Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

Invariant Transformations of Loop Transversals in Loops. 2. The Case of Isotopy

Kuznetsov Eugene

Abstract

A special class of invariant transformations of loop transversals in loops is investigated. Transformations from this class correspond to arbitrary isotopies of transversal operations in loops. Isotopies of loop transversal operations with the same unit 1 are investigated.

Keywords: quasigroups, loops, transversals in loops, invariant transformations, isotopy.

1 Introduction

In [6], the author gives a version of a natural generalization of the notion of transversal at a suitable class of loops. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloops (see [8] and the **Condition A** below).

Let a loop L and its proper subloop R be set, and $T_0 = \{t_i\}_{i\in E}$ be a fixed loop transversal in L to R. How to describe all other loop transversals in L to R? In other words, what kind of transformations are admissible over a loop transversal T_0 in order to get that the obtained sets are loop transversals too? And how to describe the set of all such admissible transformations?

©2024 by Kuznetsov Eugene

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, t hey a re transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies, and isostrophies are well studied.

In the present work, the author investigates the transformations of loop transversals that correspond to the well-known transformations of transversal operations – to isotopies. Similar investigation of transformations of loop transversals in groups can be found in [7].

2 Necessary definitions a nd statements

2.1 Quasigroups, loops, and transversals in loops

Definition 1 . *A system $\langle E, \cdot \rangle$ is called [2] a **right (left) quasigroup** if, for arbitrary $a, b \in E$, the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If a system $\langle E, \cdot \rangle$ is both a right and a left quasigroup, then it is called a **quasigroup**. If, in a right (left) quasigroup $\langle E, \cdot \rangle$, there exists an element $e \in E$ such that*

$$x \cdot e = e \cdot x = x,$$

*for every $x \in E$, then the system $\langle E, \cdot \rangle$ is called a **right (left) loop** (the element e is called a **unit** or **identity element**). If a system $\langle E, \cdot \rangle$ is both a right and left loop, then it is called a **loop**.*

Definition 2 . *([8]) Let $\langle L, \cdot \rangle$ be a loop and $\langle R, \cdot \rangle$ be its proper subloop. Then a **left coset** of R is a set of the form*

$$xR = \{xr \mid r \in R\},$$

*and a **right coset** has the form*

$$Rx = \{rx \mid r \in R\}.$$

Definition 3. *([8]) A loop L has a **left (right) coset decomposition by its proper subloop** R, if the left (right) cosets form a partition of the loop L, i.e., for some set of indexes E, we have:*

1. $\bigcup\limits_{i\in E}(a_iR) = L$;
2. *for every $i, j \in E$, $i \neq j$*

$$(a_iR) \cap (a_jR) = \varnothing.$$

Definition 4. *(**Left Condition A**) The product on the left of an arbitrary element a of a loop L and an arbitrary left coset R_i of a loop L by its proper subloop R is a left coset of the loop L by its proper subloop R too, i.e., for every $a, b \in L$, there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{1}$$

2.2 The concept of a transversal in a loop to its subloop. Elementary properties

Definition 5. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and let the left (right) **Condition A** be fulfilled. Let $\{R_x\}_{x\in E}$ be a set of all left (right) cosets in L to R that form a left (right) coset decomposition of the loop L. A set $T = \{t_x\}_{x\in E} \subset L$ is called a **left (right) transversal** in L to R if T is a complete set of representatives of the left (right) cosets R_x in L to R, i.e., there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.*

Definition 6. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and both left and right **Conditions A** are fulfilled. A set $T = \{t_x\}_{x\in E}$ is called a **two-sided transversal** in L to R, if it is a left and a right transversal in L to R.*

Remark 7. *In Definitions 5, we assume that $t_1 = e$ (for simplicity and analogously to the case of transversals in groups).*

On a set E it is possible to define correctly the following operation:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \quad (2)$$

if T is a left (non-reduced) transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \quad (3)$$

if T is a right (non-reduced) transversal in L to R.

Further we will study only the left transversals in L to R (the study of right transversals is similar).

Definition 8. *If a system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a loop, then the left transversal* $T = \{t_x\}_{x \in E}$ *is called a* ***loop transversal****.*

The following statement is known (see [6]):

Lemma 9. *A system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a left loop with the two-sided unit 1.*

2.3 Representation of loop by cosets

Let $\langle L, \cdot, e \rangle$ be a loop, $\langle R, \cdot, e \rangle$ be its subloop, and let the left **Condition A** is fulfilled in $\langle L, \cdot, e \rangle$. Let $T = \{t_x\}_{x \in E}$ be a left transversal in L to R and let define the following mapping:

$$\begin{array}{rcll} f & : & L \times E \to E, & \\ f & : & (g, x) \to y = \hat{g}\,(x)\,, & (4) \\ \hat{g}\,(x) & = & y \quad \overset{def}{\Leftrightarrow} & g \cdot (t_x \cdot R) = t_y \cdot R. \end{array}$$

By virtue of the left **Condition A**, this definition (a *left action* of the loop L on a set E) is correct.

Lemma 10. *The mapping* $\hat{g}$ *is a permutation on the set* E*, for every element* $g \in L$*.*

It follows from the previous Lemma, that we obtained a permutation representation of a loop $\langle L, \cdot, e\rangle$. Indeed, we have the function φ :

$$\begin{aligned} \varphi &: g \to \hat{g}, \\ \varphi &: L \to \hat{L} \subset S_E, \end{aligned}$$

(where S_E is the symmetric group on a set E), and multiplication of permutations from $\hat{L}$ is defined as follows:

$$\hat{g}_1 * \hat{g}_2 = \hat{g}_3 \quad \overset{def}{\Leftrightarrow} \quad g_1 \cdot g_2 = g_3 \quad \text{in the loop } \langle L, \cdot, e\rangle. \tag{5}$$

Then it is easy to see that

$$\varphi(g_1) * \varphi(g_2) = \hat{g}_1 * \hat{g}_2 = \hat{g}_3 = \widehat{g_1 \cdot g_2} = \varphi(g_1 \cdot g_2),$$

i.e., φ is a homomorphism of the loop $\langle L, \cdot, e\rangle$ to the loop $\left\langle \hat{L}, *, \mathbf{id} \right\rangle$.

Lemma 11. *For an arbitrary left transversal $T = \{t_x\}_{x\in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$ the following statements are true:*

1. *$\forall r \in R$: $\hat{r}(1) = 1$;*
2. *$\forall x, y \in E$:*

$$\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_x^{-1}(y) = x \backslash y,$$

where $\hat{t}_x^{-1}$ is the inverse of the permutation $\hat{t}_x$ in S_E, and "$\backslash$" is the left division in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\begin{aligned} \hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \\ \hat{t}_x^{-1}(1) = x\backslash 1; \quad \hat{t}_x^{-1}(x) = 1. \end{aligned}$$

Let us show how any two left transversals T and P in a loop L to its subloop R are connected.

Lemma 12 (see also [5]). *Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$- be left transversals in L to R. Then there exists a set of elements $\{r_{(x)}\}_{x\in E}$ from R such that:*

1. $p_x = t_x r_{(x)}\ \forall x \in E$;
2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{r}^*_{(x)}(y)$, *where $\hat{r}^*_{(x)}$ is some suitable permutation from $\hat{R}$.*

This set $\{r_{(x)}\}_{x\in E}$ may be called a **derivation set** for transversal T (and for transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

Definition 13 (see [2]). *A groupoid $\langle E, \circ\rangle$ is called an isotope of a groupoid $\langle E, \cdot\rangle$ if there exists three permutations α, β, γ on E such that*

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y), \quad \forall x, y \in E.$$

*In this case, the ordered triplet $\Phi = (\alpha, \beta, \gamma)$ is called an **isotopy.** If $\gamma = id$, then the isotopy $\Phi = (\alpha, \beta, id)$ is called a **principal isotopy**.*

Lemma 14. *If a loop $\langle E, \cdot, e_1\rangle$ is isotopic to a loop $\langle E, \circ, e_2\rangle$, then it is isomorfic to some principal isotop of a loop $\langle E, \circ\rangle$ (and this principal isotopy has the form $T_0 = (L_a^{-1}, R_b^{-1}, id), a \cdot b = e_2$).*

Remark 15. *If a loop $\langle E, \cdot, 1\rangle$ is principal isotopic to a loop $\langle E, \circ, 1\rangle$, then this principal isotopy has a form $T_0 = (R_b^{-1}, L^{-1}_{a\backslash 1}, id)$ for some $a \in E$ ($^{-1}a = a\backslash 1$ is the left inverse element to a in the loop $\langle E, \cdot, 1\rangle$).*

3 The transformations which correspond to isotopy of the transversal operations

As follows from Lemma 14, to investigate loop transversals transformations, which correspond to isotopy of operations $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and

$\langle E, \overset{(P)}{\cdot}, 1\rangle$, it is enough to study the case of principal isotopy $T_0 = (L_a^{-1}, R_{a\backslash 1}^{-1}, id)$ (as the transformations which correspond to isomorphism of transversal operations were studied earlier).

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a loop L to its subloop R, and $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ be transversal operations. Fix one of transversals, for example, $T = \{t_x\}_{x\in E}$.

Theorem 16. *Let the loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be principal isotopic and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id)$ (note that $a \overset{(T)}{\cdot} b = 1$). Then*

$$\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}$$

for some $a \in E$.

Theorem 17. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in L to R, and $a \in E$ be an arbitrary element from the set E. Define the following set of permutations:*

$$\widehat{p}_{x'} \overset{def}{=} \widehat{t}_x \widehat{t}_a^{-1} \quad \forall x \in E.$$

Then

1. *$P = \{p_{x'}\}_{x'\in E}$ is a left transversal in G to H;*
2. *The transversal operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is principal isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and principal isotopy S has the following form: $S = (R_{a\backslash 1}^{-1}, L_a^{-1}, id)$;*
3. *P is a loop transversal in L to R.*

Acknowledgments. The Institutional Research Program of the Moldova State University for 2024–2027 years, subprogram 011303 has supported part of the research for this paper.

References

[1] R. Baer. *Nets and groups.* Trans. Amer. Math. Soc., vol. 46, 1939, pp. 110–141.

[2] V. Belousov. *Foundations of quasigroup and loop theory.* Moscow, Nauka, 1967. (in Russian).

[3] E. Kuznetsov. *Transversals in groups. 1. Elementary properties.* Quasigroups and related systems, vol. 1, no. 1, 1994, pp. 22–42.

[4] E. Kuznetsov. *Transversals in groups. 2. Loop transversals in a group by the same subgroup.* Quasigroups and related systems, vol. 6, 1999, pp. 1–12.

[5] E. Kuznetsov. *Transversals in groups. 4. Derivation construction.* Quasigroups and related systems, vol. 9, 2002, pp. 67–84.

[6] E. Kuznetsov. *Transversals in loops. 1. Elementary properties.* Quasigroups and related systems, vol. 18, 2010, pp. 43–58.

[7] E. Kuznetsov and S. Botnari. *Invariant transformations of loop transversals. 2. The case of isotopy.* Buletinul Academiei de Stiinte a RM. Matematica, vol. 70, no. 3, 2012, pp. 72–80.

[8] H. Pflugfelder. *Quasigroups and loops: Introduction.* Sigma Series in Pure Math., vol. 7, Helderman Verlag, New York, 1972.

Kuznetsov Eugene

Vladimir Andrunachievici Institute of Mathematics and Computer Science, USM
E–mail: kuznet1964@mail.ru
ORCID: https://orcid.org/0009-0003-4910-8995