) = -a - b + c - d - e + f$$
$$h([a, b, c]) \circ h([d, e, f]) = (a + b - c) \circ (d + e - f) = -a - b + c - d - e + f$$

So h is a homomorphism, and setting $b = 0, c = 0$ shows h surjects onto $\bar{A}$, therefore $Q^{\Delta}/Q^{I} \cong \bar{A}$. □

Corollary 3.6. *Given a quasigroup Q isotopic to abelian group A, the largest commutative quotient of Q^{Δ} is isomorphic to $\bar{A}$.*

Proof. Suppose $a \bowtie b$ is some congruence on Q^{Δ} such that $Q^{\Delta}/\bowtie$ is commutative i.e.

$$[a, b, c]*[d, e, f] = [-b+f, c-d, a+e] \bowtie [d, e, f]*[a, b, c] = [-e+c, f-a, d+b]$$

Let $a = 0, b = -x-v+w+z, c = v+y, d = x+v-z, e = y, f = -v+w+z$; then

$$\begin{aligned}&[-(-x - v + w + z) + (-v + w + z), (v + y) - (x + v - z), 0 + y] \bowtie\\ &\qquad [(v + y) - y, -0 + (-v + w + z), (x + v - z) + (-x - v + w + z)]\end{aligned}$$

or $[x, -x + y + z, y] \bowtie [v, -v + w + z, w]$ meaning $\alpha \bowtie \beta \Rightarrow \alpha \sim \beta$ so any commutative quotient of Q^{Δ} must be contained within $\bar{A} \cong Q^{\Delta}/\sim$.

And $\bar{A}$ is always commutative thus we already have the converse implication. □

Proposition 3.7. *Given an abelian group isotope Q, left or right multiplication by some fixed element $\alpha \in Q^{\Delta}$ is an automorphism of Q^{Δ} if and only if α is idempotent.*

Proof. Let $L_\alpha : Q^{\Delta} \to Q^{\Delta}$ be left multiplication by α where $\alpha = [x, y, z]$. Then

$$\begin{aligned}&L_\alpha([a, b, c] * [d, e, f]) = L_\alpha([-b + f, c - d, a + e])\\ &= [x, y, z] * [-b + f, c - d, a + e] = [a + e - y, b - f + z, c - d + x]\end{aligned}$$

and

$$\begin{aligned} L_\alpha([a,b,c]) * L_\alpha([d,e,f]) &= ([x,y,z]*[a,b,c]) * ([x,y,z]*[d,e,f]) \\ &= [c-y, -a+z, b+x] * [f-y, -d+z, e+x] \\ &= [a+e+x-z, b-f+x+y, c-d-y+z] \end{aligned}$$

Quasigroup mulitplication is bijective, thus by definition L_α is an automorphism iff

$$[a+e-y, b-f+z, c-d+x] = [a+e+x-z, b-f+x+y, c-d-y+z]$$

or simply

$$[-y, z, x] = [x-z, y+x, -y+z]$$

which is equivalent to $x+y=z$, the condition for $[x,y,z]$ being idempotent.

Likewise

$$\begin{aligned} R_\alpha([a,b,c]*[d,e,f]) &= R_\alpha([-b+f, c-d, a+e]) \\ = [-b+f, c-d, a+e]*[x,y,z] &= [-c+d+z, a+e-x, -b+f+y] \end{aligned}$$

and

$$\begin{aligned} R_\alpha([a,b,c]) * R_\alpha([d,e,f]) &= ([a,b,c]*[x,y,z]) * ([d,e,f]*[x,y,z]) \\ &= [-b+z, c-x, a+y] * [-e+z, f-x, d+y] \\ &= [-c+d+x+y, a+e+y-z, -b+f-x+z] \end{aligned}$$

so R_α is an automorphism iff

$$[z, -x, y] = [x+y, y-z, -x+z]$$

which is also equivalent to $x+y=z$. □

Corollary 3.8. *Given an abelian group isotope Q, then Q^Γ is distributive.*

Proof. This follows immediately from Lemma 3.3 and Proposition 3.7. □

Theorem 3.9. *Given a quasigroup Q isotopic to abelian group A, then $Q^\Delta \cong Q^\Gamma \times \bar{A}$ if and only if $3 \nmid |Q|$.*

Proof. Given quasigroups U_1, U_2 then an element $[a, b] \in U_1 \times U_2$ is idempotent if $[a, b] * [a, b] = [a \cdot a, b \cdot b] = [a, b]$, so the number of idempotents in $U_1 \times U_2$ is equal to the number of idempotents in U_1 times the number of idempotents in U_2.

An element $a \in \bar{A}$ is idempotent if $a \circ a = a$ which means $-a - a = a$ i.e. $3a = 0$, that is to say a is either 0 or an element of order 3, so $\bar{A}$ contains nonzero idempotents if $|A| = |Q|$ is divisible by 3 [3]. Because all elements of Q^Γ are idempotent, then $Q^\Gamma \times \bar{A}$ contains more than $|Q|^2$ idempotents if $|Q|$ is divisible by 3 - but by lemmas 3.2 and 3.3 Q^Δ has exactly $|Q|^2$ idempotents, therefore if 3 divides $|Q|$ then Q^Δ cannot be isomorphic to $Q^\Gamma \times \bar{A}$.

Suppose $3 \nmid |Q|$, and define function $h : Q^\Delta \to Q^\Gamma \times \bar{A}$ by

$$[a, b, c] \mapsto [[a + c, b + c], a + b - c]$$

Then

$$\begin{aligned} &h([a, b, c] * [d, e, f]) = h([-b + f, c - d, a + e]) \\ &= [[a - b + e + f, a + c - d + e], -a - b + c - d - e + f] \end{aligned}$$

and

$$\begin{aligned} &h([a, b, c]) * h([d, e, f]) \\ &= [[a + c, b + c], a + b - c] * [[d + f, e + f], d + e - f] \\ &= [[(a+c)-(b+c)+(e+f), (a+c)-(d+f)+(e+f)], -(a+b-c)-(d+e-f)] \\ &= [[a - b + e + f, a + c - d + e], -a - b + c - d - e + f] \end{aligned}$$

so h is a homomorphism.

Further, $h([a, b, c]) = h([d, e, f])$ means

$$\begin{aligned} a + c &= d + f \\ b + c &= e + f \\ a + b - c &= d + e - f \end{aligned}$$

therefore

$$\begin{aligned} (a + b - c) - (a + c) - (b + c) &= (d + e - f) - (d + f) - (e + f) \\ -3c &= -3f \\ c &= f \end{aligned}$$

from which it straightforwardly follows that $a = d, b = e$ and so $h([a, b, c]) = h([d, e, f]) \Rightarrow [a, b, c] = [d, e, f]$ proving h is injective; division by 3 is always possible because $3 \nmid |A|$. Because Q^Δ and $Q^\Gamma \times \bar{A}$ have the same cardinality, by finiteness injectivity implies surjectivity and h is an isomorphism. □

Given any abelian group isotope Q where $3 \nmid |Q|$, define $\lambda : Q^\Delta \to Q^\Delta$ by

$$[a,b,c] \mapsto [(2a-b+c)/3, (-a+2b+c)/3, (a+b+2c)/3]$$

Proposition 3.10. *λ is an endomorphism with image Q^I such that $\lambda(\alpha) = \alpha$ for any $\alpha \in Q^I$.*

Proof. $\lambda([a,b,c] * [d,e,f]) = \lambda([-b+f, c-d, a+e])$
$= [((2(-b+f) - (c-d) + (a+e))/3, (-(-b+f) + 2(c-d) + (a+e))/3,$
$((-b+f) + (c-d) + 2(a+e))/3]$
$= [(a-2b-c+d+e+2f)/3, (a+b+2c-2d+e-f)/3, (2a-b+c-d+2e+f)/3]$

and

$\lambda([a,b,c]) * \lambda([d,e,f])$
$= [(2a-b+c)/3, (-a+2b+c)/3, (a+b+2c)/3] * [(2d-e+f)/3,$
$(-d+2e+f)/3, (d+e+2f)/3]$
$= [-(-a+2b+c)/3 + (d+e+2f)/3, (a+b+2c)/3 - (2d-e+f)/3,$
$(2a-b+c)/3 + (-d+2e+f)/3]$
$= [(a-2b-c+d+e+2f)/3, (a+b+2c-2d+e-f)/3,$
$(2a-b+c-d+2e+f)/3]$

therefore λ is an endomorphism.

Further,

$$(2a-b+c)/3 + (-a+2b+c)/3 = (a+b+2c)/3$$

so $h([a,b,c]) \in Q^I$, and given $[a,b,a+b] \in Q^I$

$\lambda([a,b,a+b])$
$= [(2a-b+(a+b))/3, (-a+2b+(a+b))/3, (a+b+2(a+b))/3]$
$= [a,b,a+b].$ □

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