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Stationary Nash Equilibria for Stochastic Positional Games

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Abstract

The problem of the existence and determining stationary Nash equilibria for stochastic positional games with discounted and average payoffs is considered. We show that, for a stochastic positional game with discounted payoffs, there exists a Nash equilibrium in pure stationary strategies and, for a stochastic positional game with average payoffs, there exists a Nash equilibrium in mixed stationary strategies. Some approaches for determining pure and mixed stationary equilibria in such games are proposed.

Keywords: stochastic positional games, average and discounted payoffs, pure and mixed strategies, Nash equilibrium.

1 Introduction and problem formulation

Stochastic games were introduced by Shapley [1]. He considered two-person zero-sum games for which he proved the existence of the value and the optimal stationary strategies of the players with respect to a discounted payoff criterion. Later this result has been extended for m-player stochastic games with discount payoffs of the players [2, 3]. For average stochastic games, stationary Nash equilibria is known to exist only for some special classes of games. In general, the problem of the existence of Nash equilibria for average stochastic games is an open problem, however, for some classes of games (as example for zero-sum average stochastic games), has been proven the existence of Nash equilibria in non-stationary strategies.

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In this contribution, we consider a special class of stochastic games with average and discounted payoffs for which Nash equilibria exist. This class of games has been considered in [4] as a generalization of deterministic positional games; we called this class of games *stochastic positional games*. A stochastic positional game with m players is determined by the following elements: A finite set of states X; a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$, where X_i represents the position set of player $i \in \{1, 2, \ldots, m$; a finite set $A(x)$ of actions in each state $x \in X$; a step reward $r^i_{x,a}$ with respect to each player $i \in \{1, 2, \ldots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$ and a transition probability function $p : X \times \prod\limits_{x \in X} A(x) \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum\limits_{y \in X} p^a_{x,y} = 1, \forall x \in X, a \in A(x)$.

The positional game starts at the moment of time $t = 0$ in a given state $x_0 \in X$, where the player $i \in \{1, 2, \ldots, m\}$, who is the owner of the state position x_0 ($x_0 \in X_i$), chooses an action $a_0 \in A(x_0)$ and determines the rewards $r^1_{x_0,a_0}, r^2_{x_0,a_0}, \ldots, r^m_{x_0,a_0}$ for the corresponding players $1, 2, \ldots, m$. After that the game passes to a state $y = x_1 \in X$ according to a probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$, the player $k \in \{1, 2, \ldots, m\}$, who is the owner of the state position x_1 ($x_1 \in X_k$), chooses an action $a_1 \in A(x_1)$ and players $1, 2, \ldots, m$ receive the corresponding rewards $r^1_{x_1,a_1}, r^2_{x_1,a_1}, \ldots, r^m_{x_1,a_1}$. Then the game passes to a state $y = x_2 \in X$ according to a probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \ldots, x_t, a_t, \ldots$ that defines a stream of stage rewards $r^1_{x_t,a_t}, r^2_{x_t,a_t}, \ldots, r^m_{x_t,a_t}$, $t = 0, 1, 2, \ldots$. *The average stochastic positional game* is the game with payoffs of the players

$$\omega^i_{x_0} = \lim_{t \to \infty} \inf \; \mathsf{E}\left(\frac{1}{t} \sum_{\tau=0}^{t-1} r^i_{x_\tau, a_\tau}\right), \quad i = 1, 2, \ldots, n,$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players

in their position sets and given starting state x_0. Each player in this game has the aim to maximize his average reward per transition. *The discounted stochastic positional game* is the game with payoffs of the players

$$\omega^i_{x_0,\gamma} = \mathsf{E}\left(\sum_{\tau=0}^{\infty} \gamma^\tau r^i_{x_\tau,a_\tau}\right), \quad i = 1, 2, \ldots, n.$$

2 The main results

A stationary strategy of player $i \in \{1, 2, \ldots, m\}$ is a mapping s^i that provides a probability distribution over the set of actions $A(x_t)$ for every state $x_t \in X_i$. If these probabilities only take values 0 and 1, then s^i is called a pure strategy; otherwise, s^i is called a mixed strategy. A pure strategy of player i is a mapping $s^i : x \to a \in A(x)\ for\ x \in X$ that determines an action $a \in A(x)$ for each state $x \in X$, i.e., $s^i(x) = a$. The set of pure stationary strategies S^i of player i is determined by the set of solutions of the following system

$$\sum_{a \in A(x)} s^i_{x,a} = 1,\ \forall x \in X_i; s^i_{x,a} \in \{0, 1\},\ \forall x \in X_i,\ \forall a \in A(x),$$

and the set of mixed stationary strategies $\mathbf{S}^i$ of player i is determined by the set of solutions of the following system

$$\sum_{a \in A(x)} s^i_{x,a} = 1,\ \forall x \in X_i;\ s^i_{x,a} \geq 0,\ \forall x \in X_i,\ \forall a \in A(x).$$

Let $s = (s^1, s^2, \ldots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players. Then the probability transition matrix $P^s = (p_{x,y})$ in the Markov process induced by s can be calculated as follows $p^s_{x,y} = \sum_{a \in A(x)} s^i_{x,a} p^a_{x,y},\ x \in X^i,\ i = 1, 2, \ldots, m$. If $Q^s = (q^s_{x,y})$ is the limiting matrix of P^s, then the average payoffs $\omega^1_{x_0}(s), \omega^2_{x_0}, \ldots, \omega^m_{x_0}(s)$ of the corresponding players in a positional game are defined as follows: $\omega^i_{x_0}(s) = \sum_{k=1}^m \sum_{y \in X_k} q^i_{x_0,s^k},\ i = 1, 2, \ldots, m$, where $r^i_{y,s^k} = \sum_{a \in A(y)} s^k_{y,a} r^i_{y,a}$, for $y \in X_k,\ k \in \{1, 2, \ldots, m\}$. The discounted payoffs $\sigma^1_{x_o}(s), \sigma^2_{x_o}, \ldots, \sigma^m_{x_o}(s)$ of the players in a positional game are defined as $\sigma^i_{x_0} = \sum_{y \in X} w_{x_0,y} r^i_{y,s},\ i = 1, 2, \ldots, m$, where $w^s_{x,y}$ represent the elements of the matrix $W^s = (I - \gamma P^s)$.

We proved the following two theorems.

Theorem 1. *The game in normal form* $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ *has a Nash equilibrium which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game.*

Theorem 2. *The game in normal form* $\langle \{S^i\}_{i=\overline{1,m}}, \{\sigma^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ *has a Nash equilibrium which is a Nash equilibrium in pure stationary strategies for the discounted stochastic positional game.*

3 Conclusion

For average and discounted stochastic positional games there exist Nash equilibria in stationary strategies.

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