MINISTRY OF EDUCATION AND RESEARCH OF THE REPUBLIC OF MOLDOVA
MOLDOVA STATE UNIVERSITY

ECONOMIC MATHEMATICS

course support

EDITION I

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CHISINAU 2024

CZU 330.4(075.8)

S 74

Recommended for publication by the Council of the Faculty of Mathematics and Computer Science and the Quality Council of the State University of Moldova.

The paper is a theoretical and practical guide for the Information and Communication Technologies course, designed for foreign students attending the Romanian Language training course, adapted to the specific needs of non-native participants in the Romanian language learning process.

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Written in accordance with the Curriculum in the discipline of Information and Communication Technologies, it is intended for foreign, non-native students, who follow the training course Language Romanian.

CIP DESCRIPTION OF THE NATIONAL BOOK CHAMBER OF THE REPUBLIC OF MOLDOVA

Sprincean, Galina.

Economic mathematics : course support / Sprincean Galina ; Ministry of Education and Research of the Republic of Moldova, Moldova State University. – Ed. 1. – Chișinău : Editura USM, 2024. – 311 p..

System requirements: PDF Reader.

Bibliographic references: p. 310 (22 tit.).

ISBN 978-9975-62-715-3 (PDF).

330.4(075.8)

S 74

ISBN 978-9975-62-715-3 (PDF)

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PREFACES

This material represents an introduction to the study of Mathematical Economics and its applications. It is intended for students in economics faculties and all those interested in using fundamental concepts of Mathematical Economics in practical applications. The purpose of the course support in "Economic Mathematics" is to provide students with a well-structured, synthesized, and accessible resource for learning relevant mathematical concepts in economics. The course support aims to facilitate understanding and application of mathematics in economic problems, providing essential information and relevant examples to support students in developing the necessary mathematical skills in an economic context.

The work has a methodical character and is organized into 5 chapters, covering knowledge related to "Economic Mathematics". It is designed as an efficient support for preparing seminars, homework, and exams in the fundamental discipline of Mathematical Economics, taught and seminarized in the first semester, second year at economics faculties.

The general objectives of the discipline of Mathematical Economics are as follows:

- To provide students with a solid foundation in mathematical techniques and tools relevant to economic analysis.
- To equip students with the ability to model and analyse various economic phenomena using mathematical methods and approaches.
- To enhance students' problem-solving skills and critical thinking abilities in the context of economic decision-making and optimization.
- To enable students to understand and interpret mathematical models used in economic theory and policy analysis.
- To foster students' ability to apply mathematical concepts to real-world economic problems and policy issues.
- To develop students' quantitative skills, enabling them to conduct empirical research and analyse economic data effectively.
- To instil an appreciation for the role of mathematics in economics and its importance in addressing complex economic challenges.
- To prepare students for further advanced studies in economics or related fields that require a strong mathematical background.

Overall, the general objectives of Mathematical Economics aim to bridge the gap between economic theory and mathematical methods, equipping students with the necessary skills and knowledge to analyse and understand economic behaviour and interactions using rigorous mathematical approaches.

The chapters include general considerations about matrix calculus, analytical geometry, linear programming, functions of multiple variables, and ordinary differential equations. Special attention is given to interpreting the introduced concepts and presenting situations in which they are used, with relevant examples from economics.

Vector Algebra Fundamentals: This section focuses on the fundamental principles and operations of vector algebra, including vector addition, scalar multiplication, dot product, cross product, with application of vectors in economy.

Elements of three-dimensional Analytical Geometry: This part explores the principles of analytical geometry in three-dimensional space. Topics include the study of points, lines, planes, distances, angles, and intersections in three dimensions.

Linear Programming: This section delves into linear programming, which involves optimizing a linear objective function subject to a set of linear constraints. Students learn how to formulate and solve linear programming problems using graphical methods and various optimization techniques.

Functions of Several Variables: This segment explores functions that depend on multiple variables. Students learn about partial derivatives, gradients, directional derivatives, optimization of multivariable functions, and applications of these concepts in economics and other fields.

Differential Equations: This part covers the study of ordinary differential equations. Topics include first-order differential equations, linear differential equations, systems of differential equations, and applications of differential equations in modelling real-world phenomena.

The course support in "Economic Mathematics" is necessary for students for several reasons: it ensures a well-defined structure of study materials, synthesizing and concentrating essential information, eliminating redundancies. It also provides relevant examples from economics and allows students to learn independently, saving time and being easily accessible, including in electronic format.

In conclusion, this course support in "Economic Mathematics" offers students a solid foundation for understanding and applying mathematical concepts in the economic domain. This material is intended to be just the beginning of exploring the fascinating world of economic mathematics and hopes to spark interest in applied mathematics, combining rigor, ingenuity, and pragmatism.

Thanks to all who contributed to this work with suggestions and complements, especially to the reviewers for their observations and appreciations made during the manuscript analysis.

1 VECTOR ALGEBRA FUNDAMENTALS

CONTENT UNITS that will be covered in the study of Vector Algebra:

Introduction to Vectors: Definition of vectors, representation of vectors, and concept of magnitude and direction.

Vector Operations: Addition and subtraction of vectors, scalar multiplication, and finding the magnitude and direction of vectors.

Dot Product: Definition of dot product, properties of dot product, geometric interpretation, and applications in vector projections, angle determination, and work calculations.

Cross Product: Definition of cross product, properties of cross product, geometric interpretation, applications in calculating areas of parallelograms and determining vector perpendicularity.

Vector Spaces: Introduction to vector spaces, vector space properties, and examples of vector spaces.

Linear Independence and Dependence: Understanding linearly independent and dependent vectors, basis and spanning sets, and determining linear dependence using row reduction.

Matrices and Systems of Linear Equations: Representing systems of linear equations using matrices and vectors, solving systems using Gaussian elimination or matrix inversion.

✍ *These content units cover the fundamentals of Vector Algebra, including operations, properties, applications, and connections to other areas*.

Keywords: Vector, Magnitude, Direction, Addition, Subtraction, Scalar multiplication, Dot product, Cross product, Linear combination, Linear independence, Linear dependence, Basis, Norm, Orthogonality, Unit vector, Coordinate system, Matrices, Matrix operations.

Learning Objectives:

Understanding Vector Concepts: Define vectors, grasp the concept of magnitude and direction, and comprehend vector representation.

Vector Operations: Perform vector addition and subtraction, apply scalar multiplication to vectors, and determine the magnitude and direction of vectors.

Dot Product: Calculate the dot product of vectors, understand its geometric interpretation, and apply it to find angles, projections, and work.

Cross Product: Compute the cross product of vectors, interpret its geometric significance, and utilize it in calculating areas and testing perpendicularity.

Vector Spaces: Define and identify vector spaces, understand the properties and operations of vector spaces, and solve problems using vector space concepts.

Linear Independence and Dependence: Determine linear independence and dependence among vectors, identify bases and spanning sets, and apply row reduction techniques.

Matrix Methods: Represent systems of linear equations using matrices and vectors, solve systems using Gaussian elimination or matrix inversion, and find general and particular solutions.

Geometric Applications: Apply vector algebra in geometric settings, solve problems related to lines, planes, intersections, and reflections. (*The subject will be developed in Chapter Two of the paper*)

Vector Calculus: Apply vector calculus concepts such as gradient, divergence, curl, line integrals, and surface integrals to solve problems. (*The subject will be developed in Chapter Four of the paper*)

Eigenvalues and Eigenvectors: Understand the concept of eigenvalues and eigenvectors, compute eigenvalues and eigenvectors of matrices, and apply them in various applications. (*The subject will be developed in Chapter Five of the paper*)

✍*These learning objectives cover the fundamental concepts and operations of Vector Algebra and provide a roadmap for understanding and applying Vector Algebra in various contexts*.

Vector Algebra fundamentals related to Economic Mathematics:

Vectors as Representations: In Economic Mathematics, vectors can be used to represent various quantities. For example, a vector could represent the input or output levels of different goods or factors of production in an economic model. The magnitude and direction of the vector can capture important information about the quantity and direction of change.

Vector Addition and Subtraction: Vector addition and subtraction can be applied in Economic Mathematics to analyse changes in quantities. For instance, adding vectors representing the changes in supply and demand can help determine the resulting equilibrium quantity and price in a market.

Scalar Multiplication: Scalar multiplication of vectors can be used in Economic Mathematics to examine the impact of scaling factors on economic variables. For example, multiplying a vector representing the quantity of a good by a scalar factor can help analyses the effect of a proportional change in demand or supply.

Dot Product: The dot product of vectors can be related to economic concepts such as utility and costs. For instance, the dot product between a vector representing the quantity of goods consumed and a vector representing their respective prices can yield the total expenditure or cost incurred.

Cross Product: While less commonly used in Economic Mathematics, the cross product can find application in specific contexts. For instance, in international trade analysis, the cross product of vectors representing the quantities of goods produced by different countries can help determine comparative advantages and trade patterns.

Linear Independence and Dependence: Economic models often involve analysing relationships between different variables. Determining whether a set of vectors is linearly independent or dependent can shed light on the existence of redundant or essential variables in a model, which can affect model validity and efficiency.

Linear Transformations: Linear transformations, such as matrices, are used in Economic Mathematics to represent various economic processes, such as production functions, input-output models, and multi-equation systems. These transformations allow for the analysis of relationships and interactions between different economic variables.

✍ *These connections demonstrate how Vector Algebra fundamentals can be applied and extended to analyses economic concepts and relationships in Economic Mathematics. By leveraging*

vector algebra tools, economists can develop mathematical models and analyses economic phenomena more rigorously.

1.0 INITIATION IN VECTOR ALGEBRA

Vector Algebra is a branch of mathematics that deals with the manipulation and properties of vectors, which are mathematical objects characterized by magnitude, direction and orientation. Vectors are commonly used in various fields. Here are some fundamental concepts and operations in Vector Algebra:

Vector Representation: A vector is represented by a directed line segment or an arrow, with the length of the segment representing the magnitude of the vector and the direction of the arrow representing its direction. Vectors are typically denoted using boldface letters or with an arrow above the letter (e.g., $\boldsymbol{v}$ or $\vec{v}$).

Coordinates: Vectors can be represented in terms of their components. In a two-dimensional Cartesian coordinate system, a vector can be expressed as $\boldsymbol{v} = (v_1, v_2)$, where v_1 and v_2 are the components of the vector along the x and y axes, respectively. In three-dimensional space, a vector can be expressed as $\boldsymbol{v} = (v_1, v_2, v_3)$, where v_1, v_2, and v_3 are the components along the x, y, and z axes, respectively.

Magnitude: The magnitude (or length) of a vector $\boldsymbol{v}$, denoted as $||\boldsymbol{v}||$ or $|\boldsymbol{v}|$, represents the scalar value corresponding to the length of the vector. In two dimensions, the magnitude of $\boldsymbol{v} = (v_1, v_2)$ is given by $||\boldsymbol{v}|| = \sqrt{{v_1}^2 + {v_2}^2}$. In three dimensions, the magnitude of $\boldsymbol{v} = (v_1, v_2, v_3)$ is given by $||\boldsymbol{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$.

Addition and Subtraction: Vectors can be added or subtracted component-wise. For two vectors $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$, the sum of the vectors is $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$, and the difference of the vectors is $\boldsymbol{v} - \boldsymbol{w} = (v_1 - w_1, v_2 - w_2)$. The same principle applies to vectors in three dimensions.

Scalar Multiplication: A vector can be multiplied by a scalar, which scales the magnitude of the vector without changing its direction. If $\boldsymbol{v} = (v_1, v_2)$ is a vector and $\boldsymbol{c}$ is a scalar, then the scalar multiplication of c and $\boldsymbol{v}$ is $c\boldsymbol{v} = (cv_1, cv_2)$.

Dot Product: The dot product (or scalar product) of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, denoted as $\boldsymbol{v} \cdot \boldsymbol{w}$, is a scalar quantity given by the product of their magnitudes and the cosine of the angle between them. In two dimensional space, $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2$. In three-dimensional space, $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. The dot product is commutative, meaning $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

Cross Product: The cross product (or vector product) of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, denoted as $\boldsymbol{v} \times \boldsymbol{w}$, is a vector that is perpendicular (orthogonal) to both $\boldsymbol{v}$ and $\boldsymbol{w}$. The magnitude of the cross product is given by $||\boldsymbol{v} \times \boldsymbol{w}|| = ||\boldsymbol{v}||\,||\boldsymbol{w}||\,sin(\theta)$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. The direction of the cross product is determined by the *right-hand rule*. The cross product is not commutative, meaning $\boldsymbol{v} \times \boldsymbol{w} = -(\boldsymbol{w} \times \boldsymbol{v})$.

✍ *These are some fundamental concepts and operations in Vector Algebra. They provide the basis for solving problems involving vectors and understanding vector-related phenomena in various fields of study.*

1.1 FUNDAMENTAL CONCEPTS OF SYSTEMS OF LINEAR EQUATIONS

Keywords: *Linear equations, Coefficients, Variables, Unknowns, System of equations, Augmented matrix, Row-echelon form, Reduced row-echelon form, Gaussian elimination, Pivot, Leading entry, Solution set, Compatible system, Incompatible system, Unique solution, General solution, Particular solution, Homogeneous system, non-homogeneous system.*

In a system of linear equations, the goal is to find values for the variables that satisfy all the equations simultaneously. The variables represent unknown quantities, and the equations represent relationships or constraints among these unknowns.

Linear Equations: The equations are called "linear" because the highest power of the variables in each equation is one (i.e., the variables are not raised to any exponent).

Systems of Linear Equations: Systems of linear equations involve multiple equations with linear relationships between variables. The goal is to find values for the variables that satisfy all the equations simultaneously.

Example 1.1.a:

1) $\begin{cases} 2x + 3y - z = 10 \\ x - 2y + 4z = -3 \\ 3x + y + 2z = 5 \end{cases}$

2) $\begin{cases} 2x + 3y - z + 4w = 10 \\ x - 2y + 4z - 3w = -3 \\ 3x + y + 2z + w = 5 \\ -x + 2y - z - w = 1 \end{cases}$

3) $\begin{cases} 2x + 3y - z = 10 \\ x - 2y + 4z = -3 \\ -7y + 11z = -15 \end{cases}$

4) $\begin{cases} 2x + y - z = 5 \\ x - 3y + 2z = -4 \\ 3x + 2y - 3z = 7 \end{cases}$

5) $\begin{cases} 2x + 3y - z + w = 10 \\ x - 2y + 4z - 3w = -3 \\ 3x + y + 2z + w = 5 \end{cases}$

6) $\begin{cases} x - y + 2z - w + 3v = 4 \\ 2x + y - 3z + w - 2v = -1 \\ -x + 3y + 4z - 2w + 5v = 6 \\ x + y - z + 2w - v = 3 \\ 3x - y + z - w + 4v = 5 \end{cases}$

Forms of representation of a System of Linear Equations:

Augmented Matrix Form: The system is represented by an augmented matrix, where the coefficients of the variables are arranged in a rectangular matrix, and the constants are represented in the last column.

Example 1.1.b:

$$\begin{cases} 2x + 3y - z = 10 \\ x - 2y + 4z = -3 \\ 3x + y + 2z = 5 \end{cases}$$

To represent this system in *augmented matrix form*, we arrange the coefficients of the variables and the constants in a matrix with an additional column for the constants. The augmented matrix will look like this:

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 10 \\ 1 & -2 & 4 & -3 \\ 3 & 1 & 2 & 5 \end{array}\right)$$

Matrix-Vector Form: The system is represented using matrix notation, where the coefficients of the variables are arranged in a matrix, and the variables and free terms are represented as vectors.

Example 1.1.c:

Consider the following system of linear equations:

$$\begin{cases} 2x + 3y - z = 7 \\ x - 2y + 4z = -2 \\ 3x + y + 2z = 4 \end{cases}$$

To represent this system in matrix-vector form, we arrange the coefficients of the variables in a matrix, the variables themselves in a vector, and the constants in another vector. The matrix-vector form will look like this:

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

Equation Form: The system is represented by listing the individual equations one after another.

Example 1.1.d:

$$\begin{cases} 2x + 3y - z = 7 \\ x - 2y + 4z = -3 \\ 3x + y + 2z = 4 \end{cases}$$

Each equation represents a linear relationship between the variables x, y, and z, with the corresponding constants on the right-hand side.

Types of solutions for a System of Linear Equations:

Unique Solution: A System of Linear Equations has a unique solution if there is exactly one set of values for the variables that satisfies all the equations simultaneously. Geometrically, this corresponds to the intersection point of lines, planes, or hyperplanes representing the equations.

No Solution: A System of Linear Equations has no solution (or is ***incompatible***) if there are no values for the variables that satisfy all the equations simultaneously. Geometrically, this corresponds to parallel lines, planes, or hyperplanes that do not intersect.

Infinity Many Solutions: A System of Linear Equations has infinity many solutions (or is ***undetermined compatible***) if there are infinitely many sets of values for the variables that satisfy all the equations simultaneously. Geometrically, this corresponds to overlapping or coinciding lines, planes, or hyperplanes. The infinite solutions indicate that system of equations allowing for a range of possible solutions parametrized by one or more free variables.

Leading entry: The "leading entry" refers to the leftmost non-zero entry in a row of a matrix in row-echelon form or reduced row-echelon form. It is the first non-zero element encountered when scanning the row from left to right.

Example 1.1.e: Let's consider the system of linear equations from the *Example 1.1.f* The leading entry for the first row is equal to 2, for the second row is equal to -7 and for the third row is equal to -2.

The row-echelon form for the augmented matrix of a system of linear equations:

Gaussian elimination method with elementary row operations (***Method 1***): To transform the augmented matrix of a system of linear equations into row-echelon form, you can apply elementary row operations. These operations include:

- ✍ Interchanging two rows: *Swap the positions of two rows in the matrix*.
- ✍ Scaling a row: *Multiply all entries in a row by a non-zero scalar*.
- ✍ Adding one row to another row: *Multiply one row by a scalar and add it to another row, replacing the entries in the second row*.

The resulting matrix (row-echelon form) will have the following properties:

Properties 1 Leading Entry: Each non-zero row has a leading entry, which is the leftmost non-zero entry in that row. All entries below the leading entry are zero.

Properties 2 Leading Entry Position: The leading entry of a row appears to the right of the leading entry of the row above it.

Properties 3 Zero Rows: Any rows consisting entirely of zeros are located at the bottom of the matrix.

Properties 4 Columns with Leading Entries: Each column containing a leading entry has zeros in all positions below the leading entry.

Properties 5 Order of Rows: The rows are ordered so that all zero rows, if any, are at the bottom of the matrix.

Let's illustrate the process by applying elementary row operations to an augmented matrix to transform it into row-echelon form.

Example 1.1.f: Consider the following system of linear equations:

$$\begin{cases} 2x + 3y - z = 7 \\ x - 2y + 4z = -3 \\ 3x + y + 2z = 4 \end{cases}$$

The augmented matrix will look like this

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 1 & -2 & 4 & -3 \\ 3 & 1 & 2 & 4 \end{array}\right)$$

By applying elementary transformations, in a few steps, we obtain the row-echelon matrix of the system of linear equations.

$$\underbrace{\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 1 & -2 & 4 & -3 \\ 3 & 1 & 2 & 4 \end{array}\right)}_{augmented\ metrix} \overset{\substack{L_2:\, 2*L_2-L_1 \\ L_3:\, 2*L_3-3*L_2}}{\sim} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & -7 & 9 & -13 \\ 0 & -7 & 7 & -13 \end{array}\right) \overset{L_3:\, L_3-L_2}{\sim} \underbrace{\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & -7 & 9 & -13 \\ 0 & 0 & -2 & 0 \end{array}\right)}_{row-echelon\ matrix}$$

Gaussian elimination method using pivoting (Method 2): To obtain the row-echelon form of a matrix, we can use the Gaussian elimination method with pivoting. Here are the steps:

Step 1: Write down the augmented matrix of the system of linear equations, where the coefficients of the variables and the constants are arranged in a matrix form.

Step 2: Choose the leftmost non-zero column as the pivot column.

Step 3: Locate the row with the largest absolute value in the pivot column. If necessary, swap rows to move this row to the top of the column. This step is known as partial pivoting.

Step 4: Divide the pivot row by the value in the pivot column to make the pivot element equal to 1 (if it is not already).

Step 5: Use row operations to eliminate all non-zero entries below the pivot in the pivot column. Subtract a suitable multiple of the pivot row from each row below it to make all the entries below the pivot equal to zero.

Step 6: Move to the next column and repeat steps 2-5 until you reach the last column or the bottom row.

Step 7: Repeat steps 2-6 for each subsequent row, ensuring that the pivot column is to the right of the pivot columns used in previous rows.

Step 8: Continue this process until you have reached the last row or the last column, resulting in a matrix in row-echelon form.

The resulting matrix (row-echelon form) will have the following properties:

Properties 1 The first non-zero entry in each row, known as a pivot, is always to the right of the pivot in the row above it.

Properties 2 The pivots form a staircase pattern from the top-left to the bottom-right of the matrix.

Properties 3 It's important to note that the row-echelon form is not unique, and different sequences of pivot rules can result in different row-echelon matrices for the same system of linear equations.

Example 1.1.g: Consider the following system of linear equations:

$$\begin{cases} 2x + 3y - z = 7 \\ x - y + 2z = 3 \\ 3x + y - z = 5 \end{cases}$$

The augmented matrix will look like this

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 3 & 1 & -1 & 5 \end{array}\right)$$

Now, let's apply the Gaussian elimination method with pivot rules:

Step 1: The leftmost non-zero column is the first column. The third row has the *largest absolute value* in this column, so we swap columns 1 and 3.

$$\left(\begin{array}{ccc|c} \boxed{3} & 1 & -1 & 5 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & -1 & 7 \end{array}\right)$$

Step 2: Divide the first row by the *pivot element*, which is 3, to make it equal to 1.

$$\left(\begin{array}{ccc|c} \boxed{1} & 1/3 & -1/3 & 5/3 \\ 1 & -1 & 2 & 3 \\ 2 & 3 & -1 & 7 \end{array}\right)$$

Step 3: We replace with zero the elements in the pivot column that are located below the pivot element. Then, applying the pivot rule, which will be described below, we recalculate the remaining elements of the matrix (in red colour):

$$\left(\begin{array}{ccc|c} \boxed{1} & 1/3 & -1/3 & 5/3 \\ 0 & -4/3 & 7/3 & 4/3 \\ 0 & 7/3 & -1/3 & 11/3 \end{array}\right)$$

The calculus, applying the partial pivoting, to recalculate the remaining elements:

-1:	$\boxed{1}\ \ \frac{1}{3}$ $1\ \ -1$	$\boxed{1} * (-1) - \left(1 * \frac{1}{3}\right) = -1 - \frac{1}{3} = -\frac{4}{3}$
2:	$\boxed{1}\ \ -\frac{1}{3}$ $1\ \ 2$	$\boxed{1} * 2 - \left(1 * \left(-\frac{1}{3}\right)\right) = 2 + \frac{1}{3} = \frac{7}{3}$
3:	$\boxed{1}\ \ \frac{5}{3}$ $1\ \ 3$	$\boxed{1} * 3 - \left(1 * \frac{5}{3}\right) = 3 - \frac{5}{3} = \frac{4}{3}$
3:	$\boxed{1}\ \ \frac{1}{3}$ $2\ \ 3$	$\boxed{1} * 3 - \left(2 * \frac{1}{3}\right) = 3 - \frac{2}{3} = \frac{7}{3}$
-1:	$\boxed{1}\ \ -\frac{1}{3}$ $2\ \ -1$	$\boxed{1} * (-1) - \left(2 * \left(-\frac{1}{3}\right)\right) = -1 + \frac{2}{3} = -\frac{1}{3}$
7:	$\boxed{1}\ \ \frac{5}{3}$ $2\ \ 7$	$\boxed{1} * 7 - \left(2 * \frac{5}{3}\right) = 7 - \frac{10}{3} = \frac{11}{3}$

Step 4: Move to the next column (second column) and repeat steps 2 & 3. The third row has the *largest absolute value* in this column, so we swap it with the second row:

$$\left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 5/3 \\ 0 & \boxed{7/3} & -1/3 & 11/3 \\ 0 & -4/3 & 7/3 & 4/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 5/3 \\ 0 & \boxed{1} & -1/7 & 11/7 \\ 0 & -4/3 & 7/3 & 4/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 5/3 \\ 0 & 1 & -1/7 & 11/7 \\ 0 & 0 & \boxed{15/7} & 24/7 \end{array}\right)$$

Step 5: Move to the next column (third column) and repeat steps 2 & 3. The third row has the largest absolute value in this column, so we keep it as it is. Now, we have reached the bottom row and obtained the row-echelon form of the matrix. The resulting matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 5/3 \\ 0 & 1 & -1/7 & 11/7 \\ 0 & 0 & 1 & 8/5 \end{array}\right)$$

✍ *The row-echelon form simplifies the system of linear equations by introducing a triangular structure to the matrix. It allows for straightforward back substitution and makes it easier to determine the solution to the system. The row-echelon form for the augmented matrix of a system of linear equations is a specific arrangement of the coefficients and constants in the matrix that simplifies the solving process.*

Pivot variables (basic variables): The variables that corresponding to leading entries in the row-echelon matrix or reduced row-echelon matrix.

Example 1.1.g: Let's consider the system of linear equations from the *Example 1.1.e*. The pivot variables are: x_1 (for the first row, and corresponds to the leading entry, which is equal to 2), x_2 (for the second row, and corresponds to the leading entry, which is equal to -7), x_3 (for the third row, and corresponds to the leading entry, which is equal to -2).

Free variables: The variables that corresponding to columns without leading entries in the row-echelon matrix.

Example 1.1.h: Let's consider the system of linear equations from the *Example 1.1.e*. The free variables are: $x_2 \; or \; x_3$ (for the first row), $x_1 \; or \; x_3$ (for the second row), $x_1 \; or \; x_2$ (for the third row).

✍ *Pivot variables and free variables are essential in determining the solutions of systems of linear equations. Pivot variables are used to find solutions dependent on their values, while free variables allow for the generation of parameterized solutions that can vary based on the chosen values for them.*

The inverse of a matrix: The inverse of a matrix is a fundamental concept in linear algebra. Given a square matrix A, the inverse is another square matrix that, denoted as A^{-1}, when multiplied by the original matrix, results in the identity matrix.

Formally, for a square matrix A of size $n \; x \; n$, if there exists another matrix B of the same size such that the product $A * B$ equals the identity matrix I $(of \; size \; n \; x \; n)$, and $B * A$ also exist and is equals I, then B is called the inverse of A. Mathematically, it can be written as:

$$A * B = B * A = I$$

The inverse of a matrix exists only if the matrix is non-singular. A non-singular matrix has a determinant that is non-zero. If the determinant of a matrix is zero, then it is called a singular matrix, and its inverse does not exist.

To find the inverse of a matrix, several methods can be used, including Gaussian elimination, Gauss-Jordan elimination, or using matrix algebraic properties. The method chosen depends on the specific characteristics of the matrix and the desired computational efficiency.

It's important to note that not all square matrices have an inverse.

The method of calculating the inverse of a matrix using pivoting (Method 1):

The method of calculating the inverse of a matrix using pivoting is as follows:

Step 1: Start with the matrix that you want to find the inverse of, let's call it A.

Step 2: Augment the matrix A with an identity matrix of the same size, resulting in the augmented matrix $[A \mid I]$.

Step 3: Apply Gaussian elimination to the augmented matrix, using pivoting to ensure numerical stability. The goal is to transform the left side (matrix A) into an identity matrix.

Step 4: Perform the same row operations on the right side (identity matrix) as you do on the left side. This ensures that the transformation on the left side is also applied to the right side.

Step 5: Once the left side becomes an identity matrix, the right side will be the inverse of the original matrix A. Extract the right side and that will be the inverse matrix A^{-1}.

Step 6: Note that if during the process, a row exchange is performed due to pivoting, you need to keep track of the row swaps and perform the same swaps on the identity matrix.

✍ *This method is used to handle cases where the original matrix may have rows or columns that are linearly dependent, resulting in a matrix that is not invertible. Pivoting helps to overcome these issues and find a valid inverse if it exists.*

Example 1.1.i Consider the following $3x3$ matrix A:

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 3 & 5 \\ 7 & 8 & 9 \end{pmatrix}$$

To calculate the inverse of matrix A using pivoting, we'll follow these steps:

Step 1: Augment the matrix with the identity matrix:

$$\bar{A} = \left(\begin{array}{ccc|ccc} 2 & 4 & -1 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right)$$

Step 2: Apply Gaussian elimination with pivoting to transform the left side into an identity matrix. Perform the same row operations on the right side as well.

$$\bar{A} = \left(\begin{array}{ccc|ccc} 2 & 4 & -1 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ \boxed{7} & 8 & 9 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} \boxed{7} & 8 & 9 & 0 & 0 & 1 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 2 & 4 & -1 & 1 & 0 & 0 \end{array}\right) \overset{L_1 \div 7}{\leftrightarrows} \left(\begin{array}{ccc|ccc} \boxed{1} & 8/7 & 9/7 & 0 & 0 & 1/7 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 2 & 4 & -1 & 1 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} \boxed{1} & 8/7 & 9/7 & 0 & 0 & 1/7 \\ 0 & 13/7 & 26/7 & 0 & 1 & -1/7 \\ 0 & 12/7 & -25/7 & 1 & 0 & -2/7 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 8/7 & 9/7 & 0 & 0 & 1/7 \\ 0 & \boxed{13/7} & 26/7 & 0 & 1 & -1/7 \\ 0 & 12/7 & -25/7 & 1 & 0 & -2/7 \end{array}\right) \overset{L_2 \div \frac{13}{7}}{\leftrightarrows}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -13/7 & 0 & -8/7 & 3/7 \\ 0 & \boxed{1} & 2 & 0 & 7/13 & -1/13 \\ 0 & 0 & -13 & 13/7 & -12/7 & -2/7 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -13/7 & 0 & -8/7 & 3/7 \\ 0 & 1 & 2 & 0 & 7/13 & -1/13 \\ 0 & 0 & \boxed{-13} & 13/7 & -12/7 & -2/7 \end{array}\right) \overset{L_3 \div (-!3)}{\leftrightarrows}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 169/49 & 572/49 & -104/49 \\ 0 & 1 & 0 & 0 & 7/13 & -1/13 \\ 0 & 0 & \boxed{1} & -26/7 & 25/7 & -11/7 \end{array}\right)$$

Step 3: Extract the right side of the augmented matrix. It is the inverse of the original matrix.

$$A^{-1} = \begin{pmatrix} 169/49 & 572/49 & -104/49 \\ 0 & 7/13 & -1/13 \\ -26/7 & 25/7 & -11/7 \end{pmatrix}$$

The method of calculating the inverse of a matrix using algebraic complements (Method 2):

Here's an explanation of the method of calculating the inverse of a matrix using algebraic complements:

Step 1: Start with the matrix you want to find the inverse of, let's call it A.

Step 2: Calculate the determinant of matrix A. If the determinant is zero, the matrix does not have an inverse.

Step 3: Create the matrix of algebraic complements of A, noted with A^*. To do this, replace each element of A with its corresponding cofactor. The cofactor of an element is the signed determinant of the matrix obtained by removing the row and column containing that element.

Step 4: Transpose the matrix of algebraic complements to obtain the adjugated matrix.

Step 5: Divide the adjugated matrix by the determinant of A. Each element of the adjugated matrix is divided by the determinant.

Step 6: The resulting matrix is the inverse of matrix A.

Let's illustrate this method with an example:

Example 1.1.j Consider the following $3x3$ matrix A:

$$A = \begin{pmatrix} 2 & 4 & -1 \\ 1 & 3 & 5 \\ 7 & 8 & 9 \end{pmatrix}$$

Step 1: Calculate the determinant of matrix A.

$$|A| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 3 & 5 \\ 7 & 8 & 9 \end{vmatrix} = 54 + 140 - 24 + 21 - 80 - 36 = 75$$

$$|A| \neq 0 \Rightarrow \exists A^{-1}$$

Step 2: Create the adjugated matrix (A^*) as transpose matrix of corresponding cofactor of A.

$A^* = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$, wher A_{ij} is the algebraic complement corresponding to position (i, j) in matrix.

The calculations for determining all cofactors are:

$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 5 \\ 8 & 9 \end{vmatrix} = -13$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 7 & 9 \end{vmatrix} = 26$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = -13$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 4 & -1 \\ 8 & 9 \end{vmatrix} = 45$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 7 & 9 \end{vmatrix} = -25$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} = 12$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 23$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = -11$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2$

$$A^* = \begin{pmatrix} -13 & 45 & 23 \\ 26 & -25 & -11 \\ -13 & 12 & 2 \end{pmatrix}$$

Step 3: Divide the adjugated matrix by the determinant of A. Each element of the adjugated matrix is divided by the determinant. Thus, the inverse matrix will be obtained.

$$A^{-1} = \frac{1}{|A|} * A^* \Rightarrow A^{-1} = \frac{1}{75} * \begin{pmatrix} -13 & 45 & 23 \\ 26 & -25 & -11 \\ -13 & 12 & 2 \end{pmatrix}$$

Rank of the matrix: The rank of a matrix refers to the maximum number of linearly independent rows or columns in the matrix. It is a fundamental concept in linear algebra and is used to determine various properties and solutions of linear systems. The rank of a matrix can be thought of as the dimension of the vector space spanned by its rows or columns.

There are several techniques to determine the rank of a matrix. Here are three common methods:

Row Echelon Form or Reduced Row Echelon Form (Variant 1): Convert the given matrix to either row echelon form (REF) or reduced row echelon form (RREF) using elementary row operations. The rank of the matrix is equal to the number of non-zero rows in the REF or RREF form.

Example 1.1.h Consider the following matrix

$$A = \underbrace{\begin{pmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 3 & 1 & 2 \end{pmatrix}}_{givin\ metrix} \overset{\substack{L_2: 2*L_2 - L_1 \\ L_3: 2*L_3 - 3*L_2}}{\sim} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -7 & 9 \\ 0 & -7 & 7 \end{pmatrix} \overset{L_3: L_3 - L_2}{\sim} \underbrace{\begin{pmatrix} 2 & 3 & -1 \\ 0 & -7 & 9 \\ 0 & 0 & -2 \end{pmatrix}}_{row-echelon\ form}$$

The rank of matrix A is equal to the number of non-zero rows in REF form of matrix, which in this case is 3. Therefore, the rank of matrix A is 3.

Row Echelon Form or Reduced Row Echelon Form (Variant 2): Convert the given matrix to either row echelon form (REF) or reduced row echelon form (RREF) using pivoting rule. The rank of the matrix is equal to the number of non-zero rows in the REF or RREF form.

Example 1.1.k Suppose we have the following matrix A:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 1 & 3 & 5 \end{pmatrix}$$

So, the calculus are:

$$A = \begin{pmatrix} \boxed{2} & 1 & 3 \\ 4 & 2 & 6 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 1/2 & 3/2 \\ 4 & 2 & 6 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 3/2 \\ 0 & 0 & 0 \\ 0 & 5/2 & 7/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 3/2 \\ 0 & 5/2 & 7/2 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of matrix A is equal to the number of non-zero rows in REF form of matrix, which in this case is 2. Therefore, the rank of matrix A is 2.

Determinants of Minors: Calculate the determinants of all possible square submatrices of increasing order within the given matrix. The rank of the matrix is equal to the largest order of a submatrix whose determinant is non-zero. This method is known as the "Minors method" or "Pivot method."

Example 1.1.l Consider the following matrix A:

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 6 & 8 \end{pmatrix}$$

Calculate the determinants of all possible square submatrices of increasing order.

Submatrices of order 1:

$$A_1 = (2) \Rightarrow |A_1| = 2 \neq 0$$

Submatrices of order 2:

$$A_2 = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \Rightarrow |A_2| = 6 - 4 = 2 \neq 0$$

Submatrices of order 3:

$$A_3 = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 2 & 6 & 8 \end{pmatrix} \Rightarrow |A_3| = 48 + 40 + 36 - 36 - 60 - 32 = -4 \neq 0$$

Determine the highest order of a submatrix with a non-zero determinant. In this case, the highest order of a submatrix with a non-zero determinant is 3. The rank of the matrix is equal to the highest order found. Therefore, the rank of matrix A is 3.

✍ *These techniques are commonly used to determine the rank of a matrix and can be applied depending on the specific requirements and available tools for calculations.*

The relationship between the rank of a matrix and its invertibility is as follows:

For a square matrix (an n x n matrix), the matrix is invertible if and only if its rank is equal to its size, which is n. In other words, a square matrix is invertible if and only if it has *full rank*.

If the rank of a square matrix is less than its size (rank < n), it is called a *singular matrix* or a *non-invertible matrix*. A singular matrix does not have an inverse.

On the other hand, if the rank of a square matrix is equal to its size (rank = n), the matrix is non-singular or invertible, and it has an inverse. An invertible matrix has the property that it can be multiplied by its inverse to produce the identity matrix.

✍ *In summary, the rank of a matrix and its invertibility are directly related. A square matrix is invertible if and only if its rank is equal to its size, and it is non-invertible (singular) if its rank is less than its size.*

Homogeneous Linear Equations: Homogeneous linear equations are systems where all the constant terms are zero. In other words, the right-hand side of each equation is zero. These systems always have at least one solution, known as the trivial solution, where all variables are zero.

Questions you should be able to answer:

What is a system of linear equations?

How can you represent a system of linear equations in matrix form?

What is the difference between a unique solution, no solution, and infinitely many solutions in a system of linear equations?

What are pivot variables and free variables in a system of linear equations?

How do elementary row operations help in solving a system of linear equations?

What is the row-echelon form of a matrix?

How can you determine if a system of linear equations is consistent or inconsistent?

What is the difference between a homogeneous and non-homogeneous system of linear equations?

What does it mean for a matrix to be invertible?

How do you determine if a matrix has an inverse?

What are the conditions for the existence of the inverse of a square matrix?

How do you calculate the inverse of a matrix?

Explain the concept of the rank of a matrix and its significance.

What is the relationship between the rank of a matrix and its invertibility?

Can a matrix with rank less than its size have an inverse? Explain.

How can you determine the rank of a matrix using the Pivot Method?

Provide an example of a matrix and demonstrate how to calculate its inverse and determine its rank.

Proposed Exercises:

1.1.1 These exercises further explore various scenarios related to systems of linear equations and their interpretation in the context of economic mathematics. Fill in the missing words to enhance your understanding of these concepts and their applications in economic models. Show all the steps involved in solving the exercises to deepen your grasp of the underlying principles:

a) Solving Systems of Linear Equations: Consider the following system of linear equations: $\begin{cases} 3x + 2y - z = 10 \\ 2x - y + 3z = 5 \\ x + 3y - 2z = 4 \end{cases}$. Fill in the missing words to complete the sentences: *a.1)* The system of equations has _____ solutions: *a unique solution, no solution, or infinitely many solutions, depending on the values of x, y, and z. a.2)* Geometrically, the three equations represent three _____ in a three-dimensional space. *a.3)* In the context of economic mathematics, x, y, and z can represent the quantities of _____ goods or resources produced or consumed in a multi-product production or consumption model.

b) Economic Interpretation of Solutions: Consider the following system of linear equations: $\begin{cases} 2x + 3y + z = 12 \\ x - 2y + 4z = 8 \\ 3x + y + 2z = 15 \end{cases}$. Fill in the missing words to complete the sentences: *b.1)* The system of equations can be used to model the _____ of goods in a three-product production system. *b.2)* Solving the system provides the values of x, y, and z, which represent the _____ of each product needed to meet the production requirements. *b.3)* If x represents the quantity of product A, y represents the quantity of product B, and z represents the quantity of product C, then the solution (x, y, z) gives the optimal _____ that maximizes production while satisfying resource constraints.

c) Economic Interpretation of No Solution: Consider the following system of linear equations: $\begin{cases} 2x - y + z = 5 \\ x + y - z = 2 \\ 3x + 2y + z = 8 \end{cases}$. Fill in the missing words to complete the sentences: *c.1)* The system of equations may have no solution when the three planes determined by the equations are _____. *c.2)* In the context of economic mathematics, this situation could indicate that the _____ constraints are infeasible or that the economic model is not realistic. *c.3)* Geometrically, no solution implies that the three planes are _____ and do not intersect at a common point.

d) Economic Interpretation of Infinite Solutions: Consider the following system of linear equations: $\begin{cases} 3x + 2y - z = 6 \\ 2x + y + 3z = 9 \\ x + y + 4z = 7 \end{cases}$. Fill in the missing words to complete the sentences: *d.1)* The system of equations may have infinitely many solutions when the three planes determined by the equations are _____. *d.2)* In the context of economic mathematics, this situation could indicate multiple combinations of _____ of goods that satisfy the production requirements. *d.3)* Geometrically, infinitely many solutions imply that the three planes are _____ and coincide with each other.

e) Underdetermined System: Consider the following system of linear equations: $\begin{cases} 2x + y + z = 8 \\ x - 2y + 3z = 5 \end{cases}$. Fill in the missing words to complete the sentences: *e.1)* The system of equations is an underdetermined system because it has _____ equations and three unknowns. *e.2)* In the context of economic mathematics, this situation could indicate an _____ problem where there are fewer constraints than variables. *e.3)* Geometrically, the underdetermined system represents two planes in a three-dimensional space that may _____ at a line.

f) Overdetermined System: Consider the following system of linear equations: $\begin{cases} x + y + z = 6 \\ 2x - y + 3z = 8. \\ 3x + y - z = 1 \end{cases}$ Fill in the missing words to complete the sentences: *f.1)* The system of equations is an overdetermined system because it has _____ equations and three unknowns. *f.2)* In the context of economic mathematics, this situation could indicate an _____ problem where there are more constraints than variables. *f.3)* Geometrically, the overdetermined system represents three planes in a three-dimensional space that may have no _____ or intersect at a unique point.

g) Economic Interpretation of Solution: Set Consider the following system of linear equations: $\begin{cases} 2x + y - z = 5 \\ x - 2y + z = 3 \end{cases}$. Fill in the missing words to complete the sentences: *g.1)* The system of equations may have a _____ solution set when the two planes determined by the equations are parallel. *g.2)* In the context of economic mathematics, this situation could indicate _____ production or consumption possibilities with limited resources. *g.3)* Geometrically, the solution set represents a _____ of points that lie on the parallel planes.

1.1.2 These exercises involve determining the rank of a matrix and finding the inverse of a matrix if it exists. Show all the steps involved in solving the exercises to understand the concepts thoroughly. These concepts are essential in linear algebra and have applications in various fields, including economic.

a) Gauss Elimination with Elementary Transformations: Consider the following system of linear equations: $\begin{cases} 2x + 3y - z = 5 \\ 2x + 3y - z = 5 \\ 6x + 7y + 3z = 11 \end{cases}$. a) Use the Gauss Elimination with elementary transformations to determine the rank of the coefficient matrix.

b) Gauss Elimination with Pivoting: Consider the following system of linear equations: $\begin{cases} 3x + 2y - z = 1 \\ 6x + 4y + 2z = 2. \\ 9x + 6y + 3z = 3 \end{cases}$ Use the Gauss Elimination with pivoting to determine the rank of the coefficient matrix.

c) Rank of a Matrix: Consider the matrix A given by: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$: *c.1)* Use elementary transformations to transform matrix *A* into its row-echelon form. *c.2)* Determine the rank of matrix *A*.

d) Inverse of a Matrix: Consider the matrix B given by $B = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$. Use the Gauss Elimination method with elementary transformations to find the inverse of matrix B, if it exists.

e) *Inverse of a Singular Matrix:* Consider the matrix C given by $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$: *e.1)* Use the Gauss Elimination method with pivoting to determine if matrix C is singular or non-singular. *e.2)* If the matrix is non-singular, find the inverse of matrix C.

f) Gaussian Elimination with Elementary Transformations: Consider the following system of linear equations $\begin{cases} 2x + y - z = 5 \\ x - 3y + z = -2 \\ 3x + 2y - 2z = 8 \end{cases}$: *f.1)* Use Gaussian elimination with elementary

transformations to convert the system into row-echelon form. *f.2)* Solve the system using back-substitution. *F.3)* Interpret the solution in the context of economic mathematics.

g) Gauss-Jordan Elimination to Find the Inverse of a Matrix: Consider the following matrix equation: $A * X = I$, where $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 0 & 2 & -4 \end{pmatrix}$ is a square matrix and *I* is the identity matrix. Use Gauss-Jordan elimination with pivoting to find the inverse matrix *X*.

h) Economic Application of Matrix: Operations Consider an economic model with the following system of linear equations representing supply and demand: $\begin{cases} 2x + y = 100 \\ x + 2y = 120 \end{cases}$: *h.1)* Use Gaussian elimination to solve the system and find the equilibrium point. *h.2)* Interpret the solution in the context of economic supply and demand.

i) Matrix Inverse and Economic Model: Consider an economic model represented by the following matrix equation: $\begin{cases} 2x + y = 1 \\ x + 3y = 5 \end{cases}$: *i.1)* Find the inverse of the coefficient matrix using Gauss-Jordan elimination. b) Solve for the variables x and y using the inverse matrix. c) Interpret the solution in the context of the economic model.

1.1.3 These exercises continue to build your understanding for matrix inversion. The applications of matrix inversion in economic models help solve various problems, such as production optimization and resource allocation. Show all the steps involved in the calculations to strengthen your grasp of these techniques:

a) Gauss Elimination with Pivoting: Consider the following matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$. Use Gauss elimination with elementary transformations to find the inverse of matrix *A*.

b) Gauss Elimination with Pivoting: Consider the following matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$. Use Gauss elimination with pivoting to find the inverse of matrix B.

c) Economic Application of Inverse Matrix: Consider an economic model with the following system of linear equations $\begin{cases} 5x + 3y = 10 \\ 2x - 4y = 5 \end{cases}$: *c.1)* Represent the system of equations in matrix form $AX = B$, where *A* is the coefficient matrix, *X* is the vector of variables (x and y), and B is the constant vector. *c.2)* Use Gauss elimination with elementary transformations to find the inverse of matrix A. *c.3)* Use the inverse of matrix A to solve for the vector X (x and y).

d) Economic Application of Inverse Matrix and Pivoting: Consider an economic model with the following system of linear equations: $\begin{cases} 3x + y + 2z = 7 \\ x - 2y + z = 1 \\ 2x + y + 3z = 10 \end{cases}$: *d.1)* Represent the system of equations in matrix form $AX = B$, where *A* is the coefficient matrix, *X* is the vector of variables (x, y, and z), and B is the constant vector. *d.2)* Use Gauss elimination with pivoting to find the inverse of matrix A. *d.3)* Use the inverse of matrix A to solve for the vector X (x, y, and z).

e) Economic Application of Inverse Matrix: Consider an economic model with the following system of linear equations: $\begin{cases} 2x + 3y - z = 10 \\ 3x - y + 2z = 5 \\ x + y + z = 8 \end{cases}$: *e.1)* Represent the system of equations in matrix form $AX = B$, where A is the coefficient matrix, X is the vector of variables (x, y, and z), and B is the constant vector. *e.2)* Use Gauss elimination with elementary transformations to find the inverse of matrix A. *e.3)* Use the inverse of matrix A to solve for the vector X (x, y, and z).

f) Economic Application of Inverse Matrix and Pivoting: Consider an economic model with the following system of linear equations: $\begin{cases} x + y + 2z = 7 \\ 2x - y + z = 3 \\ 3x + 4y - 3z = 10 \end{cases}$: *f.1)* Represent the system of equations in matrix form $AX = B$, where A is the coefficient matrix, X is the vector of variables (x, y, and z), and B is the constant vector. *f.2)* Use Gauss elimination with pivoting to find the inverse of matrix A. *f.3)* Use the inverse of matrix A to solve for the vector X (x, y, and z).

1.2 THE MATRIX METHODS FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

Keywords: *Gaussian Elimination, Row Reduction, Row Operations, Augmented Matrix, Row-Echelon Form, Reduced Row-Echelon Form, Matrix Inversion, Cramer's Rule, Determinants, Backward Substitution, System of Equations, Coefficient Matrix, Constant Vector, Unique Solution, General Solution, Particular Solution.*

The concept of systems of linear equations is fundamental in mathematics and has wide-ranging applications in fields such as physics, engineering, economics, and computer science.

Solving a system of linear equations involves finding the values of the variables that satisfy all the equations.

Methods available for solving systems of linear equations:

Gaussian Elimination / Row Reduction: This method involves performing row operations on an augmented matrix to transform it into row-echelon form or reduced row-echelon form. Row operations include scaling rows, swapping rows, and adding multiples of one row to another. Once the augmented matrix is in row-echelon or reduced row-echelon form, back-substitution is used to find the values of the unknown variables.

Here are the steps to using Gaussian elimination to solve systems of linear equations:

Step 1: Write the system of linear equations in augmented matrix form. Identify the coefficients of the variables and the constant terms. Arrange the coefficients into a matrix called the coefficient matrix. Arrange the constant terms into a column matrix called the constant matrix. Combine the coefficient matrix and the constant matrix into a single augmented matrix by placing them side by side.

Step 2: Perform row operations to transform the augmented matrix into row-echelon form. Use elementary row operations to simplify the augmented matrix. Elementary row operations

include scaling rows, swapping rows, and adding multiples of one row to another. The goal is to create zeros below the main diagonal in the augmented matrix.

Step 3: Continue row operations to transform the augmented matrix into reduced row-echelon form. If necessary, perform additional row operations to create zeros above the main diagonal, resulting in a reduced row-echelon form. The reduced row-echelon form has ones along the main diagonal and zeros both above and below the main diagonal.

Step 4: Read off the solution from the reduced row-echelon form. The variables corresponding to the pivot columns (leading ones) are the basic variables. Express the basic variables in terms of the non-basic variables, if any. The resulting values of the variables form the solution to the system of linear equations.

Step 5: Interpret the solution.

- If a unique solution is obtained, it means that the system of equations has a unique solution.
- If there are no solutions or infinitely many solutions, it means that the system is incompatible or indefinite compatible, respectively.

✍ *By following these steps, Gaussian elimination simplifies the system of linear equations by transforming the augmented matrix into row-echelon and reduced row-echelon forms. It allows you to read off the solution and determine the nature of the system of equations.*

Example 1.2.a Let's solve a three-dimensional system of linear equations using Gaussian elimination. Consider the following system of linear equations:

$$\begin{cases} 2x + y - z = 4 \\ x - 3y + 2z = -1 \\ 3x + 2y - 4z = 5 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$AX = B$, where $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 3 & 2 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$

Step 2: Perform row operations to transform the augmented matrix into reduced row-echelon form. Use row operations to create zeros below and above the pivot (leading ones) in the all columns:

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{2} & 1 & -1 & 4 \\ 1 & -3 & 2 & -1 \\ 3 & 2 & -4 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 2 \\ 0 & -7/2 & 5/2 & -3 \\ 0 & 1/2 & -5/2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 2 \\ 0 & \boxed{-7/2} & 5/2 & -3 \\ 0 & 1/2 & -5/2 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/7 & 11/7 \\ 0 & \boxed{1} & -5/7 & 6/7 \\ 0 & 0 & -15/7 & -10/7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1/7 & 11/7 \\ 0 & 1 & -5/7 & 6/7 \\ 0 & 0 & \boxed{-15/7} & -10/7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & \boxed{1} & 2/3 \end{array}\right)$$

Therefore, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} 5/3 \\ 4/3 \\ 2/3 \end{pmatrix}$$

Matrix Inversion: This method involves representing the system of equations as a matrix equation: $AX = B$, where A is the coefficient matrix, X is the column matrix of unknown variables,

and B is the constant matrix. If the coefficient matrix A is invertible, the solution can be obtained by multiplying both sides of the equation by the inverse of A: $X = A^{-1}B$. Matrix inversion can be computationally expensive for large matrices, so it may not be the most efficient method in all cases.

Here are the steps to using matrix inversion to solve systems of linear equations:

Step 1: Write the system of linear equations in matrix form. Identify the coefficients of the variables and the constant terms. Arrange the coefficients into a matrix called the coefficient matrix. Arrange the constant terms into a column matrix called the constant matrix.

Step 2: Check the invertibility of the coefficient matrix. Determine if the coefficient matrix is invertible by calculating its determinant. If the determinant is non-zero, the matrix is invertible and you can proceed to the next step. If the determinant is zero, the matrix is non-invertible, and the system of equations may not have a unique solution.

Step 3: Calculate the inverse of the coefficient matrix. If the coefficient matrix is invertible, calculate its inverse using matrix inversion techniques. The inverse of a matrix A is denoted as A^{-1}.

Step 4: Multiply both sides of the matrix equation by the inverse of the coefficient matrix. Multiply the coefficient matrix inverse on the left side of the equation and the constant matrix on the right side. This yields a solution matrix that represents the values of the unknown variables.

$$AX = B$$

$$A^{-1} * A * X = A^{-1} * B$$

$$I * X = A^{-1} * B$$

$$X = A^{-1} * B$$

Step 5: Interpret the solution.

- The resulting solution matrix provides the values of the unknown variables, solving the system of linear equations.
- If a unique solution is obtained, it means that the system of equations has a unique solution.
- If there are no solutions or infinitely many solutions, it means that the system is incompatible or indefinite compatible, respectively.

✍ *By following these steps, matrix inversion allows you to solve systems of linear equations by finding the inverse of the coefficient matrix and using it to obtain the values of the unknown variables.*

Example 1.2.b Let's solve a three-dimensional system of linear equations using matrix inversion. Consider the following system of linear equations:

$$\begin{cases} 2x + y - z = 4 \\ x - 3y + 2z = -1 \\ 3x + 2y - 4z = 5 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$AX = B$, where $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 3 & 2 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$

Step 2: Calculate the inverse of the coefficient matrix (A^{-1}). To find A^{-1}, augment the matrix with the identity matrix and apply Gaussian elimination with pivoting to transform the left side into an identity matrix.

$$\left(\begin{array}{ccc|ccc} \boxed{2} & 1 & -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 & 1 & 0 \\ 3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} \boxed{1} & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & -7/2 & 5/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & -5/2 & -3/2 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & \boxed{-7/2} & 5/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & -5/2 & -3/2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/7 & 3/7 & 1/7 & 0 \\ 0 & \boxed{1} & -5/7 & 1/7 & -2/7 & 0 \\ 0 & 0 & -15/7 & -11/7 & 1/7 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/7 & 3/7 & 1/7 & 0 \\ 0 & 1 & -5/7 & 1/7 & -2/7 & 0 \\ 0 & 0 & \boxed{-15/7} & -11/7 & 1/7 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/15 & 2/15 & -1/15 \\ 0 & 1 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & \boxed{1} & 11/15 & -1/15 & -7/15 \end{array}\right)$$

Step 3: Multiply the inverse of the coefficient matrix (A^{-1}) by the constant matrix (B)

$$X = A^{-1}B = \begin{pmatrix} 8/15 & 2/15 & -1/15 \\ 2/3 & -1/3 & -1/3 \\ 11/15 & -1/15 & -7/15 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 25/15 \\ 4/3 \\ 10/15 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 4/3 \\ 2/3 \end{pmatrix}$$

Therefore, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} 5/3 \\ 4/3 \\ 2/3 \end{pmatrix}$$

Cramer's Rule: Cramer's Rule provides a method to solve systems of linear equations using determinants. It involves calculating determinants of submatrices formed from the coefficient matrix and replacing the corresponding column with the constant matrix. The values of the unknown variables are obtained by dividing the determinants by the determinant of the coefficient matrix.

Here are the steps to using Cramer's Rule to solve systems of linear equations:

Step 1: Write the system of linear equations in matrix form. Identify the coefficients of the variables and the constant terms. Arrange the coefficients into a matrix called the coefficient matrix. Arrange the constant terms into a column matrix called the constant matrix.

Step 2: Calculate the determinant of the coefficient matrix. Determine the determinant of the coefficient matrix. Denote this determinant as D.

Step 3: Calculate the determinants of the matrices obtained by replacing each column of the coefficient matrix with the constant matrix. Replace the first column of the coefficient matrix with the constant matrix and calculate the determinant. Denote this determinant as D_x. Replace the second column of the coefficient matrix with the constant matrix and calculate the determinant. Denote this determinant as D_y. Continue this process for each column of the coefficient matrix, obtaining the determinants D_x, D_y, D_z, and so on.

Step 4: Calculate the values of the unknown variables. The value of each unknown variable can be found by dividing the determinant of the corresponding column (D_x, D_y, D_z, etc.) by the

determinant of the coefficient matrix (D). For example, the value of x can be calculated as $x = D_x / D$, the value of y can be calculated as $y = D_y / D$, and so on.

Step 5: Interpret the solution.

- The resulting values of the unknown variables provide the solution to the system of linear equations.
- If a unique solution is obtained, it means that the system of equations has a unique solution.
- If there are no solutions or infinitely many solutions, it means that the system is incompatible or indefinite compatible, respectively.

✍ *By following these steps, Cramer's Rule allows you to solve systems of linear equations by calculating determinants and dividing them to find the values of the unknown variables.*

*Example 1.2.*c Let's solve a three-dimensional system of linear equations using Cramer's Rule. Consider the following system of linear equations:

$$\begin{cases} 2x + y - z = 4 \\ x - 3y + 2z = -1 \\ 3x + 2y - 4z = 5 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$AX = B$, where $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 3 & 2 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$

Step 2: Calculate the determinant of the coefficient matrix (D):

$$D = \det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 3 & 2 & -4 \end{vmatrix} = 24 - 2 + 6 - 9 - 8 + 4 = 15$$

Step 3: Calculate the determinants obtained by replacing the columns of the coefficient matrix with the constant matrix:

$$D_x = \det(A) = \begin{vmatrix} 4 & 1 & -1 \\ -1 & -3 & 2 \\ 5 & 2 & -4 \end{vmatrix} = 48 + 2 + 10 - 15 - 16 - 4 = 25$$

$$D_y = \det(A) = \begin{vmatrix} 2 & 4 & -1 \\ 1 & -1 & 2 \\ 3 & 5 & -4 \end{vmatrix} = 8 + 24 - 5 - 3 - 20 + 16 = 20$$

$$D_z = \det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -3 & -1 \\ 3 & 2 & 5 \end{vmatrix} = -30 - 3 + 8 + 36 + 4 - 5 = 10$$

Step 4: Calculate the values of the unknown variables:

$$x = \frac{D_x}{D} = \frac{25}{15} = \frac{5}{3}$$

$$y = \frac{D_y}{D} = \frac{20}{15} = \frac{4}{3}$$

$$z = \frac{D_x}{D} = \frac{10}{15} = \frac{2}{3}$$

Therefore, the solution to the given system of linear equations is: $X = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} & \frac{2}{3} \end{pmatrix}^t$

✍ *These methods offer different approaches to solve systems of linear equations and may have advantages or disadvantages depending on the specific problem. The choice of method depends on factors such as matrix size, computational efficiency, and the presence of special characteristics in the system.*

Types of solutions: The types of solutions depend on the specific relationships and coefficients of the system of linear equations. By applying appropriate methods, such as Gaussian elimination or matrix operations, it is possible to determine the type of solution and find the corresponding values for the variables.

Unique Solution: The system has a single set of values for the variables that satisfies all the equations.

Example 1.2.d Let's solve a three-dimensional system of linear equations using Gaussian elimination. Consider the following system of linear equations:

$$\begin{cases} 2x + y - z = 4 \\ 2x - y + 3z = 1 \\ x + y - 2z = -2 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$$AX = B, \text{ where } A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Step 2: Perform row operations to transform the augmented matrix into reduced row-echelon form. Use row operations to create zeros below and above the pivot (leading ones) in the all columns:

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{2} & 1 & -1 & 4 \\ 2 & -1 & 3 & 1 \\ 1 & 1 & -2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 2 \\ 0 & -2 & 4 & -3 \\ 0 & 1/2 & -3/2 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 2 \\ 0 & \boxed{-2} & 4 & -3 \\ 0 & 1/2 & -3/2 & -4 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 5/4 \\ 0 & \boxed{1} & -2 & 3/2 \\ 0 & 0 & -1/2 & -19/4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 5/4 \\ 0 & 1 & -2 & 3/2 \\ 0 & 0 & \boxed{-1/2} & -19/4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & 41/2 \\ 0 & 0 & \boxed{1} & 19/2 \end{array}\right)$$

Therefore, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} -7/8 \\ 41/2 \\ 19/2 \end{pmatrix}$$

No Solution: The system has no set of values for the variables that satisfies all the equations simultaneously. It represents an incompatible system.

Example 1.2.f Let's solve a system of linear equations that doesn't have a solution using Gaussian elimination.

Consider the following system of linear equations:

$$\begin{cases} 3x + 2y - z = 4 \\ 2x - y + 3z = 1 \\ 6x + 4y - 2z = 10 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$$AX = B, \text{ where } A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 3 \\ 6 & 4 & -2 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 \\ 1 \\ 10 \end{pmatrix}$$

Step 2: Perform pivot rule to transform the extended augmented matrix into row-echelon form.

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{3} & 2 & -1 & 4 \\ 2 & \mathbf{-1} & \mathbf{3} & \mathbf{1} \\ 6 & \mathbf{4} & \mathbf{-2} & \mathbf{10} \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 2/3 & -1/3 & 4/3 \\ 0 & -7/3 & 11/3 & -5/3 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

The augmented matrix is now in row-echelon form.

Step 3: Analyse the row-echelon form to determine the nature of the solution.

In the row-echelon form, we can see that the third row represents the equation 0 = 2, which is *not possible*. This indicates that there are no solutions to the system of linear equations.

Step 4: Interpret the solution: *the given system of linear equations has no solutions*.

✍ *Since the row-echelon form leads to an inconsistent equation, the system of linear equations is said to be inconsistent. Geometrically, it means that the three planes represented by the equations do not intersect at a single point, and therefore, there is no common solution.*

Infinitely Many Solutions: The system has an infinite number of sets of values for the variables that satisfy all the equations. It represents an indefinite compatible system.

Example 1.2.g Let's solve a system of linear equations with no unique solution using Gaussian elimination. Consider the following system of linear equations:

$$\begin{cases} 2x + 3y - z = 4 \\ 4x + 6y - 2z = 8 \\ x + 9y - z = 12 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$$AX = B, \text{ where } A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 6 & -2 \\ 1 & 9 & -1 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}$$

Step 2: Perform row operations to transform the augmented matrix into reduced row-echelon form. Use row operations to create zeros below and above the pivot (leading ones) in the all columns:

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{2} & 3 & -1 & 4 \\ 4 & 6 & -2 & 8 \\ 1 & 9 & -1 & 12 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 3/2 & -1/2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 15/2 & -1/2 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3/2 & -1/2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{15/2} & -1/2 & 10 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1/15 & 4/3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1/15 & 4/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The augmented matrix is now in row-echelon form.

Step 3: Analyse the row-echelon form to determine the nature of the solution.

In the row-echelon form, we can see that the third row represents the equation 0 = 0. This indicates that there are infinitely many solutions to the system of linear equations.

Step 4: Interpret the solution:

Since the system has more variables than equations, it is an underdetermined system. It means that there is not enough information to determine a unique solution. Instead, the system represents a plane or a line in three-dimensional space.

The solution can be expressed in terms of one free variable. Let's denote it as $z = t$, where t is parameter. Thus, we obtain the following simplified system of linear equations:

$$\begin{cases} x - 3t = 0 \\ y - \frac{1}{5}t = \frac{4}{3} \\ z = t \end{cases} \Rightarrow \begin{cases} x = 3t \\ y = \frac{1}{5}t + \frac{4}{3} \\ z = t \end{cases}$$

Hence, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} 3t \\ \frac{t}{5} + \frac{4}{3} \\ t \end{pmatrix}$$

✍ *In summary, the system has infinitely many solutions, represented by the parametric solution for* $X = (x(t), y(t), z(t))^t, t \in R$

General and Particular Solution: The general solution of a system of linear equations expresses all possible solutions using both basic and free variables. It provides a formula or parameterization for finding any solution to the system. A particular solution is a specific solution that can be obtained by assigning specific values to parameter.

Example 1.2.h Let's solve a system of linear equations that has both a general solution and a particular solution using Gaussian elimination. Consider the following system of linear equations:

$$\begin{cases} 2x - y + z = 4 \\ 3x + 6y - 3z = 3 \\ x + 2y - z = 1 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$AX = B$, where $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 6 & -3 \\ 1 & 2 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

Step 2: Perform pivot rule to transform the extended augmented matrix into reduced row-echelon form.

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{2} & -1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ 1 & 2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & -1/2 & 1/2 & 2 \\ 0 & 15/2 & -9/2 & -2 \\ 0 & 5/2 & -3/2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 2 \\ 0 & \boxed{15/2} & -9/2 & -3 \\ 0 & 5/2 & -3/2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1/10 & 28/15 \\ 0 & \boxed{1} & -6/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The extended augmented matrix is now in row-echelon form.

Step 3: Analyse the row-echelon form to determine the nature of the solution.

In the row-echelon form, we can see that the third row represents the equation 0 = 0. This indicates that there are infinitely many solutions to the system of linear equations.

Step 4: Interpret the solution:

Since the system has more variables than equations, it is an underdetermined system. It means that there is not enough information to determine a unique solution.

The solution can be expressed in terms of one free variable (parameter). Let's denote it as $z = t$, where t is parameter. Thus, we obtain the following simplified system of linear equations:

$$\begin{cases} x - \frac{1}{10}t = \frac{28}{15} \\ y - \frac{6}{5}t = -\frac{2}{5} \\ z = t \end{cases} \Rightarrow \begin{cases} x = \frac{1}{10}t + \frac{28}{15} \\ y = \frac{6}{5}t - \frac{2}{5} \\ z = t \end{cases}$$

Hence, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} 3t \\ \frac{t}{5} + \frac{4}{3} \\ t \end{pmatrix}$$

This represents the general solution of the system:

$$X_g = \begin{pmatrix} 3t \\ \frac{t}{5} + \frac{4}{3} \\ t \end{pmatrix}$$

Step 5: Find a particular solution:

To find a particular solution, we can set values for the parameters. For example, let's set $t = 1$.

Using these values, we can find a particular solution:

$$X_{p(t=1)} = \begin{pmatrix} 3 \\ \frac{23}{15} \\ 1 \end{pmatrix}$$

✍ *In summary, the given system of linear equations has infinitely many solutions represented by the general solution, and a particular solution can also be found.*

Basic Solutions and Finding Them: Basic solutions are specific solutions to a system of linear equations where all the free variables are set to zero. They can be found by setting each free variable to zero and solving for the corresponding basic variables.

Example 1.2.i Let's solve a system of linear equations that has both a general solution and a particular solution using Gaussian elimination. Consider the following system of linear equations:

$$\begin{cases} 2x - y + z = 4 \\ 6x - 3y + 3z = 12 \\ 4x - 2y + 2z = 8 \end{cases}$$

Step 1: Write the system of equations in matrix form:

$$AX = B, \quad \text{where} \quad A = \begin{pmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 \\ 12 \\ 8 \end{pmatrix}$$

Step 2: Perform pivot rule to transform the extended augmented matrix into reduced row-echelon form.

$$\bar{A} = \left(\begin{array}{ccc|c} \boxed{2} & -1 & 1 & 4 \\ 6 & -3 & 3 & 12 \\ 4 & -2 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & -1/2 & 1/2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The extended augmented matrix is now in row-echelon form.

Step 3: Analyse the row-echelon form to determine the nature of the solution.

In the row-echelon form, we can see that the second and third row represents the equation $0 = 0$. This indicates that there are infinitely many solutions to the system of linear equations.

Step 4: Interpret the solution.

Since the system has more variables than equations, it is an underdetermined system. It means that there is not enough information to determine a unique solution.

The solution can be expressed in terms of two free variables. Let's denote it as $y = t$ and $z = s$, where t and s are parameters. Thus, we obtain the following simplified system of linear equations:

$$\begin{cases} x - \frac{1}{2}t + \frac{1}{2}s = 2 \\ y = t \\ z = s \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}t - \frac{1}{2}s + 2 \\ y = t \\ z = s \end{cases}$$

Hence, the solution to the given system of linear equations is:

$$X = \begin{pmatrix} \frac{1}{2}t - \frac{1}{2}s + 2 \\ t \\ s \end{pmatrix}$$

This represents the general solution of the system:

$$X_g = \begin{pmatrix} \frac{1}{2}t - \frac{1}{2}s + 2 \\ t \\ s \end{pmatrix}$$

Step 5: Find a basic solution. To find a basic solution, we must set values for the parameters, all equal to zero.

Using these values, we can find a unique basic solution:

$$X_{b(t=0,\ s=0)} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

✍ *In summary, the given system of linear equations has a unique basic solution represented by the particular solution with all values for the parameters equal to zero.*

Questions you should be able to answer:

What are the main methods for solving systems of linear equations?

How does Gaussian elimination work to solve systems of linear equations?

What are the key steps involved in Gaussian elimination?

What is the role of row operations in Gaussian elimination?

Explain the concept of row-echelon form and reduced row-echelon form in Gaussian elimination.

How do you interpret the solution obtained from Gaussian elimination?

What is matrix inversion, and how does it relate to solving systems of linear equations?

When can you use matrix inversion to solve a system of linear equations?

What is Cramer's Rule, and how does it differ from other methods?

What determinants are involved in Cramer's Rule?

What conditions must be met for Cramer's Rule to be applicable?

How do you interpret the solution obtained from Cramer's Rule?

What are the advantages and limitations of each method for solving systems of linear equations?

How do you determine if a system of linear equations is inconsistent or dependent?

Proposed exercises:

1.2.1 In each exercise, fill in the missing words to complete the description of the corresponding method for solving systems of equations. These exercises aim to reinforce your understanding of different techniques used in linear algebra to solve systems of equations:

a) Fill in the Missing Words - Gaussian Elimination: Gaussian elimination is a method used to solve a system of linear equations by performing _____ operations to convert the system into _____ form. The process involves eliminating one variable at a time by adding or subtracting multiples of one equation to/from another. The goal is to reduce the system to a form where each equation has a leading coefficient of _____ and all other entries in the same column are _____. Once the system is in row-echelon form, back-substitution can be used to find the values of the variables.

b) Fill in the Missing Words - Cramer's Rule: Cramer's Rule is a method used to solve a system of linear equations using _____ determinants. For a system with n variables, Cramer's Rule requires calculating n+1 determinants. Each determinant involves replacing the coefficients of one variable with the constants from the right-hand side of the equations while keeping the other coefficients unchanged. The values of the variables can be found by dividing each determinant by the _____ determinant of the system.

c) Fill in the Missing Words - Inverse Matrix Method: The inverse matrix method is used to solve a system of linear equations by first finding the _____ of the coefficient matrix. If the inverse exists, the solution to the system can be obtained by multiplying the inverse matrix with the _____ vector. The coefficient matrix must be square and non-singular for the inverse to exist.

d) Fill in the Missing Words - LU Decomposition: LU decomposition is a technique used to solve a system of linear equations by factorizing the coefficient matrix into the product of a lower triangular matrix (L) and an upper triangular matrix (U). The LU decomposition simplifies the process of solving

multiple systems with the same coefficient matrix but different _____ vectors. Once the LU decomposition is done, forward-substitution and back-substitution can be applied to find the solutions.

e) Fill in the Missing Words - Iterative Methods: Iterative methods are a class of techniques used to solve large systems of equations by iteratively improving an initial guess of the solution. Examples of iterative methods include the Jacobi method, Gauss-Seidel method, and _____ method. These methods are particularly useful for solving sparse systems and systems with a large number of equations.

1.2.2 The rule of pivoting is a method used to determine the type of system of equations (consistent, inconsistent, or dependent) and to find the solutions of the system. In the context of economic mathematics, it is used to analyses economic models with multiple variables and equations. Show all the steps involved in solving the exercises to understand the application of the rule of pivoting in economic mathematics:

a) Rule of Pivoting: Consider the following system of equations: $\begin{cases} 2x + y = 5 \\ 3x - 2y = 8 \end{cases}$: *a.1)* Use the rule of pivoting to determine whether this system of equations is consistent or inconsistent. *a.2)* If the system is consistent, find the unique solution.

b) Rule of Pivoting: Consider the following system of equations $\begin{cases} 3x - y + 2z = 5 \\ x + 2y + z = 6 \\ 2x + 3y + 4z = 15 \end{cases}$: *b.1)* Use the rule of pivoting to determine whether this system of equations is consistent or inconsistent. b) If the system is consistent, find the unique solution.

c) Rule of Pivoting: Consider the following system of equations $\begin{cases} 2x - y + 3z = 4 \\ x + 2y + z = 5 \end{cases}$: *a.1)* Use the rule of pivoting to determine whether this system of equations is consistent or inconsistent. *a.2)* If the system is consistent, find the general solution (in terms of free variables).

d) Rule of Pivoting: Consider the following system of equations $\begin{cases} x + y + z = 6 \\ 2x + 2y + 2z = 12 \end{cases}$: *a.1)* Use the rule of pivoting to determine whether this system of equations is consistent or inconsistent. *a.2)* If the system is consistent, find the infinitely many solutions (in terms of free variables).

e) Economic Application: Consider an economic model with two goods, X and Y, and two markets, A and B. The production and consumption are given by the following system of equations: $\begin{cases} X + 2Y = 10 \text{ (Production in Market A)} \\ 2X + Y = 8 \text{ (Production in Market B)} \\ X + Y = 5 \text{ (Consumption in the Overall Market)} \end{cases}$. Use the rule of pivoting to determine the type of system (determined or indeterminate) and the existence of a feasible solution in the context of the economic model.

1.2.3 Each exercise involves solving a system of linear equations using different techniques. Show all the steps involved in solving the equations. The exercises aim to demonstrate the application of various methods for solving systems with more than three variables and their relevance in economic mathematics:

a) Gauss Method: Solving a System with Four Variables Consider the following system of equations $\begin{cases} 2x + 3y - z + 4w = 7 \\ 4x - y + 3z - 2w = 8 \\ x + 2y + z + w = 1 \\ 3x - 4y + 2z - 3w = -5 \end{cases}$: *a.1)* Use the Gauss method to convert the system into row-echelon form. *a.2)* Solve the system of equations by back-substitution. *a.3)* Interpret the solutions in the context of an economic problem with four variables.

b) Cramer's Method: Solving a System with Four Variables Consider the following system of equations $\begin{cases} 3x + y - 2z + 4w = 9 \\ 2x - 2y + z + 3w = 5 \\ 4x - 3y + 3z - w = 12 \\ 5x - y + 2z + 2w = 4 \end{cases}$: *b.1)* Use Cramer's method to solve the system of equations. *b.2)* Interpret the solutions in the context of an economic problem with four variables.

c) Inverse Matrix Method: Solving a System with Five Variables Consider the following system of equations $\begin{cases} x + 2y - z + w - 2u = 5 \\ 3x - 4y + 2z + u + 5v = 12 \\ 2x + y + z - w + 3u = 8 \\ x - y + 3z - 2w + 4v = 7 \\ 4x + y - 2z + u - 3v = 3 \end{cases}$: *c.1)* Use the inverse matrix method to solve the system of equations. *c.2)* Interpret the solutions in the context of an economic problem with five variables.

d) Gaussian Elimination: Solving a Homogeneous System Consider the following homogeneous system of equations: $\begin{cases} 2x - y + 3z = 0 \\ 3x + z = 0 \\ x - 2y + z = 0 \end{cases}$: *d.1)* Use Gaussian elimination to find the non-trivial solutions to the system. *d.2)* Interpret the non-trivial solutions in the context of an economic problem with three variables.

e) Solving a System with Six Variables using Row Reduction: Consider the following system of equations: $\begin{cases} x + y + z + w + u + v = 10 \\ 2x - y + 3z - w + u - 2v = 4 \\ 3x + y - z + w - 2u + 3v = 5 \\ x + 2y + z + 3w - u - 4v = 8 \\ x + y + 2z + 4w + u + v = 12 \\ 2x + 3y + z + 2w + 5u + v = 20 \end{cases}$: *e.1)* Use row reduction (Gauss-Jordan elimination) to solve the system of equations. *e.2)* Interpret the solutions in the context of an economic problem with six variables.

1.2.4 Each exercise involves exploring the concepts of general solutions (with free variables), particular solutions (with assigned values or parameters), and basic solutions (with non-redundant variables) in systems of equations with more than two variables. Show all the steps involved in finding these solutions. Understanding these concepts is essential in linear algebra and lays the foundation for solving more complex systems of equations with multiple variables in economic mathematics:

a) <u>General, Particular, and Basic Solutions</u>: Consider the following system of equations $\begin{cases} 3x + 2y - z + w = 7 \\ 3x + 2y - z + w = 7 \\ 2x + y + 3z - 2w = 3 \\ x + y - z + 2w = 4 \end{cases}$: *a.1)* Determine the general solution to the system in terms of the free variable(s). *a.2)* Find a particular solution to the system by assigning values to the free variable(s). *a.3)* Identify the basic solutions of the system.

b) <u>General Solution of a Homogeneous System:</u> Consider the following homogeneous system of equations $\begin{cases} 2x - y + 3z - w = 0 \\ x + 2y - z + w = 0 \\ 3x - y + 4z - 2w = 0 \end{cases}$: *b.1)* Determine the general solution to the homogeneous system in terms of the free variable(s). *b.2)* Find the basic solutions of the system.

c) <u>Particular Solution with Given Coefficients:</u> Consider the following system of equations $\begin{cases} 3x + 2y - z + 4w = 12 \\ 3x + 2y - z + 4w = 12 \\ 2x + 3y - z + 2w = 10 \\ x - y + 2z - w = 4 \end{cases}$. Given that x = 2, y = 3, and z = 1, find the particular solution for the variable w.

d) <u>Basic Solutions of a Non-Homogeneous:</u> System Consider the following non-homogeneous system of equations $\begin{cases} x + 2y - z = 1 \\ 2x - y + z = -3 \\ 3x + y - 2z = 4 \end{cases}$: *d.1)* Determine the general solution to the system. *d.2)* Find the basic solutions of the system.

e) <u>Particular Solution with Parametric Variables:</u> Consider the following system of equations $\begin{cases} x + 2y - z + 3w = 7 \\ 2x - y + 3z - 2w = 4 \\ 3x + y - z + w = 6 \end{cases}$. Find a particular solution for x, y, z, and w using the parameter t, where $x = t,\ y = 2t,\ z = 3t$, and $w = 1 - t$.

1.2.5 These exercises demonstrate the application of systems of linear equations in the context of economic mathematics. Fill in the missing words to complete the questions and interpret the solutions within the given economic scenarios. Show all the steps involved in solving the exercises to gain a deeper understanding of the concepts and their economic implications:

a) <u>Solving a System of Linear Equations:</u> Consider the following system of linear equations $\begin{cases} 2x + 3y - z = 10 \\ 3x - y + 2z = 5 \\ 4x + y + 3z = 12 \end{cases}$, related to economic mathematics: *a.1)* Write the augmented matrix for the system of equations. *a.2)* Use Gaussian elimination to find the row-echelon form of the augmented matrix. *a.3)* Determine whether the system has a unique solution, no solution, or infinitely many solutions.

b) <u>Economic Interpretation of Solutions:</u> Consider an economic model with three variables, x, y, and z, representing the quantities of goods X, Y, and Z produced by three different companies. The

system of linear equations is given by $\begin{cases} 2x + y - z = 100 \\ 3x - 2y + 2z = 150 \\ x + y + 3z = 200 \end{cases}$: *b.1)* Interpret each equation in the context of the economic model. *b.2)* Solve the system of equations to find the values of x, y, and z. *b.3)* Interpret the solution in terms of the quantities of goods produced by each company.

c) Economic Equilibrium: Consider an economic model with three variables, x, y, and z, representing the supply, demand, and price of a commodity. The equilibrium conditions are given by the following system of equations $\begin{cases} x - 2y + 3z = 100 \\ 3x + y - z = 50 \\ 2x - 4y + 6z = 200 \end{cases}$: *c.1)* Interpret each equation in the context of the economic equilibrium. *c.2)* Solve the system of equations to find the equilibrium values of x, y, and z. *c.3)* Interpret the solution in terms of the commodity's equilibrium supply, demand, and price.

1.3 EXAMPLES OF LINEAR ECONOMIC MODELS

Keywords: *Supply and demand, Cost-Volume-Profit analysis, Input-Output models, Linear production functions, Linear regression models, Market equilibrium, Break-even analysis, Price elasticity of demand, Production planning, Resource allocation, Linear programming, Marginal analysis, Linear cost functions, Linear revenue functions, Linear pricing models.*

Linear Economic Models: Linear economic models use systems of linear equations to represent economic relationships and solve problems in economics. They help analyses variables such as supply and demand, production costs, and resource allocation in a linear framework.

The key assumptions of linear economic models: Linear economic models are based on several key assumptions that simplify the analysis and allow for mathematical tractability. These assumptions include:

Proportionality: Linear economic models assume that relationships between variables are linear, meaning that changes in one variable led to proportional changes in other variables.

Additivity: The models assume that the effects of different factors or variables can be added or subtracted independently. This implies that the total effect is simply the sum of individual effects.

Homogeneity: Linear economic models assume that the relationships hold uniformly across different units or individuals. In other words, the behaviour of one unit is assumed to be representative of the behaviour of all similar units.

Certainty: Linear economic models often assume certainty in terms of the information available, outcomes, and the absence of randomness or uncertainty.

Independence: The models assume that variables are independent of each other and that changes in one variable do not affect the others directly.

Constant parameters: Linear economic models assume that the parameters, such as coefficients and elasticities, are constant over the analysis period and do not change.

Continuous variables: The models assume that variables are continuous and can take any value within a given range.

✍ *It is important to note that these assumptions are simplifications of the real-world complexities and may not capture all the nuances of economic phenomena. However, they provide a useful framework for analysis and allow for analytical solutions in many cases.*

The purpose of using linear economic models: The purpose of using linear economic models is to simplify the analysis of complex economic phenomena and derive useful insights about the relationships between variables. Linear models provide a framework for understanding and quantifying the behaviour of economic systems under certain assumptions. Here are some specific purposes of using linear economic models:

Understanding Relationships: Linear models help economists understand the relationships between different economic variables. By quantifying the effects of changes in one variable on others, economists can identify patterns and make predictions about how changes in economic factors will impact the overall system.

Policy Analysis: Linear models are often used to analyses the effects of different policy interventions or changes in economic variables. They provide a way to evaluate the impact of policy decisions on various economic outcomes, such as employment, inflation, or economic growth.

Planning and Forecasting: Linear economic models can be used for planning and forecasting purposes. By estimating relationships between economic variables, economists can project future trends, assess potential scenarios, and make informed decisions based on the predicted outcomes.

Resource Allocation: Linear models assist in optimizing resource allocation by identifying the most efficient allocation of resources given certain constraints. This can be particularly useful in production planning, investment decisions, or resource allocation in public policy.

Comparative Analysis: Linear models allow for comparative analysis between different economic scenarios or policy options. By comparing the estimated outcomes under different assumptions or conditions, economists can evaluate the effectiveness or desirability of different alternatives.

Teaching and Pedagogy: Linear economic models serve as an educational tool to introduce students to economic concepts, theories, and analysis. They provide a simplified framework for understanding economic relationships and building foundational knowledge.

✍ *Overall, the purpose of using linear economic models is to provide a structured approach to analyses and understand economic phenomena, make predictions, evaluate policy options, and support decision-making processes.*

Common examples of linear economic models*: E*xamples of linear economic models that are frequently used in economic analysis, a few notable of them:

Linear Supply and Demand Model: This model represents the relationship between the quantity demanded and supplied of a good or service as linear functions. It assumes that changes in

price have a proportional effect on quantity, with a constant slope representing the elasticity of demand or supply.

Linear Production Function: The linear production function relates inputs (such as labour and capital) to the level of output produced. It assumes that each input has a linear effect on output, and the overall production function is a linear combination of these inputs.

Linear Cost Function: The linear cost function depicts the relationship between the level of production and the total cost incurred. It assumes that costs vary linearly with the level of production and can be expressed as a sum of fixed and variable costs.

Linear Regression Models: Linear regression models are commonly used in econometrics to estimate the relationship between a dependent variable and one or more independent variables. They assume a linear functional form, allowing for the estimation of coefficients that represent the marginal effects of the independent variables.

Linear Input-Output Models: Input-output models represent the interdependencies between different sectors of an economy. Linear input-output models assume that the relationships between sectors can be expressed linearly, enabling analysis of inter-industry flows and economic multipliers.

Linear Programming Models: Linear programming models are used to optimize resource allocation given a set of constraints and objectives. They assume linear relationships between variables and allow for maximizing or minimizing a linear objective function subject to linear constraints.

✍ *These are just a few examples of linear economic models commonly used in economic analysis. Linear models provide a simplified framework for understanding and quantifying economic relationships, allowing economists to analyses and derive insights from complex economic systems.*

How are variables represented in linear economic models: In linear economic models, variables are represented using symbols or letters to denote specific economic quantities or factors. These symbols can vary depending on the context and the specific model being used. However, there are some common conventions for representing variables in linear economic models:

Dependent Variable: The dependent variable, often denoted by "Y" or "Q", represents the economic outcome or quantity being analysed or predicted. For example, in a linear demand or production function, the dependent variable could represent the quantity demanded or the level of output.

Independent Variables: Independent variables, also known as explanatory variables or factors, are denoted by symbols such as "X", "Z", or specific letters associated with the variables being considered. These variables represent the factors that influence or explain changes in the dependent variable. For example, in a linear demand function, the independent variable could represent the price of a product.

Parameters or Coefficients: Parameters or coefficients, denoted by Greek letters like "α" or "β", represent the constants or numerical values associated with the variables in the model. These coefficients determine the strength and direction of the relationship between the variables.

Error Term: In econometric models or regression analysis, an error term (ε) is often included to capture unexplained variation or random disturbances that affect the dependent variable but are not accounted for by the model's independent variables.

✍ *It's important to note that the specific choice of symbols for variables in a linear economic model can vary across studies and literature. The selection of symbols is primarily driven by convention, clarity, and consistency within the specific context and field of study*.

Representation of constrains in Linear Economic Models*:* The constraints play a crucial role in determining the feasible solutions and the boundaries within which the variables can vary. In linear economic models, constraints are often represented in the form of linear equations or inequalities. Here are some common ways constraints are represented:

Resource Constraints: These constraints arise from limited availability of resources such as labour, capital, raw materials, or production capacity. They are typically represented by linear inequalities that limit the usage of resources. For example, if there is a limited amount of labour available, the constraint may be expressed as: $Total\ labor\ used \leq Available\ labor.$

Production Constraints: These constraints reflect limitations on the production process. They can be represented by linear equations or inequalities that relate the quantities of different inputs and outputs. For instance, if a company produces two products, A and B, the constraint could be expressed as:
$Quantity\ of\ A\ produced + Quantity\ of\ B\ produced \leq Production\ capacity.$

Market Constraints: These constraints relate to market conditions and demand. They can be expressed as linear inequalities that reflect limitations on the quantities bought or sold. For example, a constraint on the maximum quantity of a product that can be sold might be: $Quantity\ sold \leq Market\ Price \times Quantity\ produced.$

Technological Constraints: These constraints arise from technological limitations or requirements. They can be represented by linear equations or inequalities that define the relationships between different technological factors. For instance, if there is a fixed technological ratio between inputs and outputs, the constraint could be expressed as: $Quantity\ of\ input\ 1 \times Technological\ coefficient\ 1 + Quantity\ of\ input\ 2 \times Technological\ coefficient\ 2 = Quantity\ of\ output.$

Budget Constraints: These constraints are related to financial limitations. They are typically represented by linear inequalities that restrict the total cost or budget available for a project or decision. For example, if there is a budget constraint for a marketing campaign, it may be expressed as: $Total\ cost \leq Budget.$

Regulatory Constraints: These constraints arise from legal or regulatory requirements. They can be represented by linear equations or inequalities that reflect compliance with specific regulations or standards. For instance, if there is a maximum allowable level of pollution emissions, the constraint could be expressed as: $Quantity\ of\ emissions \leq Maximum\ allowable\ level.$

✍ *These are just a few examples of how constraints can be represented in linear economic models. The specific form and nature of the constraints depend on the particular problem being analysed and the variables involved. By incorporating constraints into economic models, analysts and decision-makers can better understand the limitations and trade-offs involved in various economic scenarios, leading to more informed and realistic decision-making.*

The main components of a linear economic model*:* The main components of a linear economic model include:

Decision Variables: These are the variables that represent the choices or decisions made in the model. Decision variables are usually denoted by symbols and are the quantities that need to be determined or optimized. They can represent production quantities, resource allocation, prices, investment levels, or any other relevant economic factors.

Objective Function: The objective function defines the goal or objective of the model. It represents the measure of success or the criteria to be optimized. The objective function is typically expressed as a linear equation or a linear combination of decision variables. It can be formulated to maximize profits, minimize costs, maximize utility, or achieve any other desired outcome.

Constraints: Constraints represent the limitations, restrictions, or conditions that must be satisfied in the model. They define the boundaries within which the decision variables can vary. Constraints are usually represented as linear equations or inequalities and can reflect resource availability, production capacity, market demand, technological limitations, budget constraints, or any other relevant constraints specific to the economic problem being modelled.

Parameters: Parameters are the fixed values that are used in the model but are not subject to optimization. They represent the data, coefficients, or constants that determine the relationships between variables or describe the characteristics of the problem. Parameters can include costs, prices, input-output ratios, technological coefficients, market data, or any other relevant numerical values.

Assumptions: Assumptions are the simplifying conditions or premises on which the model is based. They help define the scope and applicability of the model. Assumptions can include assumptions about market behaviour, production technology, consumer preferences, or any other relevant assumptions specific to the economic problem being modelled.

✍ *By incorporating these components into a linear economic model, economists and decision-makers can analyses and optimize various economic scenarios, make informed decisions, and gain insights into the relationships between variables and the potential outcomes of their decisions.*

What techniques are used to solve linear economic models*:* Linear economic models are typically solved using various optimization techniques. The specific technique used depends on the size and complexity of the model. Here are some commonly used techniques:

Linear Programming (LP): Linear programming is a widely used technique for solving linear economic models. It is designed to find the optimal solution that maximizes or minimizes the objective function, subject to the linear constraints. LP algorithms, such as the simplex method or interior point methods, iteratively search for the optimal values of decision variables that satisfy the constraints.

Integer Linear Programming (ILP): In some economic models, decision variables are required to take integer values rather than continuous values. Integer linear programming extends

linear programming by incorporating additional constraints that restrict the decision variables to integer values. ILP problems are generally more challenging to solve than LP problems and require specialized algorithms like branch and bound or cutting plane methods.

Mixed Integer Linear Programming (MILP): In certain cases, economic models may involve a combination of continuous and integer decision variables. Mixed integer linear programming (MILP) techniques are used to solve such models. MILP combines the linear programming framework with the integer programming techniques to find the optimal solution.

Network Flow Algorithms: Linear economic models that involve the flow of goods, resources, or information across a network can be solved using network flow algorithms. These algorithms, such as the maximum flow or minimum cost flow algorithms, optimize the flow of quantities subject to capacity constraints and other relevant restrictions in the network.

Decomposition Techniques: In large-scale economic models, decomposition techniques are employed to break down the problem into smaller subproblems that can be solved separately. Methods like the Benders decomposition, Dantzig-Wolfe decomposition, or Lagrange relaxation are utilized to solve the subproblems individually and iteratively converge towards the optimal solution of the entire model.

Linear Approximation: In some cases, linear economic models may be too complex to solve directly. Linear approximation techniques, such as linearization or linear regression, are used to simplify the model by approximating nonlinear relationships with linear functions. This allows for the use of linear programming or other linear techniques to solve the simplified model.

✍ *These are some of the commonly employed techniques to solve linear economic models. The choice of technique depends on the characteristics of the model and the specific objectives of the analysis. Additionally, advances in mathematical optimization algorithms and software have made it easier to solve complex linear economic models efficiently*.

The limitations of linear economic models: While linear economic models are valuable tools for analysing and understanding many economic phenomena, they have certain limitations that need to be considered. Here are some common limitations of linear economic models:

Linearity Assumption: Linear economic models assume a linear relationship between variables. This assumption may not hold in reality for many economic situations. In reality, economic relationships often exhibit non-linear behaviour, such as diminishing returns, economies of scale, or non-linear demand and supply curves. Linear models may not accurately capture these complexities, leading to potentially inaccurate predictions or policy recommendations.

Simplicity and Oversimplification: Linear economic models simplify the real-world complexity by assuming a number of simplified assumptions. While these simplifications make the models more tractable and easier to solve, they may oversimplify the economic reality. Important factors and dynamics can be overlooked, such as heterogeneity, market imperfections, asymmetric information, or behavioural biases. Oversimplification can limit the model's ability to capture the true complexity of economic phenomena.

Limited Scope of Variables: Linear models typically focus on a limited set of variables and assume ceteris paribus (all other things being equal) conditions. This can neglect important

interdependencies and feedback effects that exist in the real world. Economic systems are highly interconnected, and changes in one variable can have ripple effects throughout the system. Linear models may not capture these feedback loops and can miss important dynamic relationships.

Lack of Time Dimension: Linear economic models often assume static conditions and neglect the time dimension. They assume that relationships and variables remain constant over time. However, economic systems are dynamic, and variables can change over time due to factors such as technological progress, policy changes, or shifts in consumer preferences. Neglecting the time dimension can limit the model's ability to capture the dynamic nature of economic processes and the effects of policy interventions.

Data Requirements and Estimation Challenges: Linear economic models require reliable and accurate data to estimate model parameters. Data limitations, such as incomplete or inaccurate data, can affect the quality and reliability of the model's results. Additionally, estimating model parameters from data can be challenging, especially in cases where data is scarce or subject to measurement errors.

Boundary and Assumption Sensitivity: Linear economic models are sensitive to the boundaries and assumptions set within the model. Small changes in the model's assumptions or constraints can lead to significant changes in the results. This sensitivity to model specifications requires careful consideration and validation of the model's assumptions and robustness of the results.

✍ *It is important to acknowledge these limitations and carefully consider their implications when using linear economic models. While linear models provide valuable insights and are widely used, they are just simplifications of the complex economic reality. Complementing linear models with other modelling techniques and approaches, such as nonlinear models, agent-based models, or qualitative analysis, can help overcome some of these limitations and provide a more comprehensive understanding of economic phenomena*.

Application of linear economic models in real world scenarios: Linear economic models can be applied to various real-world scenarios to analyses and understand economic phenomena. Here are some examples of how linear economic models can be applied:

Production Planning and Resource Allocation: Linear models can be used to optimize production planning and resource allocation decisions. By considering factors such as production capacity, input-output relationships, and resource constraints, linear models can help determine the optimal quantities of inputs to use, the optimal production levels, and the allocation of resources to different production activities.

Cost Minimization: Linear models can assist in minimizing costs in production or operational processes. By considering input prices, production technologies, and resource constraints, linear models can identify the optimal combination of inputs to minimize costs while meeting production requirements.

Supply Chain Optimization: Linear models can be applied to optimize supply chain management decisions. These models can analyse the flow of goods, inventory levels, transportation costs, and demand constraints to identify the most efficient allocation of resources, minimize transportation costs, and ensure timely delivery of products.

Pricing and Revenue Maximization: Linear models can help businesses determine optimal pricing strategies and revenue maximization. By considering demand functions, cost structures, and market constraints, linear models can identify price levels that maximize revenue, taking into account factors such as price elasticity of demand, production costs, and market conditions.

Investment Analysis: Linear models can be used in investment analysis to assess the feasibility and profitability of investment projects. By considering factors such as cash flows, discount rates, and investment constraints, linear models can assist in evaluating investment opportunities and determining optimal investment decisions.

Portfolio Optimization: Linear models can be applied to optimize investment portfolios by considering risk and return trade-offs. These models can help investors allocate their investment funds across different assets to achieve the desired balance between risk and return, taking into account factors such as asset returns, correlations, and investment constraints.

✍ *These are just a few examples of how linear economic models can be applied in real-world scenarios. The versatility and simplicity of linear models make them valuable tools for decision-making, policy analysis, and understanding the relationships between variables in various economic contexts.*

Common applications of linear economic models in different industries or sectors*:* Linear economic models find applications across various industries and sectors. Here are some common examples of how linear economic models are applied in different domains:

Manufacturing: Linear economic models are used in manufacturing industries to optimize production planning, resource allocation, and inventory management. These models help in determining the optimal production levels, minimizing costs, and ensuring efficient use of resources.

Transportation and Logistics: Linear models are applied in transportation and logistics to optimize routing, scheduling, and inventory control. They help in minimizing transportation costs, optimizing delivery routes, managing warehouse inventory levels, and improving overall supply chain efficiency.

Energy and Utilities: Linear economic models are used in the energy sector for electricity generation planning, capacity expansion, and demand forecasting. These models help in determining the optimal mix of energy sources, evaluating investment decisions, and analysing the impact of policy interventions on energy systems.

Agriculture and Food Production: Linear models are applied in agriculture and food production to optimize crop planning, resource allocation, and supply chain management. They help in determining the optimal allocation of resources, minimizing costs, and maximizing agricultural productivity.

Financial Services: Linear economic models find applications in financial services for portfolio optimization, asset allocation, and risk management. They help in determining the optimal allocation of investment funds across different assets, balancing risk and return, and evaluating the performance of investment portfolios.

Healthcare: Linear models are used in healthcare for capacity planning, resource allocation, and healthcare service optimization. These models help in determining optimal staffing levels, scheduling appointments, managing hospital beds, and optimizing healthcare resource utilization.

Environmental Management: Linear economic models are applied in environmental management to assess the impact of environmental policies, optimize resource allocation, and evaluate pollution control measures. They help in determining the optimal level of emissions reductions, analysing the cost-effectiveness of environmental regulations, and evaluating the trade-offs between economic growth and environmental sustainability.

Public Policy and Government: Linear models are utilized in public policy and government sectors for economic forecasting, tax policy analysis, and public resource allocation. These models help in evaluating the impact of policy changes, optimizing tax structures, and allocating public resources efficiently.

✍ *These are just a few examples of the diverse applications of linear economic models across different industries and sectors. The versatility of linear models makes them valuable tools for decision-making, planning, and policy analysis in a wide range of economic contexts*.

✍ *However, it's important to note that real-world economic relationships can be more complex, and non-linear models may be necessary to capture more accurate representations of economic behaviour*.

Questions you should be able to answer:

What is a linear economic model?
What are the key assumptions of linear economic models?
What is the purpose of using linear economic models?
What are some common examples of linear economic models?
How are variables represented in linear economic models?
How are constraints represented in linear economic models?
What are the main components of a linear economic model?
What techniques are used to solve linear economic models?
How do you interpret the results of a linear economic model?
What are the limitations of linear economic models?
How do linear economic models differ from nonlinear economic models?
How can linear economic models be applied in real-world scenarios?
What are some common applications of linear economic models in different industries or sectors?
How do linear economic models contribute to decision-making processes?
What are some extensions or variations of linear economic models?

Proposed Exercises:

Fill in the missing words to complete the sentences and better understand the examples of linear economic models. Linear models provide a simplified representation of relationships in economics, making them useful for understanding basic economic concepts

1. Example of Linear Demand Model: In a linear demand model, the demand for a certain product is given by the equation: Q = a - bP where Q represents the quantity demanded, P is the price of the product, and a and b are the missing coefficients. The parameter 'a' represents the maximum quantity that consumers are willing to purchase when the price is zero, which is often referred to as the (1) _____. The parameter 'b' represents the change in quantity demanded for a unit change in price and is referred to as the (2) _____.

2. Example of Linear Production Function: A linear production function describes the relationship between the quantity of output produced (Q) and the quantities of inputs used in production, such as labour (L) and capital (K). The linear production function can be expressed as: Q = a + bL + cK, where a, b, and c are the missing coefficients. The coefficient 'b' represents the (1) _____, i.e., the change in output resulting from one additional unit of labor, while the coefficient 'c' represents the (2) _____, i.e., the change in output resulting from one additional unit of capital.

3. Example of Linear Cost Function: A linear cost function represents the relationship between the total cost (TC) and the quantity of output (Q) produced. The linear cost function can be expressed as: TC = a + bQ, where a and b are the missing coefficients. The constant term 'a' represents the (1) _____, which is the total cost incurred even if no units are produced. The coefficient 'b' represents the (2) _____, i.e., the increase in total cost for producing one additional unit of output.

4. Example of Linear Investment Model: In a linear investment model, the investment expenditure (I) is related to the level of income (Y) through the equation: I = a + bY, where a and b are the missing coefficients. The constant term 'a' represents the (1) _____, which represents the level of investment that occurs even in the absence of income. The coefficient 'b' represents the (2) _____, i.e., the change in investment for a unit change in income.

1.4 CONCEPTS OF VECTOR AND EUCLIDEAN SPACES

Keywords: *Vector, Magnitude, Direction, Vector Space, Linear Combination, Linear Independence, Basis, Dimension, Euclidean Space, Cartesian Coordinates, Distance, Inner Product, Norm, Orthogonality, Euclidean Transformation.*

Concepts of Vector Space:

Vector: A vector is a mathematical object that represents both magnitude and direction. It can be represented as an ordered set of elements or components.

Scalar: A scalar is a single value that represents magnitude only. Scalars can be multiplied by vectors to produce scaled vectors.

Vector Space: A vector space is a set of vectors equipped with operations of vector addition and scalar multiplication that satisfy certain properties. These properties include closure under addition and scalar multiplication, associativity, commutativity, and the existence of zero vector and additive inverses.

Linear Combination: A linear combination of vectors is the sum of scalar multiples of those vectors. It involves multiplying each vector by a scalar and adding them together.

Linear Independence: A set of vectors is linearly independent if none of the vectors can be expressed as a linear combination of the others. Linear independence implies that no vector in the set is redundant or can be removed without changing the span of the set.

Basis: A basis for a vector space is a set of linearly independent vectors that span the entire space. It provides a way to express any vector in the space as a unique linear combination of basis vectors.

Dimension: The dimension of a vector space is the number of vectors in a basis of that space. It represents the minimum number of linearly independent vectors needed to span the space.

Concepts of Euclidean Space:

Euclidean Space: Euclidean space refers to the n-dimensional space in which points are represented by n-tuples of real numbers. It is the familiar geometric space that follows Euclidean geometry, where distance, angles, and geometric properties are defined based on the Pythagorean theorem and Euclidean axioms.

Cartesian Coordinates: Cartesian coordinates provide a way to locate points in Euclidean space using a system of orthogonal coordinate axes. Each point is represented by an ordered set of coordinates $(x_1, x_2, ..., x_n)$, where each coordinate corresponds to a dimension.

Distance: In Euclidean space, the distance between two points is calculated using the Euclidean distance formula, which is derived from the Pythagorean theorem. The distance between two points $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ is given by $\sqrt{((x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2)}$.

Inner Product: The inner product (also known as dot product) is a binary operation defined on vectors in Euclidean space. It measures the similarity or orthogonality between vectors and is calculated as the sum of the products of corresponding components of the vectors.

Norm: The norm (also known as length or magnitude) of a vector in Euclidean space measures the distance or size of the vector. It is calculated using the Euclidean norm formula, which is the square root of the sum of the squares of the vector components.

Orthogonality: Two vectors in Euclidean space are orthogonal if their inner product is zero. Orthogonal vectors are perpendicular to each other and have no shared directional component.

Euclidean Transformation: Euclidean transformations are geometric transformations that preserve distances, angles, and shapes in Euclidean space. Examples include translations, rotations, reflections, and scaling.

✍ *These keywords encompass fundamental concepts related to vector spaces and Euclidean spaces, providing a foundation for understanding and working with vectors and geometric spaces.*

Questions you should be able to answer:

What is a vector, and how does it differ from a scalar?
What are the key properties that define a vector space?
What is a linear combination of vectors, and how is it computed?
What does it mean for a set of vectors to be linearly independent?
What is a basis for a vector space, and why is it important?

How is the dimension of a vector space determined?

What is Euclidean space, and how is it defined?

How are points represented in Euclidean space using Cartesian coordinates?

What is the Euclidean distance formula, and how is it used to calculate distances between points?

What is the inner product (dot product) of vectors in Euclidean space, and what does it represent?

How is the norm (length or magnitude) of a vector calculated in Euclidean space?

What does it mean for vectors to be orthogonal in Euclidean space?

What are some examples of Euclidean transformations and how do they preserve geometric properties?

How do the concepts of vector and Euclidean spaces contribute to understanding and solving real-world problems in various fields?

What are some limitations or extensions of the concepts of vector and Euclidean spaces that you have learned?

2 ELEMENTS OF THREE-DIMENSIONAL (3D) ANALYTICAL GEOMETRY

CONTENT UNITS that will be covered:

Introduction to 3D Coordinate Systems: Cartesian coordinate system in 3D space. Representation of points using coordinates (x, y, z). Visualization of 3D space and coordinate planes.

Distance and Midpoint Formulas in 3D Space: Calculation of distance between two points in 3D space. Determination of the midpoint between two points in 3D space.

Vectors in 3D Space: Representation of vectors in 3D space. Vector operations (addition, subtraction, scalar multiplication). Dot product and cross product of vectors in 3D space.

Applications of Vectors in 3D Space: Determination of vector components and magnitudes. Vector projections and scalar projections. Applications of vectors in economy.

Equations of Lines in 3D Space: Parametric equations of lines in 3D space. Vector equations of lines in 3D space. Intersection of lines in 3D space.

Planes in 3D Space: Equation of a plane in 3D space. Different forms of plane equations (normal form, general form, etc.). Intersection of planes and lines in 3D space.

✍ *These content units provide a structured framework for studying and understanding the concepts, techniques, and applications of 3D analytical geometry. The specific topics covered may vary depending on the level of study (e.g., high school, undergraduate, or graduate level) and the intended scope of the course or curriculum*.

Keywords: *Coordinate system, Cartesian coordinates, Point, Line, Plane, Distance, Midpoint, Vector, Vector operations, Dot product, Cross product, Curve, Surface, Intersection, Equations of lines, Equations of planes, Vectors in 3D space, Geometric transformations, Translation, Rotation, Reflection, Scaling, Cylindrical coordinates, Spherical coordinates, Coordinate conversions, Projections.*

Learning Objectives:

1 Understand the 3D Cartesian coordinate system and its components, including the three axes (x, y, z) and the concept of origin.

2 Recognize and apply the distance formula in 3D space to calculate the distance between two points.

3 Determine the midpoint of a line segment in 3D space using the midpoint formula.

4 Learn and apply the parametric equations and vector equations to represent lines in 3D space.

5 Identify and work with different forms of equations for planes in 3D space, such as the general form and the normal form.

6 Perform vector operations, including vector addition, subtraction, scalar multiplication, dot product, and cross product, in the context of 3D geometry.

7 Apply vector operations to solve problems related to lines, planes, and geometric objects in 3D space.

8 Solve intersection problems involving lines, planes in 3D space, including finding common points and determining whether intersections exist.

9 Apply 3D geometry concepts and techniques to real-world problems in fields such as economy.

10 Understand the connections between 3D analytical geometry and other mathematical concepts, such as vector calculus or coordinate transformations.

11 Develop critical thinking skills to analyse and evaluate geometric relationships, properties, and results in the context of 3D geometry.

✍ *These learning objectives provide a comprehensive framework for studying three-dimensional analytical geometry and serve as a roadmap for acquiring the necessary knowledge, skills, and problem-solving abilities in this field*.

Vector Algebra fundamentals related to Economic Mathematics:

Spatial Analysis: Economic activities often occur in physical space, and understanding the spatial distribution and interaction of economic variables is crucial. 3D analytical geometry provides tools and techniques to analyse spatial relationships, such as the location of economic agents, the dispersion of resources, and the accessibility of markets. Spatial analysis using techniques like distance calculations, spatial interpolation, and spatial clustering can help in studying economic phenomena such as regional development, transportation planning, and market analysis.

Input-Output Analysis: Input-output analysis is an economic modelling technique that examines the interdependencies among different sectors of an economy. In 3D analytical geometry, input-output relationships can be represented using matrices, where the rows and columns correspond to sectors and the entries represent the flow of inputs and outputs between sectors. Matrix algebra and matrix operations are utilized to analyse input-output tables, calculate multipliers, and assess the impacts of changes in one sector on other sectors. This approach helps in understanding the interconnectedness and interdependence of sectors within an economy.

Economic Modelling and Simulation: Economic models often involve complex systems with multiple variables and relationships. 3D analytical geometry provides a framework for representing and visualizing these models. Geometric objects, such as curves, surfaces, and volumes, can be used to represent economic variables and relationships. By applying analytical geometry techniques, such as parametric equations, vector operations, and geometric transformations, economists can develop models that simulate and analyse economic behaviour in three-dimensional space. This enables a more comprehensive understanding of economic dynamics and helps in making predictions and policy evaluations.

Optimization and Resource Allocation: Optimization is a fundamental concept in economic mathematics, and 3D analytical geometry provides tools for solving optimization problems in three-dimensional space. Economic models often involve optimizing resource allocation decisions, such as determining the optimal allocation of production factors or finding the best mix of inputs to maximize output. Techniques like linear programming, which utilize concepts from vector algebra and geometry, can be employed to solve these optimization problems and identify optimal resource allocation strategies.

Geospatial Economics: Geospatial economics is an emerging field that combines economic analysis with spatial data and 3D analytical geometry. It involves incorporating geospatial information, such as geographic information systems (GIS) data, satellite imagery, or location-based

data, into economic models and analysis. Geospatial economics allows economists to study the impact of geography and spatial factors on economic outcomes, such as analysing the effects of transportation networks, land use patterns, or proximity to markets on economic development and productivity.

Financial Mathematics and Risk Analysis: In finance, 3D analytical geometry techniques can be applied to asset pricing, risk analysis, and portfolio optimization. Concepts such as efficient frontiers, asset pricing models, and risk measures can be analysed using geometric representations in 3D space. For example, portfolio optimization can be framed as finding the optimal combination of assets within a three-dimensional risk-return space. By leveraging techniques from 3D analytical geometry, economists and financial analysts can better understand the relationships between risk, return, and asset allocation.

✍ *These are some of the ways in which three-dimensional analytical geometry can be related to economic mathematics. By incorporating spatial analysis, optimization techniques, modelling, and visualization, economists can gain deeper insights into economic phenomena, make more informed decisions, and develop a more comprehensive understanding of economic systems*.

2.0 INITIATION IN THREE-DIMENTIONAL (3D) ANALYTICAL GEOMETRY

In three-dimensional (3D) analytical geometry, various elements are studied to understand the properties and relationships of geometric objects in three-dimensional space. Here are the key elements of 3D analytical geometry:

Coordinate System: A 3D coordinate system is used to locate points in space. It consists of three perpendicular axes (x, y, and z) that intersect at the origin. Points are represented by three coordinates (x, y, z) that specify their position relative to the origin.

Points: Points in 3D space are represented by their coordinates (x, y, z). Each coordinate represents the distance along the respective axis from the origin. Points can be used to define vertices, positions, or locations of geometric objects.

Lines: A line in 3D space can be described using parametric equations or vector equations. It is determined by a point on the line and a direction vector that indicates the line's orientation. Lines can be straight or curved and extend infinitely in both directions.

Planes: Planes in 3D space are flat surfaces that extend infinitely. They can be defined by an equation or by specifying three non-collinear points within the plane. The equation of a plane is typically represented as $Ax + By + Cz + D = 0$, where $A, B, C, and\ D$ are constants.

Distance: The distance between two points in 3D space can be calculated using the 3D version of the Pythagorean theorem. The distance formula is derived from the differences in the coordinates of the two points.

Vectors: Vectors are essential in 3D analytical geometry. They have magnitude and direction and are represented by an ordered set of three components (x, y, z). Vectors are used to describe translations, displacements, forces, velocities, and other quantities in 3D space.

Vector Operations: Vector operations in 3D analytical geometry include addition, subtraction, scalar multiplication, dot product, and cross product. These operations allow for the manipulation, combination, and analysis of vectors in various geometric contexts.

Curves and Surfaces: Curves and surfaces in 3D space are described using equations or parametric representations. Examples include circles, ellipses, parabolas, spheres, cylinders, cones, and tori (donuts). These curves and surfaces have unique mathematical properties and can be studied and analysed using analytical techniques.

Intersections: Analysing intersections between geometric elements is a key aspect of 3D analytical geometry. This involves determining the points or sets of points where lines, planes, curves, or surfaces intersect. Intersection calculations are based on solving equations or considering geometric properties.

Geometric Transformations: Geometric transformations, such as translations, rotations, reflections, and scaling, are applicable in 3D analytical geometry. These transformations modify the positions, orientations, or sizes of geometric objects in 3D space and are often described using matrix operations.

Geometric Applications: Applications of vector algebra in geometry, such as lines, planes, intersections, and reflections.

✍ *Understanding and applying these elements of 3D analytical geometry are crucial for analysing and visualizing complex geometric structures and relationships in various fields, including mathematics, physics, engineering, computer graphics, and architecture.*

2.1 VECTORS IN 3D SPACE. APPLICATIONS AND CALCULATIONS WITH VECTORS

Keywords: *Vector, Magnitude, Direction, Vector Space, Linear Combination, Linear Independence, Basis, Dimension, Euclidean Space, Cartesian Coordinates, Distance, Inner Product, Norm, Orthogonality, Euclidean Transformation*.

In the economic context, a vector in 3D space can represent various economic quantities, such as prices, quantities, incomes, or utility levels. Vectors are used to represent and analyse these economic variables and their relationships. In economics, a vector in 3D space is typically represented using mathematical notation.

For example, a vector representing prices of goods in a 3D space might be denoted as: $P = (p1, p2, p3)$. Here, $(p1, p2, p3)$ represents the prices of three goods. Each component of the vector corresponds to the price of a specific good. The vector P captures the pricing information for the goods in the $3D$ space.

Similarly, vectors can be used to represent other economic variables. For instance, a vector representing quantities demanded might be denoted as: $Q = (q1, q2, q3)$. In this case, $(q1, q2, q3)$ represents the quantities demanded for three goods. Each component of the vector corresponds to the quantity of a specific good. The vector Q captures the demand information for the goods in the $3D$ space.

These economic vectors can be operated on using vector algebra operations such as addition, subtraction, and scalar multiplication. These operations allow for analysing relationships between economic variables and making predictions or optimizations in economic models.

Vector in 3D space, and how is it represented: In three-dimensional $(3D)$ space, a vector is represented by magnitude (size), direction and orientation. It is a mathematical object that describes a displacement or a physical quantity with both a size and a direction.

Vectors in $3D$ space are typically represented as ordered triples of real numbers. A vector in 3D space can be denoted as: $V = (x, y, z)$. Here, (x, y, z) represents the components of the vector along the x, y, and z axes, respectively. These components indicate the magnitudes of the vector in each direction. The components of a vector can be positive, negative, or zero, depending on the direction and magnitude of the vector. The sign and magnitude of the components determine the overall orientation and length of the vector.

Example 2.1.a A vector $V = (1, 4, 5)$ in $3D$ space has a magnitude of $|V| = \sqrt{1^2 + 4^2 + 5^2} = \sqrt{42}$ and a direction determined by the ratios of the components $(1;\ 4;\ 5)$.

Vectors in 3D space can also be represented geometrically using directed line segments or arrows. The length of the arrow represents the magnitude of the vector, while the direction of the arrow indicates the vector's direction in space. The starting point of the arrow can be chosen arbitrarily since only the direction and length matter.

Example 2.1.b A vector $V = (1, 4, 5)$ in $3D$ space has the next geometrical representation:

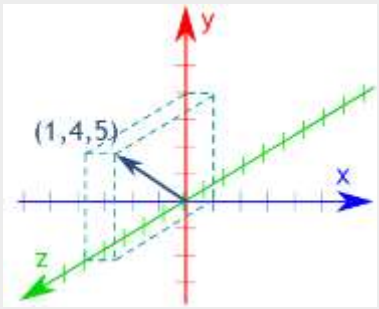


Figure 2.1 The geometric representation of the vector V =(1,4,5)

Vector notation can also be represented using boldface or underlined characters, such as $\bar{v}$ or $\boldsymbol{v}$, to distinguish vectors from other mathematical quantities like scalars.

The components of a vector in 3D space: In three-dimensional (3D) space, a vector is typically represented by three components. The components of a vector correspond to the differences in the coordinates of two points: the initial point and the terminal point. These components provide information about the magnitude and direction of the vector.

Initial Point: The initial point of the vector is often taken as the origin (0, 0, 0) in 3D space. This point serves as a reference from which the vector's components are measured.

Terminal Point: The terminal point of the vector is determined by the coordinates (x, y, z) in 3D space. These coordinates represent the displacement or difference between the initial point and the terminal point.

Components: The components of a vector in 3D space are calculated by subtracting the corresponding coordinates of the initial and terminal points. The components represent the differences in the x, y, and z coordinates

Example 2.1.c If the initial point of a vector is $A\ (1, 2, 3)$ and the terminal point is $B\ (4, 5, 6)$, the components of the vector $\overrightarrow{AB}$ would be: x-component: $4 - 1 = 3$, y-component: $5 - 2 = 3$, z-component: $6 - 3 = 3$. Thus, the vector would be represented as $\overrightarrow{AB} = (3, 3, 3)$ in $3D$ space.

Vector operations such as addition, subtraction, and scalar multiplication in 3D space: In three-dimensional (3D) space, vector operations such as addition, subtraction, and scalar multiplication can be performed using the corresponding operations on the vector components. Here's how each operation is carried out:

Vector Addition: To add two vectors in 3D space, simply add their corresponding components. For example, given two vectors $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$, the sum $\bar{a} + \bar{b}$ is calculated as: $\bar{a} + \bar{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. Each component of the resulting vector is obtained by adding the corresponding components of the two vectors.

Example 2.1.d If are given two vectors $\bar{a}\ (1, 2, 3)$ and $\bar{b}\ (4, 5, 6)$, $_3$), the sum $\bar{a} + \bar{b}$ is calculated as: $\bar{a} + \bar{b} = (1 + 4, 2 + 5, 3 + 6)$. Thus, the sum $\bar{a} + \bar{b} = (5, 7, 9)$.

Vector Subtraction: Similar to vector addition, vector subtraction is performed by subtracting the corresponding components. If $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$ are two vectors, the difference $\bar{a} - \bar{b}$ is calculated as: $\bar{a} - \bar{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$. Each component of the resulting vector is obtained by subtracting the corresponding components of the second vector from the corresponding components of the first vector.

Example 2.1.e If are given two vectors $\bar{a}\ (1, 2, 3)$ and $\bar{b}\ (4, 5, 6)$, the difference $\bar{a} - \bar{b}$ is calculated as: $\bar{a} - \bar{b} = (1 - 4,\ 2 - 5,\ 3 - 6)$. Thus, the difference $\bar{a} - \bar{b} = (-3, -3, -3)$.

Scalar Multiplication: Scalar multiplication involves multiplying a vector by a scalar (a real number). To scalar multiply a vector $\bar{a} = (a_1, a_2, a_3)$ by a scalar k, the resulting vector $\overline{ka}$ is obtained by multiplying each component of $\bar{a}$ by the scalar: $\overline{ka} = (ka_1, ka_2, ka_3)$. Each component of the resulting vector is obtained by multiplying the corresponding component of the original vector by the scalar.

Example 2.1.f If are given the vector $\bar{a}\ (1, 2, 3)$ and the scalar $k = -3$, the resulting of scalar multiplication involves the next resulting: $\overline{ka} = (-3 * 1, -3 * 2, -3 * 3)$. Thus, the scalar multiplication $\overline{ka} = (-3, -6, -9)$.

✍ *These vector operations can be used to perform calculations, manipulate vectors, and analyse relationships between vectors in 3D space. Additionally, these operations follow certain properties such as commutativity and associativity, which allow for further simplifications and algebraic manipulations when working with vectors in 3D space.*

The dot product of two vectors, and how is it calculated in 3D space: The dot product, also known as the scalar product or inner product, is an operation performed on two vectors that results in a scalar value. It measures the similarity or the extent to which two vectors are aligned in the same direction.

The dot product in 3D space: In $3D$ space, the dot product of two vectors $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$ is calculated using the following formula: $\bar{a} \cdot \bar{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The dot product involves multiplying the corresponding components of the two vectors and then summing up these products. The resulting value is a scalar quantity.

Example 2.1.g If are given two vectors $\bar{a}\,(1,2,3)$ and $\bar{b}\,(4,5,6)$, the dot product $\bar{a}\cdot\bar{b}$ is calculated as: $\bar{a}\cdot\bar{b} = 1*4 + 2*5 + 3*6$. Thus, the dot product $\bar{a}\cdot\bar{b} = 32$.

The properties of dot product: The dot product has several properties, including commutativity and distributivity, which can be useful in vector algebra. Some key properties of the dot product in 3D space are:

Commutativity: $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$ The dot product of two vectors is commutative, meaning the order of the vectors does not affect the result.

Distributivity: $\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$ The dot product distributes over vector addition. It allows for the dot product to be calculated separately for each vector and then summed.

Dot Product with the Zero Vector: $\bar{a}\cdot\bar{0} = \bar{0}$ The dot product of any vector with the zero vector $(0,0,0)$ is always zero.

The angle between two vectors: The dot product can be used to determine the angle between two vectors. If $\bar{a}$ and $\bar{b}$ are nonzero vectors, the angle θ between them can be found using the formula: $cos(\theta) = (\bar{a}\cdot\bar{b}) / (|\bar{a}|\,|\bar{b}|)$. Here, $|\bar{a}|$ and $|\bar{b}|$ represent the magnitudes (or lengths) of the vectors $\bar{a}$ and $\bar{b}$, respectively.

Example 2.1.h If are given two vectors $\bar{a}\,(1,2,3)$ and $\bar{b}\,(-3,1,-2)$, the angle between them $cos(\theta)$ is calculated as:
$cos(\theta) = (1*(-3) + 2*1 + 3*(-2))/\left(\sqrt{1^2+2^2+3^2} * \sqrt{(-3)^2+1^2+(-2)^2}\right)$. Thus, the dot product $cos(\theta) = -\frac{1}{2}$.

✍ *The dot product has various applications. It can be used to calculate the total value or total output resulting from the interaction between different sectors, total value of transactions, such as the total expenditure or total revenue. While the applications may not involve the dot product directly, they demonstrate how vector algebra and operations, including the dot product, can be employed in economic analysis to model relationships, measure changes, analyse factors of production, and evaluate investment decisions*.

The cross product of two vectors in 3D space: The cross product, also known as the vector product, is an operation performed on two vectors in three-dimensional (3D) space. Unlike the dot product, the cross product yields a new vector that is perpendicular (orthogonal) to the two input vectors. It provides a vector that is orthogonal to the plane formed by the original vectors.

The cross product of two vectors in 3D space: In $3D$ space, the cross product of two vectors $\bar{a} = (a_1,a_2,a_3)$ and $\bar{b} = (b_1,b_2,b_3)$ is calculated using the following formula: $\bar{a}\times\bar{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$. The resulting vector $(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is perpendicular to both $\bar{a}$ and $\bar{b}$ and is directed according to the *right-hand rule*.

Example 2.1.i If are given two vectors $\bar{a}\,(1,2,3)$ and $\bar{b}\,(4,5,6)$, the cross product $\bar{a}\times\bar{b}$ is calculated as: $\bar{a}\times\bar{b} = (2*6-3*5, 3*4-1*6, 1*5-2*4)$. Thus, the dot product $\bar{a}\times\bar{b} = (-3,6,-3)$.

The right-hand rule: The right-hand rule states that if you point the index finger of your right hand in the direction of $\bar{a}$ and the middle finger in the direction of $\bar{b}$, then the thumb points in the direction of the cross-product vector.

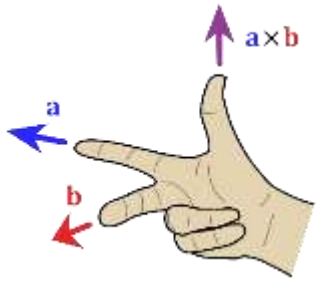


Figure 2.2 The illustration of right-hand rule

The right-hand rule is based on the orientation of the right hand and follows these guidelines:

- *Thumb*: Point your right thumb in the direction of the first vector involved in the cross product.
- *Index Finger*: Extend your right index finger in the direction of the second vector involved in the cross product.
- *Middle Finger*: Your right middle finger will then point in the direction of the resulting vector or the positive direction of rotation around the axis formed by the two vectors.

This rule applies to three-dimensional space since it involves the use of three fingers to represent the three dimensions (x, y, z) of the coordinate system.

Example 2.1.j The right-hand rule can be applied is in determining the direction of exchange rate movements in a currency pair. Consider a currency pair, such as USD/EUR. The exchange rate indicates the value of one currency in terms of the other. To determine the direction of exchange rate movements using the right-hand rule, you can follow these steps:

- Extend your right hand and position your thumb upward to represent the first currency in the pair. For example, let's say you align your thumb with USD.
- Curl your fingers toward the direction of the second currency in the pair, which would be EUR in this case.
- The direction in which your fingers curl represents the direction of exchange rate movements. If your fingers curl clockwise, it indicates an increase in the value of USD relative to EUR. If your fingers curl counter clockwise, it indicates a decrease in the value of USD relative to EUR.

✍ *By applying the right-hand rule in this scenario, you can intuitively determine the direction of exchange rate movements between the two currencies. This can be helpful in understanding the potential impact on international trade, investment flows, and exchange rate dynamics in the context of currency markets and economic analysis.*

Properties of the vector product: The cross product has several properties, including skew-symmetry and linearity, which make it useful in vector algebra. Some key properties of the cross product in 3D space are:

Skew-Symmetry: $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$. The cross product of two vectors is anti-commutative, meaning that reversing the order of the vectors changes the sign of the resulting vector.

Linearity: $(k\bar{a}) \times \bar{b} = k(\bar{a} \times \bar{b}) = \bar{a} \times (k\bar{b})$. The cross product is linear with respect to scalar multiplication. Multiplying a vector by a scalar before or after taking the cross product does not change the direction of the resulting vector.

✍ *The cross product can be used in various applications, including relative costs or productivities of different countries, the cross-sector flows, which represent the inter-industry relationships and dependencies within an economy, etc. While these applications may not involve the cross product directly, they demonstrate how vector algebra and operations, including the cross product, can be employed in economic analysis to model relationships, analyse interdependencies, evaluate factors of production, and understand economic growth and development.*

The unit vector: $\bar{u}$, and how is it related to a vector's direction and magnitude? In mathematics a unit vector is a vector that has a magnitude (or length) of exactly 1. Unit vectors are commonly used to represent directions or orientations in a vector space. They provide information about the direction of a vector without considering its magnitude.

The vector normalization: To obtain a unit vector from a given vector, you divide the vector by its magnitude. This process is known as normalization. The resulting vector will have the same direction as the original vector but with a magnitude of 1. The unit vector is typically denoted by adding a hat symbol (^) on top of the vector symbol, such as $\hat{a}$. Mathematically, the unit vector $\bar{u}$ associated with a vector $\bar{a} = (a_1, a_2, a_3)$ can be calculated as: $\bar{u} = \bar{a} \,/\, |\bar{a}|$. Here, $|\bar{a}|$ represents the magnitude of vector $\bar{a}$. By dividing each component of the vector by its magnitude, you obtain a vector that points in the same direction as $\bar{a}$ but with a length of 1. The unit vector captures the directionality of a vector while ignoring its magnitude.

Direction Representation: Unit vectors are commonly used to represent directions or orientations in vector spaces. By normalizing a vector to obtain its unit vector, you obtain a vector that captures the direction of the original vector without considering its magnitude.

Example 2.1.k Let's consider an example of calculating the direction representation using unit vectors. Suppose we have a vector $\bar{a} = (2, -3, 1)$ in three-dimensional space, and we want to find its direction representation. To calculate the direction representation, we need to normalize the vector by dividing it by its magnitude. Here are the steps involved:

Calculate the magnitude of the vector: $|\bar{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

Divide each component of the vector by its magnitude to obtain the unit vector:

$$\hat{a} = \bar{a} \,/\, |\bar{A}| = \left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$$

The resulting unit vector $\hat{a}$ represents the direction of the original vector $\bar{a}$. It maintains the same direction but has a magnitude of 1.

Coordinate Systems: Unit vectors play a crucial role in defining coordinate systems. In Cartesian coordinate systems, unit vectors along the x, y, and z axes ($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$) are used to establish a basis for the coordinate system. These unit vectors, with a magnitude of 1, indicate the positive direction along each axis and provide a reference for measuring distances and angles in space.

The analytical expression of a vector: The analytical expression of a vector in terms of unit vectors along the axes is a way to represent a vector using its components along the coordinate axes. In a three-dimensional Cartesian coordinate system with x, y, and z axes, the unit vectors along each axis are typically denoted as i, j, and k, respectively. Suppose we have a vector $\bar{a}$ in 3D space with components a_x, a_γ, and a_z. The analytical expression of vector $\bar{a}$ using unit vectors along the axes can be written as: $\bar{a} = a_x * \boldsymbol{i} + a_y * \boldsymbol{j} + a_z * \boldsymbol{k}$**.** Here, a_x represents the component of vector $\bar{a}$ along the $x - axis$, a_y represents the component of vector $\bar{a}$ along the $y - axis$, a_z represents the component of vector $\bar{a}$ along the $z - axis$. The multiplication of each component by its corresponding unit vector results in the vector A expressed in terms of the unit vectors.

Example 2.1.l If $\bar{a} = (2, 3, -1)$, the analytical expression of vector $\bar{a}$ using unit vectors would be: $\bar{a} = 2 * \boldsymbol{i} + 3 * \boldsymbol{j} - 1 * \boldsymbol{k}$.

This expression explicitly shows the contribution of each component along the respective axis in forming the vector $\bar{a}$.

✍ *Using the analytical expression of a vector, you can easily perform vector operations, such as addition, subtraction, scalar multiplication, dot product, and cross product, by operating on the corresponding components of the vectors*.

Direction Cosines: Unit vectors are used to calculate direction cosines, which represent the cosine of the angle between a vector and each of the coordinate axes. Direction cosines provide information about the orientation and alignment of a vector in relation to the coordinate system. They are used in geometric calculations, rotations, and transformations.

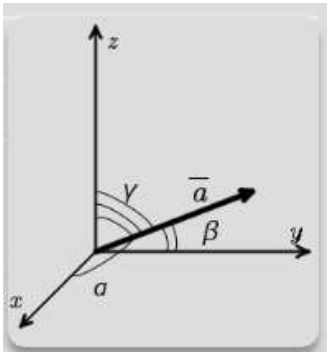


Figure 2.3 The illustration of direction cosines

Example 2.1.m To calculate the direction cosines of a vector, we need to determine the cosines of the angles between the vector and each of the coordinate axes.

Suppose we have a vector $\bar{v} = (3, 4, 5)$. To calculate the direction cosines, we divide each component of the vector by the magnitude of the vector. The steps involved are as follows:

- *Calculate the magnitude of the vector*: $|\bar{v}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$
- *Divide each component of the vector by the magnitude*: Direction cosine along the x-axis, $cos\alpha = \frac{v_x}{|\bar{v}|} = \frac{3}{5\sqrt{2}}$. Direction cosine along the y-axis, $cos\beta = \frac{v_y}{|\bar{v}|} = \frac{4}{5\sqrt{2}}$. Direction cosine along the z-axis, $cos\gamma = \frac{v_z}{|\bar{v}|} = \frac{5}{5\sqrt{2}}$.
- *Simplifying the above expressions, we get*:

  $\cos(\alpha) = 3/(5\sqrt{2})$, $\cos(\beta) = 4/(5\sqrt{2})$, $\cos(\gamma) = 5/(5\sqrt{2})$

- *Therefore, the direction cosines of the vector $\bar{v}$ are*:

$$cos\beta = \frac{4}{5\sqrt{2}} \quad cos\beta = \frac{4}{5\sqrt{2}} \quad cos\gamma = \frac{5}{5\sqrt{2}}$$

✍ *The direction cosines represent the cosines of the angles between the vector and each of the coordinate axes. They provide information about the orientation and alignment of the vector in relation to the coordinate system.*

Projection of one vector on another: The projection of a vector $\bar{a}$ on a nonzero vector $\bar{b}$ is the orthogonal projection of $\bar{a}$ onto a straight line parallel to $\bar{b}$. The projection of vector $\bar{a}$ on $\bar{b}$ is: $proj_{\bar{b}}\bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\cdot\bar{b}$. The scalar projection of $\bar{a}$ on $\bar{b}$ is the magnitude of the vector projection of $\bar{a}$ on $\bar{b}$: $|proj_{\bar{b}}\bar{a}| = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$

Example 2.1.o To calculate the projection of vector $\bar{a}$ on $\bar{b}$ and their scalar projection, if $\bar{a} = (2,1,1)$ and $\bar{b} = (3,-1,2)$.

- *Calculate the dot product of vectors $\bar{a}$ and $\bar{b}$:* $\bar{a}\cdot\bar{b} = 2*3-1*1+1*2 = 7$
- *Calculate the magnitude of the vector $|\bar{b}|$:* $|\bar{b}| = \sqrt{3^2+(-1)^2+2^2} = \sqrt{14}$
- Calculate the vector projection of $\bar{a}$ on $\bar{b}$: $proj_{\bar{b}}\bar{a} = \frac{7}{\sqrt{14}^2} * (3,-1,2) = \left(\frac{3}{2},-\frac{1}{2},1\right)$
- Calculate the scalar projection of $\bar{a}$ on $\bar{b}$: $|proj_{\bar{b}}\bar{a}| = \frac{7}{\sqrt{14}}$

✍ *The projection of one vector onto another is useful in various applications, such as understanding vector components, decomposing vectors into orthogonal components, and analysing vector relationships.*

Scalar triple product: The scalar triple product, also known as the triple scalar product or mixed product, is a mathematical operation that combines three vectors to produce a scalar value. It involves taking the dot product of the cross product of two vectors with a third vector. The scalar triple product is denoted as $[\bar{a},\bar{b},\bar{c}]$, where $\bar{a}$, $\bar{b}$, and $\bar{c}$ are vectors. The formula for the scalar triple product is as follows: $[\bar{a},\bar{b},\bar{c}] = \bar{a}\cdot\bar{b}\times\bar{c}$. To calculate the scalar triple product, follow these steps:

- *Calculate the cross product of vectors $\bar{b}$ and $\bar{c}$:* $\bar{b}\times\bar{c}$.
- *Take the dot product of vector $\bar{a}$ with the result obtained precedent step*: $\bar{a}\cdot\bar{b}\times\bar{c}$.
- *The resulting value is a scalar quantity*.

Example 2.1.p To calculate the mixt product for the next three vectors: $\bar{a} = (2,1,-3)$, $\bar{b} = (3,0,5)$, $\bar{c} = (1,2,4)$:

- *Calculate the cross product of vectors $\bar{b}$ and $\bar{c}$*:

$\bar{b}\times\bar{c} = (0-5*2, 5*1-3*4, 3*2-0) = (-10,-7,6)$.

- *Take the dot product of vector $\bar{a}$ with the result obtained from step 1*:

$\bar{a}\cdot\bar{b}\times\bar{c} = 2*(-10)+1*(-7)+(-3)*6 = -45$.

- *The mixt product of vectors $\bar{a},\bar{b},\bar{c}$ is*: $\bar{a}\cdot\bar{b}\times\bar{c} = -45$

✍ *The scalar triple product has a geometric interpretation. Its value represents six times the signed volume of the parallelepiped formed by the three vectors A, B, and C. The sign of the scalar triple product depends on the order of the vectors and determines the orientation of the volume. The*

scalar triple product is used in calculations involving moments, torque, volumes, determinants, and geometric properties of three-dimensional objects.

Collinear vectors: Collinear vectors are vectors that lie on the same straight line or are parallel to each other. In other words, two or more vectors are collinear if they have the same or proportional directions. Collinear vectors can be represented by scalar multiples of each other.

Here are a few properties and characteristics of collinear vectors:

Same Direction: Collinear vectors have the same direction, which means they either point in the same direction or are parallel to each other.

Proportional Magnitude: Collinear vectors have magnitudes that are either equal or proportional to each other. If vector $\bar{a}$ is collinear with vector $\bar{b}$, then $\bar{a} = k * \bar{b}$, where k is a non-zero scalar.

Common Line: Collinear vectors lie on the same line, either in the same or opposite directions. They can be visualized as points along a line or as vectors pointing along a line.

Linear Dependence: Collinear vectors are linearly dependent, meaning that one vector can be expressed as a linear combination of the others. For example, if vector $\bar{a}$ is collinear with vector $\bar{b}$, then $\bar{a} = k * \bar{b}$, for some non-zero scalar k.

Zero Vector: The zero vector $(0, 0, 0)$ is considered collinear with all other vectors since it has no direction and can be multiplied by any scalar to obtain the zero vector itself.

✍ *Identifying collinear vectors is useful in simplifying vector calculations, determining linear relationships between vectors, and understanding geometric or physical properties of objects.*

Vector parallel to one line or lying on one line are called collinear vectors.

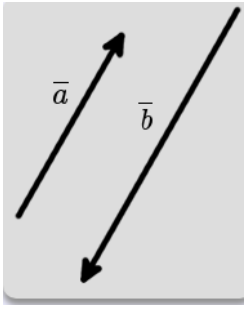


Figure 2.3 The illustration of the collinear vectors

Two vectors $\bar{a} = (a_x, a_y, a_z)$ and $\bar{b} = (b_x, b_y, b_z)$ are collinear if:

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z} \Leftrightarrow \bar{a} = k\bar{b}.$$

Example 2.1.r To determine if the vectors $\bar{a} = (2, 1, -3)$ and $\bar{b} = (3, 0, 5)$ are colliniars.

- *We check if the relationship* $\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$ *occurs:* $\frac{2}{3} = \frac{1}{1,5} = \frac{-3}{-4,5}$
- *Conclusion:* Since there exists a k such that $\bar{a} = k\bar{b}$, it follows that the vectors $\bar{a}$ and $\bar{b}$ are collinear.

✍ *In linear algebra, collinear vectors are studied in the context of vector spaces, linear dependence, and basis vectors*.

Orthogonal vectors: Orthogonal vectors, also known as perpendicular vectors, are vectors that are at right angles to each other. In other words, the dot product (scalar product) of orthogonal vectors is zero. Geometrically, orthogonal vectors form a 90-degree angle between them.

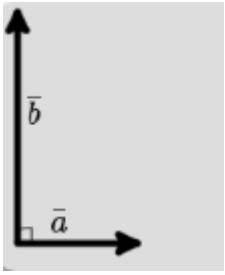


Figure 2.4 The illustration of the orthogonal vectors

Here are some properties and characteristics of orthogonal vectors:

Dot Product: The dot product of orthogonal vectors $\bar{a}$ and $\bar{b}$ is zero. Mathematically, $\bar{a} \cdot \bar{b} = 0$. This property can be used to determine if two vectors are orthogonal.

Angle: The angle between orthogonal vectors is 90 degrees or $\frac{\pi}{2}$ radians. The cosine of the angle between orthogonal vectors is zero since the dot product is zero.

Linear Independence: Orthogonal vectors are linearly independent, meaning they cannot be expressed as scalar multiples of each other. They provide a basis for constructing three-dimensional space when combined with a third linearly independent vector.

Perpendicularity: Orthogonal vectors are perpendicular to each other. If two vectors are orthogonal, the lines or planes they span are also orthogonal.

Orthogonal Projections: Orthogonal vectors are used in projection calculations. The orthogonal projection of a vector onto another vector involves decomposing the vector into two components: one parallel to the target vector and one orthogonal to it.

Orthogonal Matrices: In linear algebra, orthogonal matrices are square matrices whose columns (or rows) form an orthogonal set of vectors. The columns (or rows) of an orthogonal matrix are orthogonal unit vectors: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

✍ *Orthogonal vectors are crucial in various mathematical and scientific applications. They provide a basis for vector spaces, simplify calculations, determine perpendicularity, and enable the decomposition of vectors into orthogonal components*.

Coplanar vectors: Coplanar vectors are vectors that lie in the same plane. In three-dimensional space, a plane is a two-dimensional surface that extends infinitely in all directions. Coplanar vectors can be represented by vectors that lie within the same plane and can be expressed as linear combinations of each other.

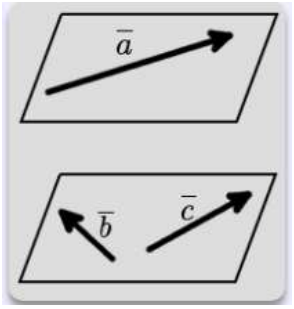


Figure 2.5 The illustration of the coplanar vectors

Here are some properties and characteristics of coplanar vectors:

Same Plane: Coplanar vectors lie in the same plane, meaning they can be contained within a two-dimensional surface that extends infinitely in all directions.

Linear Dependence: Three coplanar vectors are linearly dependent, which means that one vector can be expressed as a linear combination of the others. For example, if vectors $\bar{a}$, $\bar{b}$, and $\bar{c}$ are coplanar, then $\bar{c}$ can be written as a linear combination of $\bar{a}$ and $\bar{b}$: $\bar{c} = \lambda\bar{a} + \mu\bar{b}$, where λ and μ are scalars.

Common Normal Vector: Coplanar vectors share a normal vector that is perpendicular to the plane in which they lie. The normal vector can be obtained by taking the cross product of any two non-parallel coplanar vectors.

✍ *The concept of coplanar vectors is important. Identifying coplanar vectors is useful in simplifying vector calculations, determining linear relationships between vectors, and understanding geometric or physical properties within a plane.*

Basis of vectors: Vectors can form a basis for a vector space if they satisfy certain conditions:

- *Linear Independence*: The vectors must be linearly independent, which means that no vector in the set can be expressed as a linear combination of the others. If every vector in the set is linearly independent, it implies that no vector is redundant or unnecessary.
- *Spanning the Vector Space*: The vectors must span the entire vector space, meaning that any vector in the space can be written as a linear combination of the basis vectors. This condition ensures that no vector is left out or cannot be expressed using the basis vectors.

If both conditions are satisfied, then the set of vectors forms a basis for the vector space.

Example 2.1.s Consider the following set of three vectors in three-dimensional space:

$$W_1 = (1, 2, 0)\ W_2 = (-1, 0, 3)\ W_3 = (0, 1, -2)$$

To determine if these vectors form a basis, we need to check the conditions of linear independence and spanning the vector space:

- *Linear Independence*: To check if the vectors are linearly independent, we need to determine if any vector in the set can be expressed as a linear combination of the others. In this case, we can check for linear independence by row-reducing the matrix formed by these vectors:

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & -2 \end{pmatrix}$$

Performing row operations, we can reduce the matrix to row-echelon form:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{pmatrix}$$

Since there are no rows of zeros, we can see that the vectors are *linearly independent*.

- *Spanning the Vector Space*: To check if the vectors span the entire vector space, we need to verify if any vector in three-dimensional space can be expressed as a linear combination of the given vectors. In this case, any vector (x, y, z) can be written as: $(x, y, z) = aW_1 + bW_2 + cW_3$. The system is:

$$\begin{cases} a - b = x \\ 2a + c = y \\ 3y + 2z = z \end{cases}$$

By solving the system of equations, we can find the values of a, b, and c that satisfy the equation for any (x, y, z):

$$\begin{cases} a - b = x \\ 2a + c = y \\ 3y + 2z = z \end{cases} \Leftrightarrow \begin{cases} a = \frac{11}{7}x - \frac{2}{7}y - \frac{1}{7}z \\ b = -\frac{10}{7}x + \frac{5}{7}y - \frac{1}{7}z \\ c = \frac{6}{7}x - \frac{3}{7}y + \frac{2}{7}z \end{cases}$$

- *Conclusion*: Since we have three equations (one for each coordinate), we can find a unique solution for each vector in three-dimensional space. Therefore, the vectors $W_1 = (1, 2, 0)$, $W_2 = (-1, 0, 3)$, and $W_3 = (0, 1, -2)$ form a basis for three-dimensional space.

✍ *Bases are fundamental in linear algebra and are used to represent and analyse vectors and linear transformations. They provide a way to describe vectors in terms of a minimal set of independent vectors. By expressing vectors in terms of a basis, you can perform various calculations, transformations, and analyses in a more simplified and structured manner. It's important to note that not all sets of vectors form a basis. Some sets may be linearly dependent, meaning that one or more vectors can be expressed as linear combinations of others. In such cases, the set does not form a basis as it fails the condition of linear independence*.

Discompose the vector in the given basis: To decompose a vector into a basis, you need to find the scalar coefficients or weights that represent the vector as a linear combination of the basis vectors. The process involves solving a system of equations based on the given vector and the basis vectors.

Example 2.1.t Let's consider an example where we decompose a vector into a given basis:

Vector: $\bar{v} = (3, -2, 5)$ Basis: $\overline{b_1} = (1, 1, 0)$, $\overline{b_2} = (0, 1, 1)$, $\overline{b_3} = (1, 0, 1)$.

To decompose vector V into the given basis, we need to find the coefficients a, b, and c such that: $\bar{v} = a\overline{b_1} + b\overline{b_2} + c\overline{b_3}$. Substituting the given values, we have: $(3, -2, 5) = a * (1, 1, 0) + b * (0, 1, 1) + c * (1, 0, 1)$

This leads to a system of linear equations:

$$\begin{cases} a + c = 3 \\ a + b = -2 \\ b + c = 5 \end{cases}$$

Solving this system of equations, we find the values of a, b, and c:

$$\begin{cases} a = -2 \\ b = 0 \\ c = 5 \end{cases}$$

Therefore, the vector $\bar{v} = (3, -2, 5)$ can be decomposed as:

$$\bar{v} = -2 * (1, 1, 0) + 0 * (0, 1, 1) + 5 * (1, 0, 1)$$

Or in expanded form: $\bar{v} = (3, -2, 5)$

Thus, the vector $\bar{v}$ can be expressed as a linear combination of the given basis vectors with the coefficients (-2, 0, 5).

Applications of vectors in 3D space related to the field of economics:

International Trade Analysis: Vectors can be used to represent trade flows between countries in terms of imports and exports. Each country's import and export vectors can capture the direction and magnitude of trade for different goods or sectors. Analysing these vectors can provide insights into trade patterns, comparative advantage, and the balance of trade between countries.

Input-Output Analysis: Input-output analysis examines the interdependencies between different sectors of an economy. Vectors can be used to represent the flows of goods and services between sectors. By analysing the magnitudes and directions of these vectors, economists can study the relationships and dependencies among sectors and identify key sectors within the economy.

Portfolio Management: Vectors can be used to represent the returns and risks associated with different assets in a portfolio. Each asset's return and risk vectors capture the historical performance and volatility of the asset. Vector operations, such as dot product or vector addition, can be applied to calculate portfolio returns, diversification benefits, and risk management strategies.

Economic Networks: Vectors can represent economic networks, such as supply chains or financial networks. In supply chain analysis, vectors can represent the flow of goods and services between different nodes or entities. In financial networks, vectors can represent the linkages and interactions between financial institutions or markets. Analysing the vectors of economic networks can provide insights into system resilience, risk propagation, and the impact of disruptions.

Economic Geography: Vectors can be used to represent spatial flows of economic activity. For example, vectors can represent migration patterns, transportation routes, or the movement of capital and investment. By analysing these vectors, economists can study the spatial distribution of economic activity, regional development, and the impact of transportation infrastructure on trade and economic integration.

✍ *It's important to note that in the economic context, the interpretation and meaning of vectors in 3D space may vary depending on the specific economic scenario and the variables being represented. The representation of vectors in the economic context allows economists to analyse and understand economic relationships, optimize decisions, and make predictions about economic outcomes.*

Questions you should be able to answer:

What is a vector in 3D space, and how is it represented?

What are the components of a vector, and how are they related to the coordinates of a point in 3D space?

How do you perform vector operations such as addition, subtraction, and scalar multiplication in 3D space?

What is the dot product of two vectors, and how is it calculated in 3D space?

What is the cross product of two vectors, and how is it calculated in 3D space?

How do you find the magnitude or length of a vector in 3D space?

What is the unit vector, and how is it related to a vector's direction and magnitude?

How do you calculate the angle between two vectors in 3D space?

What are some common applications of vectors in 3D space?

Proposed Exercises:

2.1.1 For each exercise, fill in the missing words to complete the statements related to the theory of three-dimensional analytical geometry. These exercises aim to reinforce the concepts and formulas used in three-dimensional space:

a) Three-Dimensional Cartesian Coordinate: System Fill in the blanks with the missing words: *a.1)* In three-dimensional analytical geometry, a point in space is represented by its _____ coordinates (x, y, z). *a.2)* The three axes in the three-dimensional Cartesian coordinate system are _____, _____, and _____, representing the $x, y, and\ z$ directions, respectively. *a.3)* The intersection of the three axes is called the _____.

b) Distance Formula in 3D Space: Fill in the blanks with the missing words: *b.1)* The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three-dimensional space can be found using the _____ formula. *b.2)* The distance d between P_1 and P_2 is given by the formula: $d = \sqrt{((x_2 - x_1)^2 + (y_2 - __)^2 + (__ - z_1)^2}$.

c) Equation of a Sphere: Fill in the blanks with the missing words: *c.1)* In three-dimensional space, a sphere is a set of points that are equidistant from a fixed point called the _____. *c.2)* The equation of a sphere with centre (h, k, l) and radius r is given by: $(x - h)^2 + (y - ___)^2 + (___ - l)^2 = r^2$.

d) Equation of a Plane: Fill in the blanks with the missing words: *d.1)* A plane in three-dimensional space is determined by a point and a _____ that is perpendicular to the plane. *d.2)* The equation of a plane passing through point $P_0(x_0, y_0, z_0)$ with normal vector $N(a, b, c)$ is given by: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

e) Direction Cosines and Direction Ratios: Fill in the blanks with the missing words: *f.1)* The _____ of a line in three-dimensional space are the cosines of the angles that the line makes with the positive x, y, and z axes. *f.2)* The _____ of a line in three-dimensional space are the ratios of the components of the direction vector along the *x*, *y*, and *z* axes.

f) Vector Product (Cross Product): Fill in the blanks with the missing words: f.*1)* The vector product or cross product of two vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ in three-dimensional space is a vector $\boldsymbol{c} = (c_1, c_2, c_3)$ given by: $c_1 =$ _____, $c_2 =$ _____, $c_3 =$ _____.

g) Scalar Triple Product: Fill in the blanks with the missing words: g.1) The scalar triple product of three vectors ***a***, ***b***, and ***c*** in three-dimensional space is defined as: $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} =$ _____.

h) Angle between Two Vectors: Fill in the blanks with the missing words: h.1) The angle $\boldsymbol{\theta}$ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in three-dimensional space is given by: $cos\ \theta\ = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}| * |\boldsymbol{b}|}$.

2.1.2 For each exercise, carefully compute the magnitude, direction, and any required geometric representation of the given vectors. Representing the vectors graphically in a three-dimensional coordinate system will help visualize their relationships and properties. Additionally, make sure to understand the concepts of magnitude, direction, dot product, and cross product of vectors in three-dimensional space:

a) Consider the vector $\boldsymbol{v}$ in three-dimensional space, $\boldsymbol{v}\ =\ (4, -3, 6)$: *a.1)* Find the magnitude (length) of vector v. *a.2)* Interpret the magnitude of the vector in the context of three-dimensional space.

b) Consider the vector $\boldsymbol{u}$ in three-dimensional space, $\boldsymbol{u}\ =\ (-2, 5, -1)$: *b.1)* Find the direction cosines of vector $\boldsymbol{u}$. *b.2)* Interpret the direction cosines in the context of three-dimensional space.

c) Consider the vector $\boldsymbol{c}$ in three-dimensional space, $\boldsymbol{c}\ =\ (1, 2, -3)$: *c.1)* Geometrically represent vector c on a 3D coordinate system. *c.2)* Find two-unit vectors that are parallel to vector $\boldsymbol{c}$.

d) Consider the vectors $\boldsymbol{d}\ =\ (2, 1, 3)$ and $\boldsymbol{e}\ =\ (-1, 3, 2)$. Draw the geometric representation of $\boldsymbol{d}$ and $\boldsymbol{e}$ in three-dimensional space, using appropriate scales on the axes.

e) Given the vector $\boldsymbol{f}\ =\ (4, -3, 5)$, find its magnitude and the angles it makes with the positive *x, y,* and *z-axes*.

f) Given two vectors $\boldsymbol{g}\ =\ (1, 2, 3)$ and $\boldsymbol{h}\ =\ (-2, 1, 0)$, find the projection of $\boldsymbol{g}$ onto $\boldsymbol{h}$.

g) Consider the vector $\boldsymbol{v}\ =\ (2, -3, 5)$: *g.1)* Find the magnitude (length) of vector $\boldsymbol{v}$. *g.2)* Determine the direction cosines of vector $\boldsymbol{v}$. *g.3)* Represent vector $\boldsymbol{v}$ graphically in a three-dimensional coordinate system.

h) Consider the vector $\boldsymbol{u}\ =\ (4, 1, -2)$ and the vector $\boldsymbol{w}\ =\ (-3, 2, 6)$: *h.1)* Find the magnitude (length) of vector u and vector w. *h.2)* Determine the angle between vectors $\boldsymbol{u}$ and $\boldsymbol{w}$. *h.3)* Represent vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ graphically in a three-dimensional coordinate system.

2.1.3 Remember to perform the vector operations correctly by adding or subtracting corresponding components in 3D space and properly multiplying vectors with scalar values. These exercises will help reinforce your understanding of vector operations in three-dimensional analytical geometry.

a) Let vector $\boldsymbol{a}$ be *(3, -2, 5)* and vector $\boldsymbol{b}$ be *(-1, 4, 7)*: *a.1)* Find the sum of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *a.2)* Find the scalar multiplication of vector $\boldsymbol{a}$ with a scalar value of *2*. *a.3)* Find the subtraction of vector $\boldsymbol{a}$ from vector $\boldsymbol{b}$.

b) Consider the following vectors $\boldsymbol{p}$: *(2, 5, -3)*, $\boldsymbol{q}$: *(-4, 1, 6)* and $\boldsymbol{r}$: *(1, -2, 3): b.1)* Find the sum of vectors $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$. *b.2)* Find the scalar multiplication of vector $\boldsymbol{p}$ with a scalar value of *3*. *b.3)* Find the result of $\boldsymbol{p - q + 3 * r}$.

c) Let vector $\boldsymbol{x}$ be *(1, 2, 3)* and vector $\boldsymbol{y}$ be *(-2, 0, 4)*: *c.1)* Find the sum of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. *c.2)* Find the scalar multiplication of vector $\boldsymbol{y}$ with a scalar value of *-3*. *c.3)* Find the result of $5 * \boldsymbol{x - y}$.

d) Consider the following vectors: $\boldsymbol{a} = (5, -2, 1)$ and $\boldsymbol{b} = (-3, 7, -4)$*: d.1)* Find the sum of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *d.2)* Find the scalar multiplication of vector $\boldsymbol{b}$ with a scalar value of *4*. *d.3)* Find the result of $3 * \boldsymbol{a} - (-2) * \boldsymbol{b}$.

2.1.4 In these exercises, you will apply the concept of the dot product to calculate the scalar product of vectors and determine their orthogonality or efficiency in different economic contexts. Additionally, the exercises involve interpreting the results of the dot product calculations in the context of economic mathematics, such as demand, production factors, and resource allocation.

a) Dot Product Computation Consider the following vectors in 3D space $\boldsymbol{a} = (2, 3, -1), \boldsymbol{b} = (-1, 4, 5)$: *a.1)* Calculate the dot product (scalar product) of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *a.2)* Interpret the result in the context of economic mathematics.

b) Vector Projection and Economic Demand: Suppose there are two goods, X and Y, in an economy. The quantity demanded for goods X and Y are represented by the following vectors $\boldsymbol{x} = (3, 2, 4)$, $\boldsymbol{y} = (1, 5, 2)$: *b.1)* Calculate the dot product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. *b.2)* Find the magnitude of vector $\boldsymbol{x}$ and vector $\boldsymbol{y}$. *b.3)* Calculate the projection of vector $\boldsymbol{x}$ onto vector $\boldsymbol{y}$. *b.4)* Interpret the projection in the context of economic demand for goods X and Y.

c) Orthogonality in Production Factors: Consider three production factors in an economic model: Labour (*L*), Capital (*K*), and Technology (*T*). The availability of these factors is represented by the following vectors: $\boldsymbol{l} = (2, 1, 3)$, $\boldsymbol{k} = (3, 1, -2)$, $\boldsymbol{t} = (-1, 5, 4)$: *c.1)* Determine whether the factors of production (Labour, Capital, and Technology) are orthogonal (perpendicular) to each other by calculating their pairwise dot products. *c.2)* Interpret the results in the context of economic production.

d) Efficiency of Resource: Allocation In an economic system, there are three resources: Land (*L*), Labour (*A*), and Capital (*K*). The allocation of these resources to produce two goods, *X* and *Y*, is represented by the following vectors $\boldsymbol{x} = (2, 5, 3)$, $\boldsymbol{y} = (4, 2, 1)$: *d.1)* Calculate the dot product of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. *d.2)* Find the magnitude of vector $\boldsymbol{x}$ and vector $\boldsymbol{y}$. *d.3)* Determine whether the resource allocation is efficient based on the dot product and the magnitudes. *d.3)* Interpret the results in the context of economic resource allocation.

2.1.5 For each exercise, perform the cross-product calculations step-by-step and interpret the results in the context of the given economic problem. The exercises focus on understanding the properties and applications of the cross product in 3D space, particularly in economic mathematics using vectors.

a) Cross Product Computation: Given the following vectors: $\bar{a} = (2, -1, 4)$, $\bar{b} = (3, 2, -1)$. Compute the cross product $\bar{a} \times \bar{b}$.

b) Area of Parallelogram: A company operates in three markets: X and Y. The sales growth rates in each market over two years are represented by the vectors: $\bar{x} = (3, 5, -2)$, $\bar{y} = (-2, 1, 4)$, respectively. Compute the area of the parallelogram formed by vectors $\bar{x}$ and $\bar{y}$ representing the growth rates in markets X and Y.

c) Application in Economics: Production Possibility Frontier (PPF) Consider an economy that produces two goods, X and Y. The production possibilities are represented by the vectors $\bar{x} =$

$(4, 0, 3)$, $\bar{y} = (0, 6, 2)$, respectively: *c.1)* Compute the cross product of vectors $\bar{x}$ and $\bar{y}$. b) Interpret the cross product in the context of the Production Possibility Frontier (PPF) of the economy.

d) Application in Economics: Exchange Rate The exchange rate between two currencies, A and B, is represented by the vectors: $\bar{a} = (1, 0, -2)$, $\bar{b} = (2, 3, -1)$, respectively: *d.1)* Compute the cross product of vectors $\bar{a}$ and $\bar{b}$. *d.2)* Interpret the cross product in the context of the exchange rate between currencies A and B.

2.1.6 In these exercises, you will need to find the magnitude of vectors, normalize vectors, and apply the concept of vector normalization in various economic contexts. Focus on understanding the concept of vector normalization and its application in economic mathematics to solve the exercises effectively.

a) Vector Normalization: Consider a 3D vector representing the profit and cost of a product $\boldsymbol{p}\ (profit) = (6, 8, 10)$, $\boldsymbol{c}\ (cost) = (12, 15, 18)$: *a.1)* Find the magnitude (length) of vectors $\boldsymbol{p}$ and C $\boldsymbol{c}$. *a.2)* Normalize vectors $\boldsymbol{p}$ and $\boldsymbol{c}$ by dividing each component by their respective magnitudes.

b) Profit Maximization: Suppose you have two 3D vectors representing the profits and resource utilization for two products: $\boldsymbol{p_1}\ (profit) = (5, 7, 10)$, $\boldsymbol{p_2}\ (profit) = (9, 12, 6)$ and $\boldsymbol{r_1}\ (resource\ utilization) = (2, 3, 5)$, $\boldsymbol{r_2}\ (resource\ utilization) = (6, 4, 8)$: *b.1)* Normalize vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$. *b.2)* Determine the total profit if you produce 5 units of $\boldsymbol{p_1}$ and 3 units of $\boldsymbol{p_2}$ while utilizing resources $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ optimally.

c) Portfolio Optimization: Consider a portfolio of three assets with their expected returns and standard deviations: Asset A: Return = 0.08, Standard Deviation = 0.12 Asset B: Return = 0.12, Standard Deviation = 0.15 Asset C: Return = 0.10, Standard Deviation = 0.10; *c.1)* Represent each asset as a 3D vector with components (Return, Standard Deviation, 1). *c.2)* Normalize the vectors for each asset. *c.3)* Find the optimal portfolio by choosing appropriate weights for each asset, subject to the constraint that the sum of weights equals 1.

d) Production and Input Ratios Consider a production process that requires two inputs, labour and capital, to produce two products: Product X: Requires 6 units of labour and 8 units of capital. Product Y: Requires 10 units of labour and 12 units of capital: *d.1)* Represent the input requirements for products X and Y as 3D vectors. *d.2)* Normalize the vectors for products X and Y. *c.2)* If you have 300 units of labour and 400 units of capital, find the maximum combination of products X and Y that can be produced.

2.1.7 For each exercise, perform vector operations step-by-step and provide interpretations of the results in the context of the given economic scenario. These exercises will help you understand how vectors are used to represent and analyse economic quantities in three-dimensional space

a) *Vector Representation in 3D:* Space Consider two points A (2, 3, 1) and B (5, 1, 4) in 3D space: *a.1)* Find the vector representing the direction from point A to point B. *a.2)* Calculate the magnitude (length) of the vector. *a.3)* Determine the unit vector in the direction from A to B.

b) Vector Representation in 3D Space: Consider a vector in 3D space given by $\boldsymbol{v} = (3, -2, 5)$: *b.1)* Find the magnitude (length) of the vector $\boldsymbol{v}$. *b.2)* Determine the direction cosines of the vector $\boldsymbol{v}$. *b.3)* Write the vector $\boldsymbol{v}$ in the form of magnitude and direction angles.

c) Economic Application - 3D Revenue: Vector A company's revenue in three consecutive years is represented by a 3D vector $\boldsymbol{r} = (300, 400, 500)$ (in thousands of dollars). Each entry in the vector represents the revenue for the first, second, and third year, respectively: *c.1)* Find the total revenue earned by the company over the three years. *c.2)* Calculate the average annual revenue of the company. *c.3)* Determine the growth rate of revenue from the first year to the third year.

d) Vector Addition and Subtraction in 3D Space: Consider two vectors in 3D space, $\boldsymbol{a} = (2,\ 1, -3)$ and $\boldsymbol{b} = (5, -2, 4)$: *d.1)* Find the vector sum $\boldsymbol{a} + \boldsymbol{b}$. *d.2)* Calculate the vector difference $\boldsymbol{a} - \boldsymbol{b}$. *d.3)* Determine the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

e) Economic Application - 3D Displacement: Vector $\boldsymbol{a}$ delivery truck travels from the origin (point O) to deliver goods at point A with a displacement vector $\overrightarrow{OA} = (2, 4, 3)$ (in kilometres): *e.1)* Find the distance covered by the delivery truck from O to A. *e.2)* The truck then travels from A to point B with a displacement vector $\overrightarrow{AB} = (-1, 2, 1)$. Determine the new position of the truck after reaching B.

f) Economic Application - 3D Profit Vector: Consider a company's quarterly profit in three consecutive years represented by the following vectors $\overline{q_1} = (100, 150, 200)$, $\overline{q_2} = (120, 180, 240)$, $\overline{q_3} = (140, 210, 280)$, $\overline{q_4} = (160, 240, 320)$: *f.1)* Find the total profit for each year. *f.2)* Determine the average quarterly profit for each year. *f.3)* Calculate the growth rate of profit from the first to the third year.

2.1.8 For each exercise, carefully apply the concepts of triple product of vectors, collinearity, and orthogonality to solve the problems. Show all the necessary calculations and provide explanations for your answers.

a) Triple Product of Vectors: Given three vectors in 3D space: $\boldsymbol{a} = (2, 1, 3)$, $\boldsymbol{b} = (4, -2, 5)$, $\boldsymbol{c} = (1, 0, -2)$: *a.1)* Find the scalar triple product $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. *a.2)* Determine if the vectors **a**, **b**, and **c** are coplanar based on the value of the scalar triple product.

b) Collinearity of Vectors: Given three vectors in 3D space, $\boldsymbol{p} = (2, 3, -1)$, $\boldsymbol{q} = (4, 6, -2)$, $\boldsymbol{r} = (6, 9, -3)$: *b.1)* Check if the vectors $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ are collinear (lie on the same line). *b.2)* If they are collinear, find the scalar λ such that $\boldsymbol{q} = \lambda\boldsymbol{p} + \boldsymbol{r}$.

c) Orthogonality of Vectors: Given three vectors in 3D space: $\boldsymbol{u} = (1, -2, 3)$, $\boldsymbol{v} = (2, 1, 4)$, $\boldsymbol{w} = (3, 0, -1)$: *c.1)* Determine if the vectors $\boldsymbol{u}$, $\boldsymbol{v}, and\ \boldsymbol{w}$ form an orthogonal set. *c.2)* If they are orthogonal, find the scalar products $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{v} \cdot \boldsymbol{w}$, $and\ \boldsymbol{u} \cdot \boldsymbol{w}$.

d) Collinearity and Orthogonality: Given three vectors in 3D space: $\boldsymbol{x} = (1, 2, 3)$, $\boldsymbol{y} = (3, 6, 9)$, $\boldsymbol{z} = (0, 1, 2)$: *d.1)* Determine if the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are collinear. *d.2)* Determine if the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ form an orthogonal set.

2.1.9 The exercises provide opportunities to apply the concepts of direction cosines and vector projections in different economic scenarios. Use appropriate formulas and geometric interpretations to solve the exercises and relate the results to economic concepts and applications.

a) Direction Cosines: Consider a vector $\boldsymbol{v} = (3, 4, 5)$ in three-dimensional space: *a.1)* Calculate the direction cosines of vector $\boldsymbol{v}$. *a.2)* Interpret the direction cosines geometrically and explain their significance.

b) Projection of Vectors: Consider two vectors in three-dimensional space: $\boldsymbol{v} = (2, 1, 3)$, $\boldsymbol{u} = (1, 2, 1)$: *b.1)* Find the scalar projection of vector ***v*** onto vector ***u***. *b.2)* Find the vector projection of vector ***v*** onto vector ***u***.

c) Economic Application - Production Possibility Frontier: In an economy, two goods are produced: Good A and Good B. The production of Good A requires 2 units of labour and 3 units of raw materials, while the production of Good B requires 5 units of labour and 1 unit of raw materials. The economy has 100 units of labour and 60 units of raw materials available: *c.1)* Determine the production possibility frontier (PPF) of the economy. *C.2)* Find the direction cosines of the PPF at the point where the economy is producing 20 units of Good A and 10 units of Good B. *c.3)* Interpret the direction cosines in the context of the PPF.

d) Economic Application - Investment Portfolio: An investor has two investment options: Option A and Option B. The returns on investment for Option A and Option B are 8% and 12%, respectively. The investor allocates 60% of their funds to Option A and the remaining to Option B: *d.1)* Find the direction cosines of the investment portfolio vector representing the allocation of funds to Option A and Option B. *d.2)* Interpret the direction cosines in the context of the investment portfolio.

e) Economic Application - Linear Demand Function: Consider a linear demand function for a product: Q = 200 - 5P, where Q is the quantity demanded, and P is the price per unit: *e.1)* Find the price elasticity of demand at P = 20 using the concept of direction cosines. *e.2)* Interpret the price elasticity of demand in the context of the product's demand.

2.1.10 For each exercise, carefully consider the properties of linear dependence/independence, basis vectors, and decomposition of vectors in three-dimensional space. Show all the steps involved in solving the exercises. Understanding these concepts is crucial in three-dimensional analytical geometry and lays the foundation for more advanced topics in vector spaces and linear transformations.

a) Linear Dependence of Vectors: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (1, 2, 3)$, $\boldsymbol{v2} = (3, 4, 5)$, $\boldsymbol{v3} = (2, 3, 4)$: *a.1)* Determine if the vectors ***v1, v2***, and v3 are linearly dependent or linearly independent.

b) Basis of Vectors: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (1, 0, 0)$, $\boldsymbol{v2} = (0, 1, 0)$, $\boldsymbol{v3} = (0, 0, 1)$: *b.1)* Show that the vectors ***v1, v2***, and ***v3*** form a basis for three-dimensional space.

c) Decomposition of a Vector: Consider the following vectors in three-dimensional space: $\boldsymbol{v} = (2, 5, 3)$, $\boldsymbol{u1} = (1, 0, 1)$, $\boldsymbol{u2} = (0, 1, 1)$: *c.1)* Determine the scalar coefficients α and β such that $\boldsymbol{v} = \alpha\boldsymbol{u1} + \beta\boldsymbol{u2}$. *c.2)* Verify if the vector ***v*** lies in the plane spanned by ***u1*** and ***u2***.

d) Non-Standard Basis: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (1, 1, 1)$, $\boldsymbol{v2} = (1, 2, 3)$, $\boldsymbol{v3} = (2, 3, 5)$: *d.1)* Determine if the vectors ***v1, v2***, and ***v3*** form a basis for three-dimensional space. *d.2)* Express the vector *(4, 7, 9)* as a linear combination of ***v1, v2***, and ***v3***.

e) Linear Dependence of Vectors: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (2, 1, 3)$, $\boldsymbol{v2} = (4, 2, 6)$, $\boldsymbol{v3} = (1, 1, 1)$: *e.1)* Determine if the vectors ***v1, v2***, and ***v3*** are linearly dependent or linearly independent. *e.2)* If the vectors are linearly dependent, express one of them as a linear combination of the others.

f) Basis of Vectors: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (1, 2, 3)$, $\boldsymbol{v2} = (2, 4, 6)$, $\boldsymbol{v3} = (1, 1, 2)$: *f.1)* Determine if the vectors ***v1, v2***, and ***v3*** form a basis for three-dimensional space. *f.2)* If they do not form a basis, find a subset of the vectors that forms a basis.

g) Decomposition of a Vector: Consider the following vectors in three-dimensional space: $\boldsymbol{v} = (3, 1, 5)$, $\boldsymbol{u1} = (1, 0, 2)$, $\boldsymbol{u2} = (0, 1, 1)$: *g.1)* Determine the scalar coefficients α and β such that $\boldsymbol{v} = \alpha\boldsymbol{u1} + \beta\boldsymbol{u2}$. *g.2)* Find the component of vector ***v*** that lies in the direction of ***u1***.

h) Non-Standard Basis: Consider the following vectors in three-dimensional space: $\boldsymbol{v1} = (1, 2, 0)$, $\boldsymbol{v2} = (3, 1, 0)$, $\boldsymbol{v3} = (0, 0, 1)$: *h.1)* Determine if the vectors ***v1, v2***, and ***v3*** form a basis for three-dimensional space. *h.2)* If they do not form a basis, find a subset of the vectors that forms a basis.

i) Basis Conversion: Consider the following vectors in three-dimensional space: $\boldsymbol{v} = (4, 5, 1)$, $\boldsymbol{u1} = (1, 0, 2)$, $\boldsymbol{u2} = (0, 1, 1)$, $\boldsymbol{v1} = (1, 1, 0)$, $\boldsymbol{v2} = (2, 3, 1)$: *i.1)* Express vector $\boldsymbol{v}$ in terms of basis $\{\boldsymbol{u1}, \boldsymbol{u2}\}$. *i.2)* Express vector $\boldsymbol{v}$ in terms of basis $\{\boldsymbol{v1}, \boldsymbol{v2}\}$. *i.3)* Determine the change of basis matrix from $\{\boldsymbol{u1}, \boldsymbol{u2}\}$ to $\{\boldsymbol{v1}, \boldsymbol{v2}\}$.

2.2 EQUATIONS OF LINES IN 3D SPACE

Keywords: *Line in 3D space, Parametric representation, Vector form, Symmetric form, Direction vector, Point on a line, Equation given by two points, Parametric equations, Intersection of lines, Shortest distance, Angle between lines, Skew lines, Parallel lines, Perpendicular lines, Direction ratios, Direction cosines, Normal vector, Coordinate system in 3D space, Orthogonality, Euclidean Transformation.*

Equations of lines in 3D space refer to mathematical representations that describe the position and direction of lines in a three-dimensional coordinate system. These equations allow us to analyse and manipulate lines in 3D space for various applications.

While equations of lines in 3D space are not directly applicable to all aspects of economics, they can still find some applications in economic analysis and modelling. Here are a few examples:

Production Possibility Frontier (PPF): The PPF is a graphical representation of the combinations of two goods that an economy can produce given its resources and technology. It can be represented as a curved or linear line in 2D or 3D space. Equations of lines in 3D space can be used to analyse and model the PPF, enabling economists to determine trade-offs between different goods and study efficient allocation of resources.

Supply and Demand Analysis: While supply and demand curves are typically represented in 2D space, in certain cases, 3D representations may be useful. For instance, in analysing a market with multiple factors affecting supply or demand, a 3D space can be used to incorporate additional variables. Equations of lines in 3D space can help model these relationships and analyse the interactions between multiple factors.

Input-Output Analysis: Input-output models describe the interdependencies between different sectors of an economy. These models often involve multiple variables and can be represented in multi-dimensional space. Equations of lines in 3D space can aid in visualizing and analysing the relationships between different sectors and variables in input-output models.

Economic Growth and Investment: Economic growth models often involve the projection of trends and trajectories over time. In some cases, these trajectories can be represented in 3D space, with time as one dimension and economic indicators as the other two dimensions. Equations of lines in 3D space can be employed to study the paths and projections of economic variables over time.

✍ *While the direct use of equations of lines in 3D space may not be as common in economics compared to other fields, their concepts of analysing relationships, trajectories, and trade-offs can still be relevant in certain economic applications. It is important to note that economics predominantly relies on mathematical models, statistical analysis, and economic theory for its frameworks and methodologies.*

Line in 3D space: In three-dimensional (3D) space, a line is a geometric object that extends infinitely in two opposite directions. It is a one-dimensional object that can be described by its position, direction. A line in 3D space is characterized by its orientation and position relative to the coordinate system. Some key characteristics of a line in 3D space:

Direction: A line in 3D space has a specific direction indicated by a vector that represents the line's orientation. The direction vector, often denoted as $\boldsymbol{d}$, specifies the line's slope and determines how the line extends in space.

Position: A line in 3D space can be located anywhere within the 3D coordinate system. Its position can be defined by a point that lies on the line or by an equation that describes the line's position relative to the coordinate axes.

Infiniteness: A line in 3D space extends infinitely in both directions. It has no endpoints and continues indefinitely in its determined direction.

Straightness: A line in 3D space is a straight path and does not deviate from a straight trajectory. It does not curve or bend unless influenced by external factors or other geometric objects.

Intersection: Two lines in 3D space may intersect at a point if they cross each other. Alternatively, two lines may be parallel (never intersecting), or skew lines (do not lie on the same plane).

✍ *Lines in 3D space are fundamental elements. They play a crucial role in describing the position, direction, and relationships between points, objects, and geometric structures. The equations and representations of lines in 3D space allow for the analysis, manipulation, and visualization of objects and phenomena in three dimensions.*

There are several ways to express the equations of lines in 3D space:

Parametric Form: The parametric form represents a line as a set of three parametric equations that determine the coordinates of points on the line based on a parameter, typically denoted as $\boldsymbol{t}$. The parametric form is written as:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

where (x_0, y_0, z_0) is a known point on the line, and $a, b, and\ c$ are the direction ratios or components of the direction vector of the line.

Example 2.2.a Consider a line L in $3D$ space that passes through the point $P(1, 2, 3)$ and has a direction vector $\boldsymbol{d} = (2, -1, 4)$. The parametric form of the equation of the line L can be written as:

$$L: \begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 3 + 4t \end{cases}$$

In this example, the parameter t represents a scalar that determines different points on the line L. By varying the value of t, we can obtain various points along the line. For instance, when $t = 0: \begin{cases} x = 1 + 2 * 0 \\ y = 2 - 0 \\ z = 3 + 4 * 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases}$. So, when $t = 0$, the point on the line L is $(1, 2, 3)$, which is the given point P. When $t = 1: \begin{cases} x = 1 + 2 * 1 \\ y = 2 - 1 \\ z = 3 + 4 * 1 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 1 \\ z = 7 \end{cases}$. So, when $t = 1$, we obtain the point $(3, 1, 7)$, which lies on the line L. By varying the value of t, we can generate an infinite number of points that lie on the line L.

✍ *The parametric form allows us to easily determine different points on the line by manipulating the parameter. It provides a concise representation of the line's position and direction, making it useful for calculations involving the line's geometry, intersections with other lines or surfaces, and other applications in 3D space.*

Vector Form: The vector form represents a line using vector notation. It is expressed as the sum of a position vector (point on the line) and a direction vector. The vector form is written as:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t * \boldsymbol{v}$$

where $\boldsymbol{r}$ is the position vector of any point on the line, r_0 is a known point on the line, t is a scalar parameter, and **v** is the direction vector of the line.

Example 2.2.b Consider a line L in $3D$ space that passes through the point $P(2, 1, -3)$ and has a direction vector $\boldsymbol{d} = (1, -2, 4)$. The vector form of the equation of the line L can be written as: $\boldsymbol{r} = P + t * \boldsymbol{d}$, where $\boldsymbol{r}$ is the position vector of any point on the line, P is the known point $(2, 1, -3)$, t is a scalar parameter, and $\boldsymbol{d}$ is the direction vector with coordinates $(1, -2, 4)$.

Using this equation, we can obtain the position vector of any point on the line L by varying the value of t. For instance, when $t = 0: r = (2, 1, -3) + 0 * (1, -2, 4) = (2, 1, -3)$. So, when $t = 0$, the point on the line L is the given point P. When $t = 1: r = (2, 1, -3) + 1 * (1, -2, 4) = (3, -1, 1)$. So, when $t = 1$, we obtain another point $(3, -1, 1)$ on the line L. By varying the value of t, we can obtain an infinite number of points that lie on the line L.

✍ *The vector form provides a concise representation of the line's position and direction. It allows us to easily determine different points on the line by manipulating the scalar parameter t. This form is particularly useful for vector operations, intersection calculations, and other geometric analyses involving lines in 3D space.*

Symmetric Form: The symmetric form represents a line using a set of equations that relate the coordinates of points on the line. The symmetric form is written as:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a known point on the line, and a, b, and c are the direction ratios or components of the direction vector of the line.

Example 2.2.c Consider a line L in 3D space with a known point $P(2,-1,3)$ and a direction vector $\boldsymbol{d} = (3,2,-4)$. The symmetric form of the equation of the line L can be written as:

$$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-4}$$

In this form, the equation relates the coordinates (x,y,z) of any point on the line to the known point P and the direction ratios or components of the direction vector $\boldsymbol{d}$.

By manipulating this equation, we can find various points on the line L. For example, if we let $x = 5$, we can solve for y and z: $\frac{5-2}{3} = \frac{y+1}{2} = \frac{z-3}{-4} \Rightarrow \frac{5-2}{3} = \frac{y+1}{2} = \frac{z-3}{-4} \Rightarrow 1 = \frac{y+1}{2} = \frac{z-3}{-4}$.

From here, we can solve for y and z: $\begin{cases} \frac{y+1}{2} = 1 \\ \frac{z-3}{-4} = 1 \end{cases} \Rightarrow \begin{cases} y+1 = 2 \\ z-3 = -4 \end{cases} \Rightarrow \begin{cases} y = 1 \\ z = -1 \end{cases}$. Therefore, when $x = 5$, the point on the line L is $(5,0,-1)$.

✍ *The symmetric form of the equation of a line provides a concise representation that relates the coordinates of any point on the line to the known point and the direction of the line. It is useful for analysing the relationship between the coordinates and understanding the line's position and orientation in 3D space. These equations allow us to describe lines in 3D space, determine points on a line, find the intersection of lines, calculate distances between points and lines, and analyse the geometric properties of lines in 3D space. They are fundamental tools for working with lines in three dimensions and are utilized in various fields such as geometry, physics, engineering, and computer graphics.*

The direction vector of a line: The direction vector of a line in three-dimensional (3D) space can be determined using two points on the line or by coefficients in the parametric equations of the line.

Method 1 (Using two Points on the Line): To determine the direction vector using two points on the line, follow these steps:

Step 1: Choose two distinct points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ that lie on the line L.

Step 2: Calculate the vector $\overrightarrow{AB}$ by subtracting the coordinates of point A from the coordinates of point B. This vector represents the displacement from point A to point B.

Step 3: The vector $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is the direction vector of the line.

Example 2.2.d Consider a line L in 3D space that passes through two points $A(1,2,3)$ and $B(4,6,8)$. To find the direction vector of line L, we calculate the vector $\overrightarrow{AB}$ by subtracting the coordinates of point A from the coordinates of point B: $\overrightarrow{AB} = B - A = (4,6,8) - (1,2,3) = (3,4,5)$. Therefore, the direction vector of line L is $\bar{d} = (3,4,5)$.

✍ *This means that as we move along the line L, the x-coordinate increases by 3 units, the y-coordinate increases by 4 units, and the z-coordinate increases by 5 units. The direction vector (3, 4, 5) represents the slope and orientation of the line L in 3D space. It provides valuable information about the line's direction, allowing for calculations related to intersections, angles, and projections with other lines or objects.*

Method 2 (Using parametric Equations): To determine the direction vector using the parametric equations of the line, consider the coefficients of the variables in the parametric equations.

If the parametric equations of the line are: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$, the coefficients a, b, and c represent the direction ratios or components of the direction vector of the line. Thus, the direction vector is given by: $\bar{d} = (a, b, c)$.

By identifying the coefficients in the parametric equations, you can directly obtain the direction vector.

Example 2.2.e Consider a line L in 3D space with the following parametric equations:

$$\begin{cases} x = 2 + t \\ y = 1 - 2t \\ z = 3 + 3t \end{cases}$$

To find the direction vector of line L, we can identify the coefficients of the variables in the parametric equations:

The coefficient of t in the equation $x = 2 + t$ is 1. The coefficient of t in the equation $y = 1 - 2t$ is -2. The coefficient of t in the equation $z = 3 + 3t$ is 3. Therefore, the direction vector of line L is $\bar{d} = (1, -2, 3)$.

✍ *Both methods allow you to determine the direction vector of a line in 3D space. The direction vector indicates the orientation and slope of the line and can be used for various calculations and analyses involving the line's geometry and relationships with other objects or lines in 3D space*.

The equation of a line given by two points on the line: To determine the equation of a line given two points on the line, you can follow these steps:

Step 1: Identify the coordinates of the two points on the line. Let's denote them as point A with coordinates (x_1, y_1, z_1) and point B with coordinates (x_2, y_2, z_2).

Step 2: Calculate the direction vector of the line using the two points. Subtract the coordinates of point A from the coordinates of point B to obtain the vector $\overrightarrow{AB}$, which represents the direction of the line: $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Step 3: Choose one of the two points, let's say point A, and plug in its coordinates along with the direction vector into the vector form of the equation of a line: $\bar{r} = A + t * \overrightarrow{AB}$. Here, $\boldsymbol{r}$ represents the position vector of any point on the line, t is a scalar parameter, and $\overrightarrow{AB}$ is the direction vector calculated in step 2.

Step 4: Expand the equation by substituting the coordinates of point A and the components of the direction vector:

$$\begin{cases} x = x_1 + t * (x_2 - x_1) \\ y = y_1 + t * (y_2 - y_1) \\ z = z_1 + t * (z_2 - z_1) \end{cases}$$

Step 5: Simplify and rearrange the equations if necessary to obtain the final equation of the line in the desired form, such as the parametric form, symmetric form, or vector form.

Example 2.2.f Consider two points $A(1, 2, 3)$ and $B(4, 6, 8)$ that lie on a line.

Step 1: Identify the coordinates of the two points:
Point A: $(x_1, y_1, z_1) = (1, 2, 3)$
Point B: $(x_2, y_2, z_2) = (4, 6, 8)$.

Step 2: Calculate the direction vector of the line using the two points:
$\overrightarrow{AB} = B - A = (4,6,8) - (1,2,3) = (3,4,5)$.

Step 3: Choose one of the two points (let's choose point A) and write the vector form of the equation of the line: $\bar{r} = A + t * \overrightarrow{AB}$. Substituting the values, we have: $\bar{r} = (1,2,3) + t * (3,4,5)$.

Step 4: Expand the equation: $x = 1 + 3t,\ y = 2 + 4t,\ z = 3 + 5t$. These equations represent the parametric form of the equation of the line.

Step 5: Hence, the equation of the line that passes through the points $A(1,2,3)$ and $B(4,6,8)$ is: $x = 1 + 3t,\ y = 2 + 4t,\ z = 3 + 5t$.

✍ *The resulting equation represents the line that passes through the two given points. It allows you to describe and work with the line, calculate other points on the line, determine intersections, and perform various geometric analyses. Remember, it's important to ensure that the two points chosen are distinct and not coincident, as this would result in an invalid equation.*

The intersection of two lines in 3D space: To determine the intersection of two lines in 3D space, you can follow these steps:

Step 1: Express the equations of both lines in either parametric, symmetric, or vector form.

Step 2: Set the equations of the two lines equal to each other to find the values of the parameter(s) where the lines intersect.

Step 3: Solve the resulting system of equations to find the values of the parameter(s) that satisfy the intersection condition.

Step 4: Substitute the values of the parameter(s) back into one of the line equations to obtain the coordinates of the point(s) of intersection.

Example 2.2.g Consider two lines in 3D space L1 and L2:

Step 1: Express the equations of both lines in parametric form:

$L1: x = 2 + t, y = 1 - 2t, z = 3 + 3t$
$L2: x = 3 - s, y = 4 + 2s, z = 2 - s.$

Step 2: Set the equations of the two lines equal to each other:

$$\begin{cases} 2 + t = 3 - s \\ 1 - 2t = 4 + 2s \\ 3 + 3t = 2 - s \end{cases}$$

Step 3: Solve the system of equations. In this case, solving for t and s:

$$\begin{cases} t = -1, \\ s = -1 \end{cases}$$

Step 4: Substitute the values of t and s back into one of the line equations, let's use

$$L1: x = 2 + (-1) = 1, y = 1 - 2(-1) = 3, z = 3 + 3(-1) = 0$$

Therefore, the lines intersect at the point $(1,3,0)$.

✍ *If the system of equations has no solution or infinite solutions, it indicates that the lines do not intersect or are coincident/overlapping, respectively. By following these steps, you can determine the point(s) of intersection between two lines in 3D space. The process involves setting the equations equal to each other, solving the resulting system of equations, and finding the values that satisfy the intersection condition.*

The shortest distance between a point and a line in 3D space: To calculate the shortest distance between a point and a line in 3D space, you can follow these steps:

Step 1: Identify the coordinates of the given point and the equations of the line.

Step 2: Determine a vector that represents the direction of the line. This can be done by taking the coefficients of the variables in the line equation (if in vector or symmetric form) or by subtracting the coordinates of two points on the line.

Step 3: Find a vector connecting a point on the line to the given point. This can be done by subtracting the coordinates of the given point from the coordinates of a point on the line.

Step 4: Calculate the cross product of the direction vector of the line and the vector connecting the point on the line to the given point. This cross product gives a vector that is perpendicular to both vectors.

Step 5: Determine the magnitude of the cross-product vector, which represents the length of the perpendicular distance between the given point and the line (the shortest distance between a point and a line).

Example 2.2.h Consider a point $P(2,3,1)$ and a line L given by the equation:

$$L:\ x = 1 + t,\ y = 2 - t,\ z = 3 + 2t.$$

Step 1: Point: $P(2,3,1)$ Line: $x = 1 + t, y = 2 - t, z = 3 + 2t$.

Step 2: Direction vector of the line: $\bar{d} = (1,-1,2)$.

Step 3: Vector $(\bar{v})$connecting a point on the line to the given point: $P_0 = (1,2,3)$,

$$\bar{v} = P - P_0 = (2,3,1) - (1,2,3) = (1,1,-2)$$

Step 4: Cross product of the direction vector and the vector connecting the point on the line to the given point: $\bar{n} = \bar{d}\ x\ \bar{v} = (1,-1,2) \times (1,1,-2) = (0,4,2)$

Step 5: Magnitude of the cross-product vector:

$$Distance = |\bar{n}| = \sqrt{0^2 + 4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

Therefore, the shortest distance between the point $P(2,3,1)$ and the line L is $2\sqrt{5}$ units.

The angle between two lines in 3D space: To calculate the angle between two lines in 3D space, you can follow these steps:

Step 1: Identify the direction vectors of the two lines. These direction vectors represent the orientations of the lines.

Step 2: Calculate the dot product of the two direction vectors.

Step 3: Find the magnitudes of the direction vectors.

Step 4: Use the dot product formula to calculate the angle between the lines:

$angle = arccos((dot\ product)\ /\ (magnitude\ of\ direction\ vector\ 1 * magnitude\ of\ direction\ vector\ 2))$.

Example 2.2.h Consider two lines in 3D space with the direction vectors: Line 1: $\overline{d_1} = (1,2,-1)$ and Line 2: $\overline{d_2} = (3,-1,4)$.

Step 1: Direction vectors:

Line 1: $\overline{d_1} = (1,2,-1)$ and Line 2: $\overline{d_2} = (3,-1,4)$

Step 2: Dot product of the direction vectors:

$$\bar{d}_1 \cdot \bar{d}_2 = (1 * 3) + (2 * -1) + (-1 * 4) = 3 - 2 - 4 = -3.$$

Step 3: Magnitudes of the direction vectors:

Magnitude of $\overline{d_1}$: $\left|\overline{d_1}\right| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

Magnitude of $\overline{d_2}$: $\left|\overline{d_2}\right| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}.$

Step 4: Calculate the angle between the lines:

$$angle = arccos\left(\frac{-3}{\sqrt{6} * \sqrt{26}}\right)$$

Using a calculator or math software, you can find the angle to be approximately $1.606\ radians$ or $92.104\ degrees$.

Therefore, the angle between the two lines in 3D space is approximately 1.606 radians or 92.104 degrees.

The parallel lines in 3D space: To determine if two lines are parallel using vector operations in 3D space, you can compare their direction vectors. If the direction vectors are proportional or scalar multiples of each other, then the lines are parallel. Here's a step-by-step process:

Step 1: Express the equations of the two lines in either parametric, symmetric, or vector form.

Step 2: Calculate the direction vectors of the two lines. These direction vectors represent the orientations of the lines.

Step 3: Compare the direction vectors to check if they are proportional. To do this, divide the components of one direction vector by the corresponding components of the other direction vector. If the ratios are equal for all components, the lines are parallel.

Step 4: Conclusion.

Example 2.2.i Consider two lines in 3D space:

Line 1: $x = 1 + t,\ y = 2 + t,\ z = 3 + t,$

Line 2: $x = 2 - s,\ y = 3 - s,\ z = 4 - s.$

Step 1: Express the equations of the lines in parametric form:

$$L_1: \begin{cases} x = 1 + t \\ y = 2 + t \\ z = 3 + t \end{cases} \qquad L_2: \begin{cases} x = 2 - s \\ y = 3 - s \\ z = 4 - s \end{cases}$$

Step 2: Calculate the direction vectors of the lines:

Direction vector of Line 1: $\overline{d_1} = (1, 1, 1)$

Direction vector of Line 2: $\overline{d_2} = (-1, -1, -1)$

Step 3: Compare the direction vectors: Divide the components of one direction vector by the corresponding components of the other direction vector: $\frac{\overline{d_2}}{\overline{d_2}} = (\frac{1}{-1}, \frac{1}{-1}, \frac{1}{-1}) = (-1, -1, -1).$

Step 4: The ratios of the components are equal for all three components, indicating that the direction vectors are proportional. Based on this observation, we can conclude that the lines are parallel.

Skew lines: Skew lines are a pair of lines that are non-intersecting, non-parallel, and non-coplanar. This implies that skew lines can never intersect and are not parallel to each other. For lines to exist in two dimensions or in the same plane, they can either be intersecting or parallel. As this

property does not apply to skew lines, hence, they will always be non-coplanar and exist in three or more dimensions.

Identifying skew lines: Skew lines in 3D space are lines that do not intersect and are not parallel to each other. To identify skew lines, you can compare the direction vectors of the lines and check for conditions of non-intersection and non-parallelism. Here's a step-by-step process:

Step 5: Express the equations of the two lines in either parametric, symmetric, or vector form.

Step 6: Calculate the direction vectors of the two lines. These direction vectors represent the orientations of the lines.

Step 7: Compare the direction vectors to determine if they are proportional or parallel to each other. If the direction vectors are proportional (i.e., scalar multiples of each other), the lines are parallel, not skew. If the direction vectors are not proportional, proceed to the next step.

Step 8: Check for non-intersection. This can be done by comparing the position vectors or any points on the lines. If the position vectors are not equal and the lines do not intersect at any point, then they are skew lines.

Step 9: Next, check if these non-intersecting and non-parallel lines are non-coplanar. If yes then the chosen pair of lines are skew lines.

Example 2.2.i: Consider two lines in 3D space:

Line 1: $x = 1 + 2t, y = 3t, z = -2 - 5t$ and Line 2: $x = s, y = 4 + s,\ z = 3s$.

Step 1: Express the equations of the lines in parametric form:

$$L_1: \begin{cases} x = 1 + 2t \\ y = 3t \\ z = -2 - 5t \end{cases} \qquad L_2: \begin{cases} x = s \\ y = 4 + s \\ z = 3s \end{cases}$$

Step 2: Calculate the direction vectors of the lines:

Direction vector of Line 1: $\overline{d_1} = (2, 3, -5)$

Direction vector of Line 2: $\overline{d_2} = (1, 1, 3)$.

Step 3: Compare the direction vectors:

The direction vectors are not proportional, so they are not parallel to each other since they are not scalar multiples of each other.

Step 4: Check for non-intersection: Equating the corresponding components of the direction vectors we obtain the system:

$$\begin{cases} 1 + 2t = s \\ 3t = 4 + s \\ -2 - 5t = 3s \end{cases} \Rightarrow \begin{cases} 1 + 2t = s \\ t = 5 \\ -2 - 5t = 3s \end{cases} \Rightarrow \begin{cases} s = 11 \\ t = 5 \\ s = -9 \end{cases}$$

The system has no solutions. This implies that the two given lines do not intersect.

Step 5: Thus, as the lines are non-parallel, non-intersecting and non-coplanar hence, they are skew lines.

The distance between skew lines: The distance between two skew lines is given by the line that is perpendicular to the two lines as opposed to any line joining both the skew lines. The vector equation is given by

$$distance = \left| \frac{(\overline{n_1} \times \overline{n_2}) \cdot (\overrightarrow{P_1P_2})}{|\overline{n_1} \times \overline{n_2}|} \right|$$

is used when the lines are represented by parametric equations. Here, $\overline{n_1}$, $\overline{n_2}$ are the direction vectors and P_1, P_2 are the points that belong to the lines, accordingly.

Example 2.2.j Consider two lines in 3D space:
Line 1: $x = 1 + 2t, y = 3t, z = -2 - 5t$ and Line 2: $x = s, y = 4 + s,\ z = 3s$. The Line 1 and Line 2 a skew lines, which was demonstrated in the example 2.2.i.

The direction vectors of the lines are:
Direction vector of Line 1: $\overline{d_1} = (2, 3, -5)$
Direction vector of Line 2: $\overline{d_2} = (1, 1,\ 3)$.

The points that belong to the lines, obtained for t=0 and s=0:
The point on the Line 1, obtained for t=0: $P_1 = (1,0,-2)$
The point on the Line 2, obtained for s=0: $P_2 = (0,4,0)$

$$distance(L_1, L_2) = \left|\frac{\big((2,3,-5) \times (1,1,3)\big)(0-1,4-0,0+2)}{|(2,3,-5) \times (1,1,3)|}\right| = \left|\frac{(14,-11,-1)(-1,4,2)}{\sqrt{318}}\right| = \left|\frac{-60}{\sqrt{318}}\right| = \frac{60}{\sqrt{318}}$$

The perpendicular lines in 3D space: To determine if two lines are perpendicular using vector operations in 3D space, you can check if their direction vectors are orthogonal or perpendicular to each other. This can be done by calculating the dot product of the direction vectors and verifying if the dot product is zero. Here's a step-by-step process:

Step 1: Express the equations of the two lines in either parametric, symmetric, or vector form.
Step 2: Calculate the direction vectors of the two lines. These direction vectors represent the orientations of the lines.
Step 3: Take the dot product of the direction vectors. This can be done by multiplying the corresponding components of the direction vectors and summing the results.
Step 4: Check if the dot product is zero. If the dot product is zero, the direction vectors are perpendicular, and hence the lines are perpendicular. If the dot product is nonzero, the lines are not perpendicular.
Step 5: Conclusion.

Example 2.2.k Consider two lines in 3D space: Line 1: $x = 1 + t, y = 2 - t, z = 3 + t$ and Line 2: $x = 2 - s, y = 3 + s, z = 4 + 2s$.

Step 1: Express the equations of the lines in parametric form:

$$L_1: \begin{cases} x = 1 + t \\ y = 2 - t \\ z = 3 + t \end{cases} \qquad L_2: \begin{cases} x = 2 - s \\ y = 3 + s \\ z = 4 + 2s \end{cases}$$

Step 2: Calculate the direction vectors of the lines:
Direction vector of Line 1: $\overline{d_1} = (1, -1, 1)$
Direction vector of Line 2: $\overline{d_2} = (-1, 1, 2)$

Step 3: Calculate the dot product of the direction vectors:

$$\overline{d_1} \cdot \overline{d_2} = (1 * -1) + (-1 * 1) + (1 * 2) = -1 - 1 + 2 = 0$$

Step 4: Check if the dot product is zero: The dot product equal to zero.
Step 5: Based on this observation, we can conclude that the lines are perpendicular.

Questions you should be able to answer

What is a line in three-dimensional (3D) space?
How can a line in 3D space be represented parametrically?
What is the vector form of the equation of a line in 3D space?
What is the symmetric form of the equation of a line in 3D space?
How can the direction vector of a line be determined?
How can a point on a line be determined using the vector form of the equation?
How can the equation of a line be determined given two points on the line?
What is the relationship between the direction vector and the parametric equations of a line?
How can the intersection of two lines in 3D space be determined?
How can the shortest distance between a point and a line in 3D space be calculated?
How can the angle between two lines in 3D space be calculated?
How can the skew lines in 3D space be identified?
How can the parallel and perpendicular nature of lines be using vector operations?

Proposed Exercises

2.2.1 In each exercise, fill in the missing words to complete the sentences related to lines in 3D space in the context of economic mathematics. Understanding lines in 3D space is essential for various applications, such as modelling economic relationships, analysing regression data, and understanding geometric concepts in economic contexts:

a) In economic mathematics, lines in 3D space are often represented in the form of ... and are expressed as a linear combination of a ... and a ... . The direction vector represents the ... of the line, while the ... vector represents the ... of the line. For example, a line L can be defined as $L: \ldots + \ldots t$, where ... and ... are the components of the direction vector, and ... and ... are the components of the position vector.

b) To determine if a point lies on a line in 3D space, we can substitute the ... of the point into the equation of the line. If the resulting equation is ..., then the point lies on the line; otherwise, it does not.

c) Parallel lines in 3D space have the same ... vector. If two lines, *L1* and *L2*, are parallel, their direction vectors, denoted by ..., are ... to each other.

d) To find the point of ... between two skew lines in 3D space, we can set their ... equations equal to each other and solve for the values of ... and ... that satisfy both equations. The resulting point represents the point of closest approach between the two lines.

f) In economic mathematics, lines in 3D space can also be used to represent ... relationships between variables. For example, in a ... model, a line can be used to show the ... relationship between two variables. The slope of the line represents the ... rate of change between the variables, while the ... represents the value of one variable when the other variable is zero.

g) In regression analysis, lines in 3D space are used to fit a ... line through data points to model the ... relationship between two variables. The equation of the line, in this case, is usually represented as $\ldots + \ldots x$, where ... is the ... and ... is the ... of the regression line.

2.2.2 These exercises explore different types of equations for lines in 3D space (parametric, symmetric, Cartesian), their transition between each other, and how they can be applied in the context of economic mathematics. Show all the steps involved in solving the exercises and provide interpretations in the economic context to enhance the understanding of the concepts:

a) Parametric Equations of a Line: Consider the line *L1* passing through the points $A(1, 2, 3)$ and $B(3, 4, 5)$: *a.1)* Find the parametric equations of the line *L1*. *a.2)* Interpret the parametric equations in the context of economic mathematics.

b) Symmetric Equations of a Line: Consider the line *L2* with the vector equation $r = (1, 2, 1) + t(2, 1, 3)$, where *t* is a parameter: *b.1)* Find the symmetric equations of the line *L2*. *b.2)* Describe the meaning of the parameters in the context of economic mathematics.

c) Cartesian Equation of a Line: Consider the line *L3* passing through the points $C(1, 3, 2)$ and $D(2, 1, 4)$: *c.1)* Find the Cartesian equation of the line *L3*. *c.1)* Explain how the Cartesian equation represents the line in 3D space in the context of economic mathematics.

d) Transition Between Equations: Consider the line *L4* with the symmetric equations: $x - 1 = y + 2 = z - 3$: *d.1)* Convert the symmetric equations of *L4* to the parametric form. *d.2)* Convert the parametric equations of *L4* to the Cartesian form.

e) Intersection of Lines: Consider the lines *L5* and *L6* with the following equations: $L5: x = 2 + 3t, y = 1 - 2t, z = 4t$ and $L6: x - 1 = 2(y + 3) = z + 2$: *e.1)* Find the point of intersection (if any) between lines *L5* and *L6*. *e.2)* Interpret the intersection point in the context of economic mathematics.

f) Linear Independence of Lines: Consider the lines *L7* and *L8* with the following vector equations: $L7: r = (2, 1, 3) + t(1, 2, 1)$ and $L8: r = (0, -1, 2) + s(2, 4, 2)$: *f.1)* Determine whether lines *L7* and *L8* are parallel, intersect, or neither. *f.2)* Discuss the significance of linear independence of lines in the context of economic mathematics.

2.2.3 These exercises involve studying the intersection of two lines in three-dimensional space and interpreting the results in the context of economic mathematics. They aim to apply the concepts of lines and intersections to economic models, supply-demand functions, and market equilibrium analysis. Show all the steps involved in solving the exercises and provide economic interpretations of the results.

a) Intersection of Two Lines: Consider the following two lines in three-dimensional space: $Line\ 1: x = 2 + t, y = 1 - t, z = 3t$ and $Line\ 2: x = 3 - s, y = 2s, z = 1 + 2s$: *a.1)* Determine the point of intersection, if any, between *Line 1* and *Line 2*. *a.2)* Interpret the geometric meaning of the intersection point in the context of economic mathematics.

b) Parallel or Skew Lines: Consider the following two lines in three-dimensional space: $Line\ 1: x = 1 + 2t, y = 3 - t, z = 4t$ and $Line\ 2: x = 3 + s, y = 2s, z = 5 - s$: *b.1)* Determine if *Line 1* and *Line 2* are parallel, intersecting, or skew lines. *b.2)* Discuss the significance of parallel or skew lines in the context of economic mathematics.

c) Economic Interpretation of Lines: Consider the following two lines in three-dimensional space, representing two economic models: $Line\ 1: C = 50 + 2P, D = 30 - P, E = 10 + 3P$ and $Line\ 2: C = 40 + 3P, D = 20 + P, E = 12 - 2P$, where C, D, and E represent the quantities

of three different commodities, and P represents the price: *c.1)* Interpret the slopes and intercepts of *Line 1* and *Line 2* in the context of the economic models. *c.2)* Determine if the two economic models are consistent, intersecting, or diverging based on the lines' equations.

d) Economic Analysis of Intersection: Consider the following two lines in three-dimensional space, representing supply and demand functions for a product: $Line\ 1: Q_s = 50 + 2P, Q_d = 100 - P$ and $Line\ 2: Q_s = 40 + 3P, Q_d = 90 + 2P$, where Q_s represents the quantity supplied, Q_d represents the quantity demanded, and *P* represents the price: *d.1)* Determine the equilibrium price and quantity for the product based on the intersection of *Line 1* and *Line 2*. *d.2)* Discuss the implications of the equilibrium point in the context of the product's market.

e) Intersection of Economic Models: Consider the following two lines in three-dimensional space, representing two economic models: $Line\ 1: C = 100 - 2P, D = 50 + P, E = 80 - P$ and $Line\ 2: C = 80 - P, D = 70 + 2P, E = 60 + P$, where C, D, and E represent the quantities of three different commodities, and P represents the price: *e.1)* Determine the point of intersection, if any, between *Line 1* and *Line 2*. *e.2)* Interpret the geometric meaning of the intersection point in the context of economic mathematics.

f) Supply and Demand Analysis: Consider the following two lines in three-dimensional space, representing supply and demand functions for a product: $Line\ 1: Qs = 40 + P, Qd = 120 - 2P$ and $Line\ 2: Qs = 60 - 2P, Qd = 80 + P$, where *Qs* represents the quantity supplied, *Qd* represents the quantity demanded, and *P* represents the price: *f.1)* Determine the equilibrium price and quantity for the product based on the intersection of *Line 1* and *Line 2*. *f.2)* Analyse the impact of changes in supply and demand on the equilibrium point in the context of economic mathematics.

g) Economic Interpretation of Parallel Lines: Consider the following two lines in three-dimensional space, representing economic models: $Line\ 1: C = 50 + 2P, D = 20 - P, E = 80 + P$ and $Line\ 2: C = 30 + P, D = 10 + 2P, E = 70 - P$, where *C*, *D*, and *E* represent the quantities of three different commodities, and *P* represents the price.: *g.1)* Determine if *Line 1* and *Line 2* are parallel, intersecting, or skew lines. *g.2)* Discuss the economic implications of parallel lines in the context of the economic models.

h) Intersecting Economic Lines: Consider the following two lines in three-dimensional space, representing two economic models: $Line\ 1: C = 100 - P, D = 50 + P, E = 80 - P$ and $Line\ 2: C = 80 - P, D = 70 + P, E = 60 + P$, where *C*, *D*, and *E* represent the quantities of three different commodities, and P represents the price: *h.1)* Determine if *Line 1* and *Line 2* are intersecting lines. *h.1)* Interpret the geometric meaning of the intersection point in the context of economic mathematics.

2.2.4 For each exercise, carefully analyse the equations of the lines and apply the relevant formulas to find the distance between lines when applicable. Interpret the results in the context of economic mathematics to understand the implications of the position and relationship between different entities or objects represented by the lines.

a) Line Equations: Consider the following lines in three-dimensional space: $Line\ L1: r = (1, 2, 3) + t(2, 1, 0)$ and $Line\ L2: r = (3, -1, 2) + s(-1, 2, 1)$: *a.1)* Write the

parametric equations for lines *L1* and *L2*. *a.2)* Determine if the lines *L1* and *L2* are parallel, intersecting, or skew.

b) Distance between Parallel Lines: Consider the following lines in three-dimensional space:

$Line\ L1: r = (2,1,3) + t(3,2,1)$ and $Line\ L2: r = (-1,0,1) + s(3,2,1)$: *b.1)* Write the parametric equations for lines *L1* and *L2*. *b.2)* Find the distance between the parallel lines *L1* and *L2*.

c) Distance between Intersecting Lines: Consider the following lines in three-dimensional space: $Line\ L1: r = (1,2,3) + t(2,1,0)$ and $Line\ L2: r = (3,-1,2) + s(1,1,1)$: *c.1)* Write the parametric equations for lines *L1* and *L2*. *c.2)* Determine if the lines *L1* and *L2* intersect. *c.3)* If the lines intersect, find the distance between them.

d) Distance between Skew Lines: Consider the following lines in three-dimensional space: $Line\ L1: r = (1,2,3) + t(2,1,0)$ and $Line\ L2: r = (3,-1,2) + s(1,-1,1)$: *d.1)* Write the parametric equations for lines L1 and L2. *d.2)* Determine if the lines *L1* and *L2* are skew. *d.3)* If the lines are skew, find the distance between them.

e) Economic Application: Suppose two companies, A and B, operate in three-dimensional space. The position of company A is given by the line $L1: r = (2,3,1) + t(1,2,1)$, and the position of company *B* is given by the line $L2: r = (-1,2,4) + s(2,1,-1)$: *e.1)* Determine if the companies A and B are located on parallel, intersecting, or skew lines. *e.2)* If the companies are on intersecting lines, find the distance between them and interpret the result in the context of economic mathematics.

f) Distance between Two Lines: Consider the following two lines in three-dimensional space: $L1: x = 2 + t,\ y = 3 - t,\ z = 4t$ and $L2: x = 1 - 2s,\ y = 2 + s,\ z = 3 + 3s$: *f.1)* Find the distance between lines L1 and L2. *f.2)* Interpret the result in the context of economic mathematics. For example, you can consider the lines as production possibilities frontiers for two goods and discuss the implications of the distance between these frontiers.

g) Distance from a Point to a Line: Consider the line $L: x = 2 + t,\ y = 3 - t,\ z = 4t$, and the point $P(3,1,6)$: *g.1)* Find the distance between the point P and the line L. *g.2)* Interpret the result in the context of economic mathematics. For example, you can consider the line L as a production possibilities frontier and the point P as a specific combination of two goods. Discuss the economic implications of the distance between the point P and the production possibilities frontier.

h) Parallel or Intersecting Lines Consider the following two lines in three-dimensional space: $L1: x = 2 + t, y = 3 - t, z = 4t$ and $L2: x = 1 + 2s, y = 2 - s, z = 3 + 3s$: *h.1)* Determine if lines L1 and L2 are parallel, intersecting, or skew (neither parallel nor intersecting). *h.2)* If the lines are skew, find the distance between them. *h.3)* Interpret the result in the context of economic mathematics. For example, you can consider the lines as production possibilities frontiers for two goods and discuss the economic implications of their relationship.

i) Shortest Distance to Reach a Point A company has two production facilities located at points *A (2, 4, 6)* and *B (5, 2, 1)*. The company wants to transport goods from point A to point B. However, there is a river flowing in three-dimensional space, and the company can only transport goods along the river. The path along the river can be described by the line $L: x = 4 - 2t,\ y = 3 + t,\ z = 2t$: *i.1)* Find the shortest distance the company needs to transport goods from point A to point B using the

river route. *i.2)* Interpret the result in the context of economic mathematics. Discuss the economic advantages or disadvantages of using the river route for transportation.

2.3 PLANES IN 3D SPACE

Keywords: *Plane, 3D space, Cartesian coordinates, Normal vector, Point-normal form, Scalar equation of a plane, Intercept form of a plane, Distance from a point to a plane, Angle between two planes, Parallel planes, Coplanar points, Coplanar lines, Intersection of planes, Plane equations, Plane geometry in 3D.*

Planes in 3D space are two-dimensional flat surfaces that extend infinitely in all directions. They are defined by their orientation, which is determined by a normal vector perpendicular to the plane, and a point that lies on the plane. The normal vector is crucial in determining the orientation and properties of the plane.

In economics, problems involving multiple dimensions often arise when analysing complex systems and making decisions based on various variables. Here are some examples of problems in economics that involve multiple dimensions:

Portfolio Optimization: When constructing an investment portfolio, economists consider multiple dimensions such as risk, return, and diversification. They aim to find the optimal combination of assets that maximizes returns while minimizing risk.

Production Planning: Firms need to determine the optimal production levels for multiple products, taking into account factors such as resource constraints, demand, and production costs. This involves optimizing production across different dimensions to maximize profitability.

International Trade: Analysing trade patterns and determining comparative advantages involve considering multiple dimensions, including production costs, exchange rates, market demand, and trade policies. Economists assess the impact of these factors on trade flows and economic welfare.

Consumer Choice: Studying consumer behaviour involves considering multiple dimensions such as price, income, preferences, and the availability of substitute goods. Economists analyse how consumers make choices and allocate their budgets across different goods and services.

Economic Growth: Understanding economic growth requires considering various dimensions such as investment, technological progress, labour market dynamics, and productivity. Economists analyse how these factors interact and influence long-term economic growth rates.

Environmental Economics: Addressing environmental challenges involves analysing the trade-offs between economic growth, environmental sustainability, and resource conservation. Economists evaluate policies that aim to balance these dimensions and promote sustainable development.

✍ *These are just a few examples of how problems involving multiple dimensions arise in economics. By considering various factors and dimensions, economists can provide insights and develop strategies for effective decision-making and policy formulation.*

Planes in 3D space have various properties and characteristics, including:

- They divide 3D space into two half-spaces.

- They are parallel to each other if their normal vectors are parallel.
- The angle between two planes is determined by the angle between their respective normal vectors.
- The distance between a point and a plane can be calculated using the formula involving the normal vector and the coordinates of the point.
- The intersection of two planes can be a line if the planes are not parallel or coincident.
- The orientation of a plane can be determined using the direction cosines of its normal vector.
- Planes can be used to model and solve problems in economy.

✍ *Studying planes in 3D space allows for a deeper understanding of spatial relationships, geometric properties, and the ability to solve problems involving multiple dimensions.*

A plane: A plane in 3D space is a two-dimensional flat surface that extends infinitely in all directions. It can be thought of as an infinitely large sheet that has no thickness but covers an entire 2D space within the 3D coordinate system. A plane is defined by its orientation and position in space.



Figure 2.6 The illustration of a plane

To define a plane, you need two essential pieces of information:

Normal Vector: The normal vector is a vector that is perpendicular (orthogonal) to the plane. It determines the orientation or direction in which the plane is facing. Any vector parallel to the normal vector lies entirely within the plane.

Point on the Plane: To specify the position of the plane in 3D space, you need a single point that lies on the plane. This point is often referred to as a "reference point" or "base point."

✍ *The normal vector and the reference point together define the orientation and position of the plane in 3D space. The normal vector determines the direction in which the plane faces, while the reference point specifies its location within the coordinate system*.

The forms of equations used to represent planes in 3D space: There are several forms of equations used to represent planes in 3D space. These forms provide different ways to describe the orientation and position of a plane:

Scalar equation of a plane: The scalar equation of a plane is given by

$$ax + by + cz + d = 0$$

where *(a, b, c)* is the normal vector to the plane, and (x, y, z) are the coordinates of any point on the plane. The constant term d represents the distance of the plane from the origin along the direction of the normal vector. Distance d is calculated as follows; $d = -a * x_0 - b * y_0 - c * z_0$, where $P_0 = (x_0, y_0, z_0)$ is a point from this plane. The normal vector (a, b, c) is perpendicular to the plane and

determines its orientation in space. The magnitude of the normal vector represents the rate of change of the plane's equation with respect to the x, y, and z axes.

Example 2.3.a Suppose we have a plane with a normal vector of $(2, -1, 3)$ and passing through the point $P_0 = (1, 2, -1)$. The scalar equation of this plane is

$$2x - y + 3z + 3 = 0$$

here, the coordinates of any point are (x, y, z), the normal vector $\bar{n} = (2, -1, 3)$ and the distance d is calculate as $d = -a * x_0 - b * y_0 - c * z_0$.

✍ *The equation represents all the points (x, y, z) that satisfy this equation and lie on the plane.*

Vector equation of a plane: The vector equation of a plane is expressed as

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{p} \cdot \boldsymbol{n}$$

where $\boldsymbol{r}$ is the position vector of any point on the plane, $\boldsymbol{n}$ is the normal vector to the plane, and $\boldsymbol{p}$ is the position vector of a known point on the plane. This equation states that the dot product of the position vector of any point on the plane and the normal vector is equal to the dot product of the known point's position vector and the normal vector.

Example 2.3.b Consider a plane with a normal vector of $(1, 2, -3)$ and passing through the point $(2, -1, 3)$. The vector equation of this plane is given by

$$(x, y, z) \cdot (1, 2, -3) = (2, -1, 3) \cdot (1, 2, -3).$$

here, the position vector of any point $\bar{r} = (x, y, z)$, the normal vector $\bar{n} = (1, 2, -3)$ and the position vector of the point $(2, -1, 3)$ is the vector $\bar{r} = (2, -1, 3)$.

✍ *This equation represents all the points (x, y, z) that satisfy this equation and lie on the plane.*

Cartesian equation of a plane: The Cartesian equation of a plane is given by

$$ax + by + cz + d = 0$$

where (a, b, c) is the normal vector to the plane $(\bar{n})$, and (x, y, z) are the coordinates of any point on the plane. The constant term d represents the distance of the plane from the origin along the direction of the normal vector. The general form allows for a more flexible representation of the plane equation. The coefficients a, b, and c determine the orientation of the plane, while the constant term d affects the distance of the plane from the origin.

Example 2.3.c Let's say we have a plane with a normal vector of $(3, -2, 1)$ and passing through the point $(-1, 2, 4)$. The Cartesian equation of this plane is: $3x - 2y + z + 11 = 0$.

✍ *This equation represents all the points (x, y, z) that satisfy this equation and lie on the plane.*

✍ *In each example, the equations provide a different representation of the same plane. The scalar equation emphasizes the coefficients of x, y, and z and their relationship to the constant term. The vector equation shows the dot product relationship between the position vector of any point on the plane and the normal vector. The Cartesian equation expresses the plane in terms of the coefficients of the Cartesian coordinates. These forms of equations allow us to describe and analyse planes in 3D space from different perspectives, providing insight into their orientation, position, and relationships with other geometric elements.*

The distance between a point and a plane: The distance between a point and a plane can be calculated using the formula: $Distance = |ax + by + cz + d| / \sqrt{a^2 + b^2 + c^2}$ where (a, b, c) is the normal vector of the plane $(\bar{n})$, (x, y, z) are the coordinates of the point P, and d is the constant term of the plane equation.

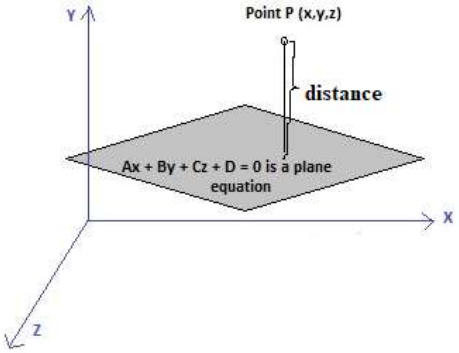


Figure 2.7 The illustration of the distance between a point and a plane

Example 2.3.d Suppose we have a plane with the equation $2x - 3y + z - 5 = 0$, and we want to find the distance between this plane and the point $P(1, -2, 3)$.

To calculate the distance, we need to determine the normal vector of the plane, which in this case is $\bar{n} = (2, -3, 1)$. The coordinates of the point P are $(1, -2, 3)$.

Using the formula, we can substitute the values into the formula:

$$Distance = \frac{|(2 * 1) + (-3 * -2) + (1 * 3) - 5|}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{|2 + 6 + 3 - 5|}{\sqrt{4 + 9 + 1}} = \frac{|6|}{\sqrt{14}} = \frac{6}{\sqrt{14}}$$

So, the distance between the point $P(1, -2, 3)$ and the plane $2x - 3y + z - 5 = 0$ is $\frac{6}{\sqrt{14}}$ units.

✍ *This distance represents the shortest distance between the point and the plane, and it is measured along a perpendicular line from the point to the plane. By calculating the distance between a point and a plane, we can determine how "close" or "far" the point is from the plane, providing insight into spatial relationships and geometric properties.*

The angle between two planes: The angle between two planes can be calculated using the normal vectors of the planes. The formula to find the angle between two planes is given by:

$$\theta = arccos\left(\left(\frac{|(a_1, b_1, c_1) \cdot (a_2, b_2, c_2)|}{(a_1^2 + b_1^2 + c_1^2) * (a_2^2 + b_2^2 + c_2^2)}\right)\right)$$

where (a_1, b_1, c_1) and (a_2, b_2, c_2) are the normal vectors of the two planes.

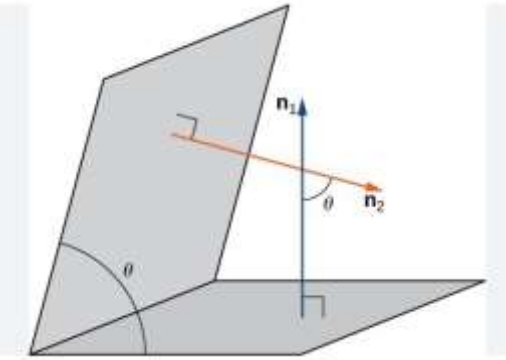


Figure 2.8 The illustration of the angle between two planes

Example 2.3.e Consider two planes with the following normal vectors: Plane 1: $\overline{n_1} = (2, 1, -3)$ and Plane 2: $\overline{n_2} = (3, -2, 1)$. To find the angle between these two planes, we can use the formula above:

$$\theta = arccos\left(\frac{|(2,1,-3)\cdot(3,-2,1)|}{\sqrt{(2^2 + 1^2 + (-3)^2) * (3^2 + (-2)^2 + 1^2)}}\right)$$

Calculating the dot product and magnitudes, we have:

$$\theta = arccos\left(\frac{\left|\left((2*3) + (1*-2) + (-3*1)\right)\right|}{\sqrt{(4+1+9)*(9+4+1)}}\right) = \theta = arccos\left(\frac{|6-2-3|}{\sqrt{14*14}}\right) = arcos\left(\frac{1}{14}\right)$$

Using a calculator, we find that $arcos\left(\frac{1}{14}\right)$ is approximately 85.41 degrees.

Therefore, the angle between the two planes is approximately 85.41 degrees.

✍ *The angle between two planes provides information about the spatial relationship between them. It helps us understand how the planes are oriented relative to each other in 3D space.*

The parallel planes: To determine if two planes are parallel, you need to compare their normal vectors. If the normal vectors are parallel or proportional, the planes are parallel. Here's a step-by-step process:

Step 1: Obtain the normal vectors of the two planes. For Plane 1, let the normal vector be $\overline{n_1} = (a_1, b_1, c_1)$. For Plane 2, let the normal vector be $\overline{n_2} = (a_2, b_2, c_2)$.

Step 2: Check if the normal vectors are parallel or proportional. To determine this, divide the components of one normal vector by the corresponding components of the other normal vector. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$ (some constant), then the normal vectors are parallel or proportional.

Step 3: Conclusion: If the normal vectors are parallel or proportional, the planes are parallel. This means the two planes are oriented in the same direction or are coincident.

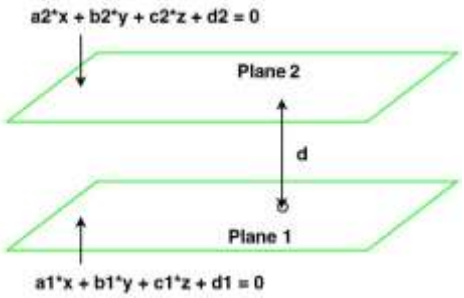


Figure 2.9 The illustration of the two parallel planes

Example 2.3.e Consider two planes with the following equations and normal vectors:

Plane 1: $2x + 3y - z = 4$ ⇒Normal vector: $\overline{n_1} = (2, 3, -1)$.

Plane 2: $4x + 6y - 2z = 8$ ⇒Normal vector: $\overline{n_2} = (4, 6, -2)$.

To determine if these two planes are parallel, we compare their normal vectors:

$$\frac{2}{4} = \frac{3}{6} = \frac{-1}{-2}$$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$, the normal vectors are proportional. Therefore, the planes are parallel.

✍ *In this example, the normal vectors of the two planes are parallel, indicating that the planes are also parallel. This means that the two planes have the same orientation and are either coincident or at a constant distance from each other.*

The intercept form of a plane equation: The intercept form of a plane equation is a specific representation of a plane that expresses its equation in terms of its intercepts on the coordinate axes. The intercept form is useful for understanding the position and orientation of the plane in relation to the coordinate axes. The intercept form of a plane equation in 3D space is given by:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are the intercepts of the plane on the $x - axis$, $y - axis$, and z-axis, respectively.

In this form, the intercepts represent the points where the plane intersects the coordinate axes. The reciprocals of the intercepts $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ determine the slopes of the lines connecting the origin to the intercept points. These slopes provide information about the plane's orientation and its relationship with the coordinate axes.

Example 2.3.f Consider a plane that intercepts the $x - axis$ at 3, the $y - axis$ at -2, and the $z - axis$ at 5. We want to find the intercept form of the plane equation. The intercept form of the plane equation is:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Given that the plane intercepts the $x - axis$ at 3, the $y - axis$ at -2, and the $z - axis$ at 5, we have the intercepts: a = 3, b = -2, c = 5

Substituting these values into the intercept form equation, we get:

$$\frac{x}{3} + \frac{y}{-2} + \frac{z}{5} = 1$$

This equation represents the plane with intercepts $(3, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 5)$ on the $x-axis$, $y-axis$, and $z-axis$, respectively.

In this example, the intercept form of the plane equation $\frac{x}{3} + \frac{y}{-2} + \frac{z}{5} = 1$ *indicates that the plane intersects the* $x-axis$ *at 3, the* $y-axis$ *at -2, and the* $z-axis$ *at 5. This representation provides a clear understanding of the position and orientation of the plane with respect to the coordinate axes.*

✍ *The intercept form of a plane equation is particularly useful when studying planes in relation to coordinate systems, as it provides a straightforward representation of the intercept points. It helps visualize the position and orientation of the plane within the 3D space and facilitates understanding of its geometric properties and relationships with other objects*.

The intersection of two planes: To find the intersection of two planes in 3D space, you need to solve their equations simultaneously.

To verify the relative position of two planes, having their equations, we can use some specific criteria. Whether

Plane 1: $a_1x + b_1y + c_1z + d_1 = 0$

Plane 2: $a_2x + b_2y + c_2z + d_2 = 0$

plane equations.

1. **Distinct parallel planes:** If normal vectors are proportional, and not proportional to free terms then the planes are parallel: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$.
2. **Identical plan:** If normal vectors and free terms are proportional, then the planes are identical: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$.
3. **Concurrent planes:** Planes are competing if normal vectors are not proportional. In this case, the planes intersect in a line.

Example 2.3.g Consider two planes with the following equations:

Plane 1: $2x - 3y + 4z = 7$

Plane 2: $4x - 6y + 8z = 14$

We check the proportionality of normal vectors and free terms:

$$\frac{2}{4} = \frac{-3}{-6} = \frac{4}{8} = \frac{7}{14}$$

We conclude: The plans coincide.

✍ *The concept of planes intersecting along a line is important in understanding geometric relationships in 3D space and can have implications in various fields, including mathematics and economy.*

The conditions for coplanarity of points or lines: The conditions for coplanarity of points or lines can vary depending on the specific situation. Here are the conditions for coplanarity:

Coplanarity of Points:

a) Three points are always coplanar. Any three points in 3D space lie on a single plane.

b) If more than three points lie on the same plane, they are coplanar.

Coplanarity of Lines:

a) If two lines are parallel or coincident, they are coplanar.

b) If two lines intersect, they are always coplanar because the point of intersection lies on both lines.

Coplanarity of Multiple Lines:

a) If two or more lines lie on the same plane, they are coplanar.

b) If lines are parallel or coincident, they are coplanar.

c) If lines are defined by a common point and intersect each other, they are coplanar.

✍ *In general, for points or lines to be coplanar, they must lie within the same plane. This means that they can be contained in a flat surface without any of them extending out of the plane. Coplanarity is determined by analysing the relationships between the points or lines and confirming if they exist on the same plane or parallel planes.*

Converting plane equations from one form to another: A plane equation can be converted between different forms, such as the general form, the normal form, the intercept form, and the parametric form. Here's an overview of the conversions between these forms:

General Form to Normal Form: Given a plane equation in general form, $ax + by + cz + d = 0$, you can convert it to the normal form by extracting the normal vector from the coefficients of *x*, *y*, and *z*. The normal vector is (a, b, c), and the equation becomes $ax + by + c * z = d$.

Example 2.3.h Consider a plane with general equation $2x + 3y - z + 4 = 0$. We want to convert this equation to normal form.

To convert to normal form, we extract the coefficients of x, y, and z to obtain the normal vector. The normal vector will have the form (a, b, c).

For the equation $2x + 3y - z + 4 = 0$, we have: $a = 2, b = 3, c = -1$

Now, we can write the equation in normal form as:

$$2x + 3y - 1z = 4$$

The resulting equation is in normal form, where the coefficients of x, y, and z represent the normal vector of the plane.

The conversion between general form and normal form allows us to extract the normal vector, which provides information about the orientation and direction of the plane.

Normal Form to General Form: To convert a plane equation to general form, you may need to rearrange the normal equation to have the coefficients and the constant term on the same side, typically with all terms on the left-hand side equal to zero. The equation becomes $ax + by + cz + d = 0$.

Example 2.3.i Consider a plane with the equation $3x - 2y + 5z = 1$. We want to convert this equation from normal form to general form.

Now, rearrange the normal equation to have the coefficients and the constant term on the same side, typically with all terms on the left-hand side equal to zero. The equation becomes $3x - 2y + 5z - 1 = 0$.

✍ *The conversion between normal form and general form allows us to express the equation of the plane in a more traditional and familiar form, facilitating further analysis and manipulation if needed.*

General Form to Intercept Form: Converting a plane equation from general form to intercept form involves determining the intercepts of the plane on the coordinate axes. Divide each term in the equation by the opposite free coefficient (d) to obtain the intercepts. For example, if the equation is $ax + by + cz + d = 0$, the intercept form would be $\underbrace{\frac{x}{-d/a} + \frac{y}{-d/b} + \frac{z}{-d/c} - 1 = 0}_{\text{The general equation after dividing by opposite free term}} \Rightarrow \underbrace{\frac{x}{-\frac{d}{a}} + \frac{y}{-\frac{d}{b}} + \frac{z}{-\frac{d}{c}} = 1}_{\text{The Intercept Form of equation}}$.

Example 2.3.j Consider a plane with the equation $2x + 3y - z + 4 = 0$. We want to convert this equation from general form to intercept form.

In the general form, the equation is given as ax + by + cz + d = 0, where a, b, c, and d are the coefficients.

To convert to intercept form, we need to determine the intercepts of the plane on the coordinate axes. Divide each term in the equation by the corresponding coefficient to obtain the intercepts.

For the equation $2x + 3y - z + 4 = 0$, we have: $\frac{x}{-4/2} + \frac{y}{-4/3} + \frac{z}{-4/-1} - 1 = 0$

Now, we can write the equation in intercept form as: $\frac{x}{-2} + \frac{y}{-\frac{4}{3}} + \frac{z}{4} = 1$

✍ *The resulting equation is in intercept form, where the intercepts of the plane on the coordinate axes are represented by the denominators of the terms.*

The conversion between general form and intercept form allows us to understand the intercepts of the plane on the coordinate axes and provides a more intuitive representation of the plane's position and orientation.

Intercept Form to General Form: To convert a plane equation from intercept form to general form, you multiply each term by the respective denominator and simplify the equation. For example, if the equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, you would multiply each term by the common denominator of the numbers a, b and c to obtain the general form of the equation, $a'x + b'y + c'z + d' = 0$.

*Example 2.3.*k Consider a plane with the equation $\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1$. We want to convert this equation from intercept form to general form.

To convert to general form, we need to multiply each term of equation by the common denominator.

For the equation $\underbrace{\frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 1}_{\text{The Intercept Form}}$, we multiply each term by the respective common denominator, equal to 12: $\underbrace{4x + 3y + 6z = 12}_{\text{the equation after multiplying by common denominator}} \Rightarrow \underbrace{4x + 3y + 6z - 12 =}_{\text{The General Form}}$

✍ *The equation is now expressed without the denominators, allowing for a more traditional representation. The conversion between intercept form and general form allows us to express the equation of the plane in a standard form that facilitates further analysis and manipulation if needed*.

General Form to Parametric Form: Converting a plane equation from general form to parametric form involves expressing the equation in terms of two independent parameters (such as u and v) and finding the corresponding x, y, and z values. This conversion requires solving a system of equations involving the parameters and the plane equation.

Example 2.3.k Consider a plane with the equation $2x - y + 3z = 4$. We want to convert this equation from general form to parametric form.

In the general form, the equation is given as $ax + by + cz + d = 0$, where a, b, c, and d are the coefficients.

To convert to parametric form, we need to express the equation in terms of two independent parameters (such as u and v) and find the corresponding x, y, and z values.

Solve one variable in terms of the other variables:

From the equation $2x - 3y + z + 4 = 0$, we can solve for z: $z = -2x + 3y - 4$.

Choose two independent parameters, let's say u and v. Assign the remaining variables as functions of u and v: $x = u$ and $y = v$.

Now we have the parametric equations: $x = u,\ y = v,\ z = 2u - 3v + 4$

These equations represent the plane in parametric form, where the variables u and v can take any values, and corresponding values of x, y, and z can be obtained using the parametric equations.

✍ *In this example, we converted the plane equation* $2x - 3y + z + 4 = 0$ *from general form to parametric form by expressing x, y, and z in terms of independent parameters u and v. The resulting parametric equations allow us to generate points on the plane by varying the values of u and v.*

Parametric Form to General Form: To convert a plane equation from parametric form back to general form, you substitute the parameter expressions into the equation and simplify it. This may involve algebraic manipulation to eliminate the parameters and express the equation solely in terms of x, y, and z.

Example 2.3.l Consider a plane with the parametric equations:

$$x = 2u + v, \qquad y = 3u - 2v, \qquad z = u + 4v$$

To convert these equations to general form, we need to eliminate the parameters (u and v) and express the equation solely in terms of x, y, and z.

Solve the first equation for u:

$$u = \frac{x - v}{2}$$

Substitute the value of u into the other equations:

$$y = 3 * \frac{x - v}{2} - 2v$$

$$z = \frac{x - v}{2} + 4v$$

Simplify and eliminate the parameters: Multiply both sides of the equation for y by 2 and both sides of the equation for z by 2:

$$2y = 3x - 3v - 4v$$
$$2z = x - v + 8v$$

Combine like terms:

$$\begin{cases} 2y = 3x - 3v - 4v \\ 2z = x - v + 8v \end{cases} \Rightarrow \begin{cases} 2y = 3x - 7v \\ 2z = x + 7v \end{cases} \Rightarrow \begin{cases} 7v = 3x - 2y \\ 7v = 2z - x \end{cases} \Rightarrow 3x - 2y = 2z - x$$

Rearrange the equations to match the general form $ax + by + cz + d = 0$:

$$4x - 2y - 2z = 0$$

✍ *The conversion from parametric form to general form allows us to express the equation of the plane in a standard form that facilitates further analysis and manipulation if needed.*

The orientation of a plane: The orientation of a plane can be determined using the direction cosines of its normal vector. The direction cosines of a vector represent the cosine of the angles between the vector and each of the coordinate axes. In the case of a plane, the normal vector is perpendicular to the plane and determines its orientation. The direction cosines of the normal vector provide information about the angles between the normal vector and the coordinate axes. Specifically, the direction cosines represent the cosines of the angles between the normal vector and the positive *x-axis*, positive *y-axis*, and positive *z-axis*.

Let's denote the direction cosines of the normal vector as *l*, *m*, and *n*, respectively. The direction cosines can be calculated as follows:

$l = cos(\alpha)$, where α is the angle between the normal vector and the positive *x-axis*.

$m = cos(\beta)$, where β is the angle between the normal vector and the positive *y-axis*?

$n = cos(\gamma)$, where γ is the angle between the normal vector and the positive *z-axis*.

The values of *l*, *m*, and *n* can range between *-1* and *1*, inclusive. By examining the signs and magnitudes of the direction cosines, we can determine the orientation of the plane:

- If all direction cosines are positive, the normal vector points in the positive octant, indicating that the plane is facing away from the origin.
- If all direction cosines are negative, the normal vector points in the negative octant, indicating that the plane is facing towards the origin.
- If some direction cosines are positive and others are negative, the normal vector spans different octants, indicating that the plane is tilted or inclined relative to the coordinate axes.

Example 2.3.m Consider a plane with the normal vector $\bar{n} = (2, -3, 1)$.

To determine the orientation of the plane using the direction cosines. We calculate the direction cosines *l*, *m*, and *n* as follows:

$$l = cos(\alpha) = cos(angle\ between\ \bar{n}\ and\ positive\ x - axis) = \frac{2}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{2}{\sqrt{14}}$$

$$m = cos(\beta) = cos(angle\ between\ \bar{n}\ and\ positive\ y - axis) = \frac{-3}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{-3}{\sqrt{14}}$$

$$n = cos(\gamma) = cos(angle\ between\ \bar{n}\ and\ positive\ z - axis) = \frac{1}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{1}{\sqrt{14}}$$

By examining the signs and magnitudes of the direction cosines, we can determine the orientation of the plane:

- The value of *l* is positive, indicating that the normal vector points in the positive *x-axis* direction.

- The value of *m* is negative, indicating that the normal vector points in the negative *y-axis* direction.
- The value of *n* is positive, indicating that the normal vector points in the positive *z-axis* direction.

✍ *Based on these direction cosines, we can conclude that the plane is oriented such that its normal vector points towards the positive x-axis, negative y-axis, and positive z-axis directions. This information provides insights into the orientation and position of the plane in three-dimensional space. Please note that the direction cosines are calculated based on the given normal vector. If you have a different normal vector, the direction cosines and the resulting orientation of the plane may vary.*

The equation of a plane given three non-collinear points on the plane: To find the equation of a plane given three non-collinear points on the plane, you can use the following steps:

Step 1: *Identify the three non-collinear points*: Let's denote them as $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$.

Step 2: *Find* two *vectors on the plane:* Determine two vectors by subtracting one point from another. For example, vector $\overrightarrow{AB} = B - A$ and vector $\overrightarrow{AC} = C - A$.

Step 3: *Calculate the normal vector:* Find the cross product of the two vectors obtained in step 2. The cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ will give a vector that is perpendicular to the plane: normal vector $\bar{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

Step 4: *Determine the coefficients of the equation*: Use the components of the normal vector to determine the coefficients a, b, and c in the plane equation $ax + by + cz + d = 0$. These coefficients correspond to the components of the normal vector $\boldsymbol{n}$: $a = n_x,\ b = n_y,\ c = n_z$

Step 5: *Find the constant term:* Substitute the coordinates of any of the three points (A, B, or C) into the plane equation and solve for the constant term *d*: $d = -(a * x_1 + b * y_1 + c * z_1)$

Step 6: *Write the equation of the plane*: Using the values of a, b, c, and d obtained in steps 4 and 5, write the equation of the plane in the form $ax + by + cz + d = 0$.

Example 2.3.n Consider three non-collinear points: $A(1, 2, 3)$, $B(2, 4, 6)$, and $C(3, 1, 5)$.

Step 1: Identify the three non-collinear points: $A(1, 2, 3)$, $B(2, 4, 6)$, and $C(3, 1, 5)$.

Step 2: Find two vectors on the plane:

Vector $\overrightarrow{AB} = B - A = (2 - 1, 4 - 2, 6 - 3) = (1, 2, 3)$

Vector $\overrightarrow{AC} = C - A = (3 - 1, 1 - 2, 5 - 3) = (2, -1, 2)$

Step 3: Calculate the normal vector: normal vector

$\bar{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (1, 2, 3) \times (2, -1, 2) = (4, -1, -5)$

Step 4: Determine the coefficients of the equation; $a = n_x = 4,\ b = n_y = -1,\ c = n_z = -5$

Step 5: Find the constant term: using point $A(1, 2, 3)$:

$d = -(a * x_1 + b * y_1 + c * z_1) = -(4 * 1 + (-1 * 2 + (-5) * 3) = -(4 - 2 - 15) = 13$

Step 6: Write the equation of the plane. The equation of the plane is: $4x - y - 5z + 13 = 0$.

In this example, we found the equation of a plane given three non-collinear points A (1, 2, 3),

B (2, 4, 6), and C (3, 1, 5). The resulting equation is $4x - y - 5z + 13 = 0$, which represents the plane that passes through these three points.

✍ *The resulting equation represents the equation of the plane that passes through the three non-collinear points A, B, and C. It's important to note that the points chosen must be non-collinear to uniquely define a plane. If the points are collinear, they will not form a plane, and the equation cannot be determined.*

The properties of the plane in 3D space: Planes in three-dimensional (3D) space possess several properties and characteristics that define their geometric behaviour. Here are some key properties:

Properties 1 Infinite extent: A plane extends infinitely in all directions, both horizontally and vertically.

Properties 2 Flat surface: A plane is a flat, two-dimensional surface with no curvature.

Properties 3 Determined by three non-collinear points: A unique plane can be determined by three non-collinear points. These points should not lie on the same line.

Properties 4 Normal vector: Each plane has a normal vector that is perpendicular to the plane.

Properties 5 Distance from the origin: The distance from the origin to a plane is given by the magnitude of the normal vector.

Properties 6 Parallel planes: Two planes are parallel if their normal vectors are parallel or proportional to each other.

Properties 7 Intersecting planes: Two planes intersect in a line if they are not parallel. The line of intersection is perpendicular to the normal vectors of both planes.

Properties 8 Angle between planes: The angle between two planes is determined by the angle between their normal vectors. It represents the deviation or inclination between the planes.

Properties 9 Equations of planes: Planes can be represented using various forms of equations, such as the general form ($Ax + By + Cz + D = 0$), the parametric form, or the normal form.

Properties 10 Coplanarity: Points or lines that lie in the same plane are said to be coplanar. They share the property of being contained within the same flat surface.

✍ *These properties and characteristics allow us to study and analyse the behaviour of planes in 3D space. They provide a foundation for solving problems involving planes, such as finding intersections, distances, angles, and relationships between planes in geometric and mathematical applications.*

Applications of the plan to solve problems in the economic field: Planes can be used as a modelling tool in the economic field to represent and solve various real-world problems. Here are some examples of how planes can be applied:

Supply and demand analysis: In economics, supply and demand curves can be represented as planes. The supply curve shows the relationship between the quantity supplied and the price, while the demand curve represents the relationship between the quantity demanded and the price. The intersection of these planes can be analysed to determine the equilibrium price and quantity in a market.

Production possibilities frontier (PPF): The PPF represents the maximum combination of goods or services that can be produced with limited resources. It can be represented as a plane in a three-dimensional space, where different points on the plane represent different production combinations. By analysing the PPF, economists can understand the trade-offs between different goods or services and make decisions regarding resource allocation.

Portfolio optimization: In finance and investment analysis, planes can be used to represent risk and return trade-offs for different portfolios. The expected return and risk of a portfolio can be plotted on a plane, and efficient portfolios lie on a specific plane called the efficient frontier. By analysing this plane, investors can determine the optimal portfolio allocation that maximizes returns for a given level of risk.

Cost and production analysis: Cost functions and production functions can be represented as planes in the economic field. Cost functions relate the cost of production to factors such as labour, capital, and inputs. Production functions represent the relationship between inputs and the quantity of output produced. These planes can be used to analyse cost minimization, production efficiency, and optimal resource allocation.

Economic modelling and simulations: Planes can be utilized in economic modelling and simulations to represent various economic relationships and interactions. For example, in macroeconomic models, planes can be used to depict the relationships between variables such as inflation, unemployment, and GDP growth. These planes can help economists analyse the impacts of policy changes and make predictions about the economy.

✍ *By using planes to model and solve real-world economic problems, economists can gain insights into complex relationships, make informed decisions, and develop strategies to address economic challenges.*

Questions you should be able to answer:

What is a plane in 3D space?

How can a plane be represented using different forms of equations?

How is a plane defined by a normal vector and a point on the plane?

What is the distance between a point and a plane?

How can the angle between two planes be calculated?

How do you determine if two planes are parallel?

What is the intercept form of a plane equation?

How do you find the intersection of two planes?

What are the conditions for coplanarity of points or lines?

How can a plane equation be converted between different forms?

How can the orientation of a plane be determined using the direction cosines of its normal vector?

What are the properties and characteristics of planes in 3D space?

How can you find the equation of a plane given three non-collinear points on the plane?

How can planes be used to model and solve real-world problems in various fields?

Proposed Exercises:

2.3.1 These exercises aim to reinforce the understanding of planes in three-dimensional space in the context of economic mathematics. Use your knowledge of vector algebra and geometry to complete the missing words. Understanding planes in 3D space is crucial in economic applications, as they are often used to model constraints and relationships in optimization and allocation problems.

a) In 3D space, a plane can be uniquely determined by a point on the plane and _____.

b) The equation of a plane in 3D space can be represented as _____.

c) If the coefficients of the variables *x, y,* and *z* in the equation of a plane are proportional to each other, the plane is said to be a _____.

d) Two planes in 3D space are _____ if their normal vectors are parallel.

e) The angle between two planes in 3D space can be found using the dot product of their _____.

f) If a line is parallel to a plane in 3D space, then the direction vector of the line is _____ to the normal vector of the plane.

g) The intersection of two planes in 3D space is a _____.

h) If two planes in 3D space are not parallel, they will intersect in a _____.

i) In economic mathematics, planes in 3D space are used to represent _____ and _____ in optimization problems and resource allocation models.

j) The normal vector of a plane in 3D space can be found using the _____ coefficients of the variables *x, y,* and *z* in the plane's equation.

2.3.2 In these exercises, fill in the missing words to complete the statements related to planes in 3D space in the context of economic mathematics. Understanding planes in 3D space is crucial for various applications, including optimization problems and economic modelling.

a) Equation of a Plane: Consider a plane in 3D space that passes through the point *(2, 3, 1)* and is perpendicular to the vector *(1, -2, 3)*. The equation of the plane is given by: __ + __ + __ = __

b) Intersection of Two Planes: Suppose there are two planes in 3D space with the following equations: Plane A: $3x - y + 2z = 5$ Plane B: $2x + 4y - 6z = 8$. Determine whether the two planes are __, __, or __.

c) Distance from a Point to a Plane: Consider a plane in 3D space with the equation $2x + 3y - z = 6$. Calculate the distance from the point *(1, -2, 3)* to the plane. The distance from the point to the plane is ___.

d) Angle between Two Planes: Let *Plane P1* have the equation $2x + y - 3z = 4$, and *Plane P2* have the equation $x - 2y + 5z = 7$. Find the ___ between *Plane P1* and *Plane P2*. The angle between *Plane P1* and *Plane P2* is ___ degrees.

e) Parallel and Perpendicular Planes: Consider two planes in 3D space with the following equations: Plane C: $3x + 2y - 4z = 9$ Plane D: $6x + 4y - 8z = 3$. Determine whether Plane C and Plane D are ___ or ___ to each other.

f) Economic Interpretation of Planes: In economic mathematics, planes in 3D space can represent various situations. For example, a plane defined by the equation $2x + 3y - z = 12$ could represent a ___ constraint in an optimization problem.

g) Finding the Normal Vector: Given a plane in 3D space with the equation $4x - 2y + 5z = 10$, determine the ___ vector of the plane. The normal vector of the plane is ___.

2.3.3 In these exercises, pay attention to the different forms of representing planes in 3D space: general form ($ax + by + cz + d = 0$) and normal form ($ax + by + cz = d'$). Also, consider the application of planes in economic mathematics, such as determining intersections, distances, and linear independence in economic models. Understanding planes in 3D space is essential in various applications, including economic modelling and optimization. Show all the steps involved in solving the exercises to fully grasp the concepts.

a) Equation of a Plane: Consider a plane P in 3D space that contains the points $A(1,2,3), B(3,4,5)$, and $C(2,1,6)$: *a.1)* Find the equation of the plane P in the general form ($ax + by + cz + d = 0$). *a.2)* Rewrite the equation of the plane P in the normal form ($ax + by + cz = d'$).

b) Distance from Point to a Plane: Consider a plane Q in 3D space with the equation $2x - y + 3z = 6$. Find the distance from the point $P(1,4,2)$ to the plane Q.

c) Intersection of Planes: Consider two planes P1 and P2 in 3D space with the equations: $P1: 2x - y + 3z = 6$ and $P2: x + 3y - 2z = 4$: *c.1)* Determine if the planes P1 and P2 are parallel or intersecting. *c.2)* If the planes intersect, find the line of intersection.

d) Plane through Points: Consider three non-collinear points $A(1,2,3), B(3,4,5)$, and $C(2,1,6)$: *d.1)* Find the equation of the plane that passes through points A, B, and C in the general form ($ax + by + cz + d = 0$). *d.2)* Rewrite the equation of the plane in the normal form ($ax + by + cz = d'$).

e) Linear Independence in Economic: Models In the context of economic mathematics, consider three vectors representing the variables in an economic model: $\boldsymbol{v1} = (2,1,3)$, $\boldsymbol{v2} = (4,2,6)$, $\boldsymbol{v3} = (1,1,1)$: *e.1)* Determine if the vectors ***v1, v2***, and ***v3*** are linearly dependent or linearly independent. *e.2)* How would the linear independence of these vectors affect the economic model they represent?

f) Plane Equations: Consider a plane in three-dimensional space with the following properties: The plane contains the point $P(2,3,1)$ and is parallel to the vector $\boldsymbol{v} = (1,-1,2)$: *f.1)* Find the equation of the plane in scalar product form. *f.2)* Find the equation of the plane in vector form. *f.3)* Find the equation of the plane in Cartesian form.

g) Intersection of Planes: Consider two planes in three-dimensional space with the following equations, Plane1: $2x - y + 3z = 7$ and Plane2: $x + 2y - z = 4$: *g.1)* Determine whether the two planes are parallel, intersecting, or coincident. *g.2)* If they intersect, find the line of intersection.

h) Transition between Plane Forms: Consider a plane in three-dimensional space with the following equation in Cartesian form: $2x - 3y + z = 6$: *h.1)* Convert the equation of the plane to scalar product form. *h.2)* Convert the equation of the plane to vector form.

i) Economic Application: Suppose a company produces two types of goods, X and Y, using two types of resources, A and B. The production of X requires 3 units of resource A and 2 units of resource B, while the production of Y requires 1 unit of resource A and 4 units of resource B. The company

has 15 units of resource A and 20 units of resource B available: *i.1)* Formulate the system of inequalities representing the resource constraints for producing goods X and Y. *i.2)* Plot the feasible production region in three-dimensional space. *i.3)* Determine the optimal production point for maximizing the total production of goods X and Y.

2.3.4 These exercises explore the concept of coplanarity of vectors and planes in three-dimensional space and their application in economic mathematics. It allows you to understand how geometric interpretations can have economic significance in various real-world scenarios. Show all the steps involved in solving the exercises and interpret the results in the context of economic applications:

a) Coplanarity of Vectors: Consider the following three vectors in three-dimensional space: $\boldsymbol{v1} = (1, 2, 3)$, $\boldsymbol{v2} = (4, 5, 6)$, $\boldsymbol{v3} = (7, 8, 9)$: *a.1)* Determine if the vectors ***v1, v2***, and ***v3*** are coplanar (lie in the same plane). *a.2)* If the vectors are coplanar, find the equation of the plane they lie in.

b) Economic Interpretation of Coplanarity: Suppose you are an economist studying the production of two goods, X and Y. You have three data points representing the production levels (in thousands of units) of the two goods over three consecutive quarters as follows: $Quarter\ 1\text{:}\ (X1, Y1) = (2, 3)$, $Quarter\ 2\text{:}\ (X2, Y2) = (4, 5)$, $Quarter\ 3\text{:}\ (X3, Y3) = (6, 7)$: *b.1)* Interpret geometrically the economic meaning of coplanarity in this context. *b.2)* Use the data points to check if the production levels of goods X and Y are coplanar.

c) Coplanarity of Economic Vectors: Consider the following three vectors representing economic variables in three-dimensional space: $\boldsymbol{v1} = (100, 200, 300)$, $\boldsymbol{v2} = (400, 500, 600)$, $\boldsymbol{v3} = (700, 800, 900)$: *c.1)* Determine if the vectors ***v1, v2***, and ***v3*** are coplanar (lie in the same plane). *c.2)* If the vectors are coplanar, find the equation of the plane they lie in.

d) Economic Interpretation of Coplanarity: Suppose you are a financial analyst studying the investment portfolios of three companies, *X*, *Y*, and *Z*. You have three data points representing the returns on investment (ROI) of these companies over three consecutive years as follows: $Year\ 1\text{:}\ (X1, Y1, Z1) = (10\%, 5\%, 8\%)$, $Year\ 2\text{:}\ (X2, Y2, Z2) = (15\%, 12\%, 9\%)$ and $Year\ 3\text{:}\ (X3, Y3, Z3) = (18\%, 10\%, 7\%)$: *d.1)* Interpret geometrically the economic meaning of coplanarity in this context. *d.2)* Use the data points to check if the ROI of companies X, Y, and Z are coplanar.

2.3.5 These exercises involve working with planes in three-dimensional space and apply the concepts of orientation and feasibility in the context of economic mathematics. Show all the steps involved in solving the exercises to demonstrate your understanding of the plane equations and their applications in economic problems

a) Plane Equations: Consider the following planes in three-dimensional space: Plane P1: $2x - 3y + z = 4$, Plane P2: $x + 2y - 3z = 5$: *a.1)* Determine the normal vector for each plane. *a.2)* Write the equation of each plane in the form $ax + by + cz = d$. *a.3)* Find the angle between the two planes.

b) Intersection of Planes: Consider the following planes in three-dimensional space: Plane P1: $3x + y - 2z = 6$, Plane P2: $2x - y + z = 3$: *b.1)* Determine if the two planes are parallel, intersecting, or coincident. *b.2)* If the planes intersect, find the line of intersection.

c) Equations of Planes: Consider the following planes in three-dimensional space: Plane P1 with the equation $2x + 3y - z = 5$, Plane P2 with the equation $x - 2y + 4z = 8$, Plane P3 with the equation $3x + y + 2z = 1$. Determine the normal vectors and point-normal form equations for each plane.

d) Parallel and Perpendicular Planes: Consider the following planes in three-dimensional space: Plane P1: $2x + 3y - z = 5$ Plane P2: $x - 2y + 4z = 8$ Plane P3: $3x + y + 2z = 1$: *d.1)* Determine whether planes P1 and P2 are parallel, perpendicular, or neither. *d.2)* Determine whether planes P1 and P3 are parallel, perpendicular, or neither. *d.3)* Determine whether planes P2 and P3 are parallel, perpendicular, or neither.

e) Intersection of Planes: Consider the following planes in three-dimensional space: Plane P1: $2x + y - z = 3$ Plane P2: $x - y + 2z = 4$: *e.1)* Determine the point of intersection between planes P1 and P2. *e.2)* Show that the intersection forms a line and find the direction vector of the line.

f) Orientation of a Plane: Consider the following plane in three-dimensional space: Plane P: $2x - 3y + z = 6$: *f.1)* Determine the normal vector of the plane P. *f.2)* Consider a vector $v = (4, 2, 1)$. Determine whether vector v is in the same direction or opposite direction of the normal vector, and interpret the orientation of the plane in the context of economic mathematics.

g) Economic Application of Planes: Consider an economic model where two goods, X and Y, are produced using labour and capital. The production possibilities are given by the following equations: Good X production: $2X + 3Y = 12$, Good Y production: $X + 4Y = 8$. *g.1)* Interpret the equations as planes in the three-dimensional space. *g.2)* Determine the point of intersection of the planes and interpret it in the context of economic production.

3 LINEAR PROGRAMMING

CONTENT UNITS that will be covered:

***Introduction to Linear Programming*:** Definition and basic concepts of LP, Applications and significance of LP in optimization problems, Assumptions and limitations of LP.

Formulating LP Problems: Identification of decision variables, objective function, and constraints, Understanding the linear nature of the objective function and constraints, Classification of LP problems as maximization or minimization problems.

Mathematical Representation of LP Problems: Writing the objective function and constraints in mathematical form, Notation and symbols used in LP, Interpretation of LP problem components in the context of real-world applications.

Geometric Interpretation of LP Problems: Graphical representation of LP problems in two dimensions, Understanding feasible regions, objective function lines, and optimal solutions, Graphical sensitivity analysis and interpretation of alternative solutions.

LP Solution Techniques: Simplex method: introduction and step-by-step solution procedure, Sensitivity analysis and interpretation of solution changes, Introduction to other solution techniques like graphical method, dual simplex method, etc.

LP Applications: Resource allocation problems, Production planning and scheduling, Inventory management and optimization, Transportation and distribution problems, Financial and investment planning, Project management and optimization.

✍ *These content units provide a comprehensive understanding of Linear Programming, its principles, techniques, and applications. They equip students with the knowledge and skills to formulate, solve, and interpret LP problems in various real-world scenarios.*

Keywords: *Linear Programming, Optimization, Objective function, Decision variables, Constraints, Feasible region, Optimal solution, Simplex method, Sensitivity analysis, Resource allocation, Production planning, Inventory management, Transportation problems*.

Learning Objectives:

Understand the basic concepts of Linear Programming: Define Linear Programming and its significance in optimization problems. Identify decision variables, objective function, and constraints in LP problems. Recognize the assumptions and limitations of LP.

Formulate LP problems: Formulate real-world problems as LP problems by identifying decision variables, objective functions, and constraints. Classify LP problems as maximization or minimization problems.

***Solve LP problems using appropriate techniques*:** Apply the simplex method to solve LP problems. Perform sensitivity analysis to interpret the impact of changes in the LP solution. Understand alternative solution techniques such as graphical method, dual simplex method, etc.

Interpret LP solutions: Interpret LP solutions in the context of the problem domain. Analyse and interpret sensitivity analysis results to understand the effects of changing parameters.

Apply LP to real-world applications: Apply LP to resource allocation problems, production planning, inventory management, transportation problems, and other relevant areas. Formulate LP models for specific economic scenarios and interpret the results.

***Apply critical thinking and problem-solving skills*:** Analyse real-world problems and identify opportunities for LP application. Apply LP techniques to make informed decisions and optimize solutions.

✍ *These learning objectives provide a framework for studying Linear Programming and developing a strong understanding of its principles, techniques, and applications*.

3.0 INTRODUCTION TO LINEAR PROGRAMMING (LP)

Linear Programming (LP) is a mathematical optimization technique that is used to solve problems involving the allocation of limited resources to achieve an optimal outcome. It provides a framework for making decisions in situations where there are multiple competing factors and constraints.

In LP, the problem is represented as a mathematical model with an objective function and a set of constraints. The objective function defines the goal or objective to be maximized or minimized, such as maximizing profit or minimizing costs. The constraints represent the limitations or restrictions on the decision variables, which are the quantities to be determined.

LP assumes that the objective function and constraints can be expressed as linear equations or inequalities. This means that the relationship between the decision variables and the objective function or constraints is linear. The decision variables represent the quantities that are being optimized.

The main goal of LP is to find the values of the decision variables that optimize the objective function while satisfying all the constraints. This is done by defining a feasible region, which is the set of all possible solutions that satisfy the constraints. The optimal solution is then determined by finding the point within the feasible region that maximizes or minimizes the objective function.

LP problems can be solved using various algorithms, with the simplex method being one of the most commonly used techniques. The simplex method iteratively improves the solution by moving from one feasible solution to another, gradually approaching the optimal solution.

LP has numerous applications in various fields, including operations research, supply chain management, finance, marketing, and production planning. It is used to optimize resource allocation, production schedules, inventory levels, transportation routes, and many other aspects of decision-making.

Linear Programming (LP) has some assumptions and limitations that are important to consider when using this optimization technique. Here are some common assumptions and limitations of LP:

Assumptions:

- ✍ *Linearity*: LP assumes that the objective function and constraints can be represented by linear relationships. Non-linear relationships may not be accurately modelled using LP.
- ✍ *Additivity*: LP assumes that the contributions of individual decision variables to the objective function and constraints are additive. Interactions or non-additive effects may not be properly captured in LP models.

- ✍ *Certainty*: LP assumes that all parameters and data values are known with certainty and do not have any variability or uncertainty. In real-world situations, uncertainty and variability are common, which may affect the accuracy of LP solutions.
- ✍ *Proportional Relationships:* LP assumes that the relationships between variables are proportional and do not exhibit non-proportional or non-linear behaviour. Deviations from proportional relationships may lead to inaccurate LP solutions.
- ✍ *Divisibility*: LP assumes that decision variables can take any real value within a given range. It does not account for situations where variables must be integer or binary values, which require the use of Integer Linear Programming (ILP) techniques.

Limitations*:*

- ✍ *Model Complexity*: LP may struggle to handle complex problems with a large number of decision variables, constraints, or non-linear relationships. The computational complexity increases exponentially as the problem size grows.
- ✍ *Limited Solution Space*: LP is limited to finding solutions within the feasible region defined by the constraints. If the feasible region is empty or very small, LP may not be able to provide meaningful solutions.
- ✍ *Single Objective*: LP typically focuses on optimizing a single objective function. Real-world problems often involve multiple conflicting objectives, which may require more advanced techniques like Multi-Objective Programming.
- ✍ *Sensitivity to Input Data*: LP solutions are sensitive to changes in the input data, such as coefficients in the objective function or constraints. Small changes in the data can result in significant changes in the optimal solution.
- ✍ *Assumptions of Rational Behaviour*: LP assumes that decision-makers act rationally and have complete information. In reality, decision-makers may have limited information, face cognitive biases, or make non-rational choices, which can impact the effectiveness of LP solutions.

✍ *By using LP, decision-makers can make more informed choices, improve efficiency, and achieve better outcomes in complex situations where multiple factors and constraints are involved. It provides a systematic and quantitative approach to decision-making and helps organizations find the most effective and efficient solutions to their problems.*

3.1 FORMULATING LP PROBLEMS

Keywords: *Decision variables, Objective function, Constraints, Feasible region, Maximization, Minimization, Resource allocation, Production planning, Scheduling, Cost minimization, Revenue maximization, Linear relationships, Non-negativity constraints, Upper and lower bounds, Linear inequalities, Integer variables, Mixed constraints, Production capacities, Demand constraints.*

Formulating LP problems involves the process of identifying and defining the components of the problem in a mathematical model. Here are some key aspects of formulating LP problems:

Objective Function: The objective function in Linear Programming (LP) represents the goal or objective that needs to be maximized or minimized. It is a mathematical expression that quantifies

the measure of success or effectiveness in achieving the desired outcome. The objective function is typically defined in terms of the decision variables.

Mathematical Description: The objective function is represented as a linear combination of the decision variables, where each decision variable is multiplied by a coefficient. The coefficients represent the relative importance or contribution of each variable to the objective. The objective function can be written in the general form:

$$Z = c_1x_1 + c_1x_2 + c_3x_3 + \ldots + c_nx_n$$

In this equation, Z represents the value of the objective function, $c_1, c_2, c_3, \ldots, c_n$ are the coefficients of the decision variables $x_1, x_2, x_3, \ldots, x_n$, respectively.

Example 3.1.a: Consider a production planning problem where a company produces two products, *A* and *B*. The objective is to maximize the total profit. Let x_1 represent the number of units of product *A* produced and x_2 represent the number of units of product *B* produced. The profit per unit of product *A* is *$10* and the profit per unit of product *B* is *$15*.

The objective function for this problem can be expressed as:

$$Z = 10x_1 + 15x_2$$

Here, *Z* represents the total profit, $10x_1$ represents the profit contribution from product *A*, and $15x_2$ represents the profit contribution from product *B*.

The objective is to find values for x_1 and x_2 that maximize the total profit *Z*, subject to any relevant constraints.

✍ *By formulating the objective function mathematically, the problem can be solved using LP techniques to determine the optimal values of the decision variables that maximize the objective, in this case, the total profit*.

Decision Variables: Decision variables in a linear programming problem represent the quantities or values that the decision-maker can control or manipulate to achieve the desired objective. They are the unknowns in the problem that need to be determined.

Mathematical Description: In a linear programming problem, decision variables are typically denoted by symbols such as *x*, *y*, or *z*. They are usually non-negative real numbers, but in some cases, they can be integer or binary variables.

Example 3.1.b: Consider a production planning problem where a company manufactures two products, *Product A* and *Product B*. Let's denote the number of units of Product A to be produced as *x* and the number of units of Product B as *y*. The decision variables can be described as follows:

Decision Variable *x*: Number of units of Product A to be produced

Decision Variable y: Number of units of Product B to be produced

In this example, *x* and *y* are the decision variables representing the quantities of Product A and Product B, respectively, that the company can produce. These variables can be adjusted or optimized to achieve the desired objective, such as maximizing profit or minimizing cost.

✍ *The values of these decision variables will be determined during the solution process of the linear programming problem. They will provide insights into the optimal production quantities or resource allocation strategies that can help the company achieve its goals effectively*.

Constraints: Constraints in Linear Programming (LP) problems are mathematical descriptions that define the limitations, restrictions, or requirements that must be satisfied by the decision variables. They play a crucial role in shaping the feasible region and finding the optimal solution.

Mathematical Description: A constraint can be expressed as a linear equation or inequality involving the decision variables. In general form, a constraint can be written as:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$$

where $x_1, x_2, \ldots, x_n$ are the decision variables, $a_1, a_2, \ldots, a_n$ are the coefficients, and b is the right-hand side constant.

Example 3.1.c: Consider a production planning problem where a company produces two products, *A* and *B*. The constraint can be formulated as: $3A + 4B \leq 240$, where *A* represents the quantity of product *A* produced, *B* represents the quantity of product *B* produced, and *240* represents the maximum production capacity.

In this example, the constraint states that the combined production of products *A* and *B* should not exceed *240* units. The coefficients *3* and *4* represent the resource requirements for each product, and the constraint ensures that the total resource usage remains within the available capacity.

✍ *By incorporating such constraints into the LP problem, decision-makers can enforce specific requirements or limitations based on the problem context. These constraints help define the feasible region and guide the optimization process towards finding the best solution that satisfies the given conditions.*

Feasible Region: The feasible region in Linear Programming (LP) represents the set of all possible solutions that satisfy the constraints of the problem. It is defined by the intersection of the constraints and determines the region in which the optimal solution exists.

Mathematical Description of Feasible Region: The feasible region is typically described using a system of linear inequalities or equations. Each constraint is represented as an inequality or equation, and the feasible region is the region where all the constraints are satisfied simultaneously.

To graphically represent the feasible region, plot the lines corresponding to each constraint on an *x-y coordinate plane*. Then, shade the region that satisfies all the constraints. The resulting shaded region represents the feasible region.

Example 3.1.d: For example, consider the following LP problem with two decision variables *x* and *y*:

Maximize: $3x + 4y$ subject to $2x + y \leq 5, x - y \geq 1, x, y \geq 0$.

The constraints can be represented as:

$$\begin{cases} 2x + y \leq 5 \\ x - y \geq 1 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

To determine the feasible region, we plot the lines $2x + y = 5$ and $x - y = 1$ on a coordinate plane and shade the region that satisfies both inequalities. The feasible region is the shaded region in the graph.



Figure 3.1 The illustration of the Feasible Region for Example 3.1.d

In this example, the feasible region is the region below the line $x - y = 1$ and below the line $2x + y = 5$, including the non-negative values of x and y (*the surface marked in red*).

✍ *The feasible region is a crucial concept in LP as it contains all the potential solutions that can be considered for optimization. The optimal solution lies within this region and is determined based on the objective function and constraints*.

Formulation Techniques: In formulation techniques, mathematical descriptions are used to represent the objective function, decision variables, and constraints in an LP problem.

Example 3.1.e: Production Planning Problem

A manufacturing company produces two products, *Product A* and *Product B*. The company wants to determine the optimal production quantities to maximize profit, considering the following constraints:

Limited resource availability:

- Product *A* requires *3* units of Resource *X* and *2* units of Resource *Y* per unit produced.
- Product *B* requires *4* units of Resource *X* and *1* unit of Resource *Y* per unit produced.
- There are *120* units of Resource *X* available and *80* units of Resource *Y* available.

Demand constraints:

- The demand for Product A is at most 50 units.
- The demand for Product B is at most 60 units.

Objective Function: Let's define the decision variables as follows:

- Let x be the number of units of Product A produced.
- Let y be the number of units of Product B produced.
- The objective is to maximize profit, which can be represented as:

$$Maximize: Profit = 10x + 15y$$

Constraints: Considering the resource availability and demand constraints, we can formulate the following linear inequalities:

- Resource *X* constraint: $3x + 4y \leq 120$
- Resource *Y* constraint: $2x + y \leq 80$
- Demand constraint for Product *A*: $x \leq 50$
- Demand constraint for Product *B*: $y \leq 60$
- Non-negativity constraint: $x \geq 0, y \geq 0$

Feasible Region: The feasible region represents the set of all possible solutions that satisfy the constraints. In this case, it is the intersection of the resource and demand constraints.

The LP problem can be formulated as follows:

Maximize: $Profit = 10x + 15y$

Subject to: $\begin{cases} 3x + 4y \leq 120 \\ 2x + y \leq 80 \\ x \leq 50 \\ y \leq 60 \\ x \geq 0 \\ y \geq 0 \end{cases}$

✍ *By solving this LP problem using appropriate techniques, such as the simplex method, the optimal production quantities for Products A and B can be determined, maximizing the profit while satisfying the given constraints. In this example, the formulation techniques involve clearly defining the decision variables, expressing the objective function and constraints as mathematical expressions, and considering the feasible region based on the intersection of constraints*.

Problem-specific Considerations: Problem-specific considerations in formulating LP problems involve tailoring the mathematical description to the specific problem domain and context.

Example 3.1.f:...Problem:

Production Planning Description: A manufacturing company produces two types of products, *A* and *B*, using limited resources. The company wants to determine the optimal production quantities of each product to maximize profit while satisfying various constraints. let's consider a specific scenario: The company has 100 units of resource available. The demand for products A and B is limited to 60 units.

Mathematical Description:

- Decision Variables:
  - ✓ Let x represent the quantity of product A to produce.
  - ✓ Let y represent the quantity of product B to produce.
- Objective Function:

  Maximize the objective function: $Z = 5x + 8y$ (where 5 is the profit per unit of product A and 8 is the profit per unit of product B).
- Constraints:
  - ✓ Resource constraint: $2x + 3y \leq 100$ (representing the total available resource limit).
  - ✓ Demand constraint: $x + y \leq 60$ (representing the demand limit for the products).
  - ✓ Non-negativity constraint: $x \geq 0, y \geq 0$ (since production quantities cannot be negative).
- Feasible Region:

  The feasible region is defined by the intersection of the constraints.

✍ *With these specifications, we can solve the LP problem using appropriate techniques (e.g., simplex method) to find the optimal production quantities of products A and B that maximize the objective function (profit) while satisfying the constraints. The specific coefficients, limits, and objectives in the mathematical formulation will vary depending on the problem context and requirements.*

Sensitivity Analysis: Sensitivity analysis in the context of Linear Programming (LP) involves studying how changes in the parameters, coefficients, or constraints of an LP problem affect the optimal solution and related outcomes. It helps assess the robustness and stability of the solution in response to variations in the problem inputs.

Mathematical Description: Sensitivity analysis involves examining the impact of changes in the coefficients of the objective function and constraints on the optimal solution and related variables. It typically includes the following components:

- *Shadow Prices (Dual Variables)*: Shadow prices represent the rate of change in the objective function value per unit change in the right-hand side of a constraint. Each constraint has an associated shadow price indicating the marginal value of relaxing or tightening that constraint. Shadow prices can be obtained as part of the LP solution using sensitivity analysis techniques.
- *Allowable* Ranges: Allowable ranges determine the range of values within which the coefficients of the objective function or constraints can change without affecting the current optimal solution. For objective function coefficients, the allowable range is given by the range of shadow prices. For constraint coefficients, the allowable range is determined by the range within which the right-hand side can change without impacting the current optimal solution.

Example 3.1.g: Consider a production planning problem where a company produces two products, *A* and *B*. The objective is to maximize profit. The LP formulation is as follows:

$Maximize: 2A + 3B$

Subject to:

$3A + 2B \leq 10$

$A + B \leq 6$

$A, B \geq 0$

After solving the LP problem, we obtain an optimal solution with $A = 2$ and $B = 4$, yielding a *profit of 16*.

Now, let's assume we want to assess the sensitivity of the solution to changes in the profit coefficients. If the coefficient for product A changes from 2 to 2.5, and the coefficient for product B changes from 3 to 2.8, we can analyse the impact on the optimal solution and profit.

By re-solving the LP problem with the updated coefficients, we find that the optimal solution remains the same ($A = 2, B = 4$), but the *profit changes to 17.2*. This indicates that the optimal solution is fairly robust to changes in the profit coefficients within this range.

Sensitivity analysis helps decision-makers understand how changes in the problem inputs impact the optimal solution and related outcomes, allowing them to make informed decisions and evaluate the stability of the solution under different scenarios.

✍ *Formulating LP problems requires a thorough understanding of the problem domain, its constraints, and the desired objectives. It involves translating the problem into a mathematical framework that can be solved using LP techniques. Proper formulation is crucial for obtaining accurate and meaningful solutions to optimize resource allocation, decision-making, and planning.*

Questions you should be able to answer

What is the objective function in a Linear Programming (LP) problem, and what does it represent?

How are decision variables identified and defined in LP formulations?

What are constraints in LP, and how are they represented mathematically?

How is the feasible region defined in an LP problem, and what does it represent?

What are the assumptions and limitations of LP formulations?

How do you formulate LP problems based on real-world scenarios and requirements?

What are the different techniques used to represent LP problems mathematically?

How can you identify the optimization type (maximization or minimization) in LP problems?

What is the significance of sensitivity analysis in LP, and how does it affect the optimal solution?

How can you perform sensitivity analysis in LP and interpret the results?

How do you interpret the shadow prices (dual variables) obtained from LP sensitivity analysis?

How can you assess the impact of changes in the problem parameters on the LP solution?

What are some specific applications of LP formulations in various industries and sectors?

How can you tailor the formulation of an LP problem to suit specific problem domains?

How can you use LP to optimize resource allocation, production planning, and decision-making processes?

Proposed Exercises:

3.1.1 To complete these sentences, you can fill in the missing words related to Linear Programming, such as "scarce", "maximized", "minimized", "constraints", "feasible region", "optimal solution", "boundary", "optimal value", "two", "extreme points", and "unbounded":

a) Linear Programming is a mathematical technique used to optimize the allocation of _____ resources, such as time, money, and labour, to achieve the best possible outcomes.

b) In Linear Programming, the objective function represents the quantity that needs to be _____ or _____.

c) The set of restrictions or limitations on the decision variables in Linear Programming is known as the _____.

d) In Linear Programming, the feasible region is the region that satisfies all the _____ and _____ constraints.

e) The values of the decision variables that maximize or minimize the objective function in Linear Programming are known as the _____.

f) The process of finding the optimal solution in Linear Programming involves identifying the _____ and _____ points of the feasible region.

g) The _____ of the objective function in Linear Programming represents the maximum or minimum value that can be achieved.

h) The graphical method of Linear Programming is suitable for problems with _____ decision variables.

i) In Linear Programming, the corner points of the feasible region are also known as _____.

j) The condition in Linear Programming where the objective function can be further improved without violating any constraints is called _____.

3.1.2 The missing words are represented by blank spaces in the exercises. Please fill in the appropriate words based on the context of Linear Programming and Economic Mathematics:

a) In linear programming, the objective function is the function that we either want to _____ or _____. The goal of linear programming is to optimize this objective function subject to a set of _____ and _____.

b) In a linear programming problem, the constraints are represented as a set of _____. Each constraint is a linear _____ of the decision variables. The feasible region is the set of all points that satisfy all the _____.

c) The optimal solution to a linear programming problem can be found at a _____ point in the feasible region. At this point, the objective function has either the _____ or the _____ value, depending on whether it is a maximization or minimization problem.

d) The graphical method is a technique used to solve _____ linear programming problems. In this method, we graph the constraints and objective function on a _____, and the optimal solution corresponds to the _____ point of the feasible region.

e) When solving a linear programming problem using the graphical method, if the objective function line is _____ to one of the constraint lines, the problem has an _____ number of optimal solutions.

f) In linear programming, the _____ method is a widely used technique to find the optimal solution when dealing with a large number of variables and constraints. This method is based on the concept of _____ and _____.

g) In the simplex method, the initial feasible solution is found at one of the _____ of the feasible region. The algorithm then iteratively moves to the _____ adjacent corner that improves the objective function value until it reaches the optimal solution.

h) The optimal solution to a linear programming problem can occur at a _____, in which the objective function has the same value for more than one feasible solution.

i) The _____ is a measure of how much the objective function value will change if the right-hand side of a constraint is increased by one unit while keeping all other variables constant.

j) In sensitivity analysis, the _____ range indicates how much the objective function coefficient can change without changing the optimal solution. The _____ range indicates how much the right-hand side of a constraint can change without changing the optimal solution.

3.1.3 For each exercise, carefully define the decision variables, set up the objective function (either maximization or minimization), and express the constraints in terms of the variables. Make sure to use the appropriate mathematical notation to formulate the LP problems

a) Production Planning: A company produces two types of products (P1 and P2). The production of P1 requires 2 units of labour and 3 units of raw materials, while the production of P2 requires 4 units of labour and 1 unit of raw materials. The company has 100 units of labour and 60 units of raw materials available. The profit per unit of P1 is $8, and the profit per unit of P2 is $6. The company wants to maximize its profit:

a.1) Formulate the LP problem to maximize the company's profit. *a.2)* Identify the decision variables, objective function, and constraints.

b) Resource Allocation: A non-profit organization is distributing food packets to the homeless. Each food packet contains 2 loaves of bread and 3 cans of soup. The organization has received 400

loaves of bread and 300 cans of soup as donations. The organization wants to maximize the number of food packets distributed to the homeless:

b.1) Formulate the LP problem to maximize the company's profit. *b.2)* Identify the decision variables, objective function, and constraints.

c) Investment Portfolio: An investor has $50,000 to invest in two types of assets (A and B). The return on investment per dollar for asset A is $0.05, and for asset B, it is $0.03. The investor wants to maximize the total return on investment:

c.1) Formulate the LP problem to maximize the company's profit. *c.2)* Identify the decision variables, objective function, and constraints.

d) Diet: Planning A nutritionist wants to create a diet plan for a patient. The diet plan must contain a minimum of 2000 calories, 50 grams of protein, and 30 grams of fat. The nutritionist has two food options (F1 and F2), and the nutritional content per serving is as follows:

	Calories	**Protein (g)**	**Fat (g)**
Food F1	500	20	15
Food F2	300	10	10

The cost per serving of F1 is $2, and for F2, it is $1. The nutritionist wants to minimize the cost of the diet plan:

d.1) Formulate the LP problem to maximize the company's profit. *d.2)* Identify the decision variables, objective function, and constraints.

3.2 Mathematical Representation of LP Problems

Keywords: *Linear Programming, Optimization, Objective function, Decision variables, Constraints, Feasible region, Feasible solutions, Non-negativity constraints, Linear equations, Linear inequalities, Slack variables, Surplus variables, Artificial variables, Equality constraints, Inequality constraints, Formulating LP problems, Mathematical modelling, Coefficients, Resource allocation, Production planning*.

Linear Programming (LP): Linear Programming (LP) is a mathematical technique used for optimization problems where a linear objective function is maximized or minimized subject to linear constraints.

Optimization: The process of finding the best solution (maximum or minimum) for a given objective, subject to certain constraints.

Objective function: The linear function that needs to be maximized or minimized in an LP problem. The objective function represents the goal of the optimization problem, either to maximize or minimize a specific quantity. It is typically denoted by "Z" and expressed as a linear combination of decision variables $(x_1, x_2, \ldots, x_n)$ multiplied by their respective coefficients $(c_1, c_2, \ldots, c_n)$:

$$Z = c_1 * x_1 + c_2 * x_2 + \ldots + c_n * x_n$$

Example 3.2.a Let's consider an example related to a manufacturing company that produces two types of products: Product A and Product B. The objective of the company is to maximize its total profit from the production and sale of these products.

Let:

- x_1, number of units of product A to be produced and sold.
- x_2, number of units of product B to be produced and sold.

The objective function represents the total profit and is defined as the revenue from selling Product A and Product B minus the production costs:

$$Objective\ Function\ (Z): Maximize\ Z\ =\ 10x_1\ +\ 8x_2$$

Explanation:

- The coefficient 10 represents the profit per unit of Product A.
- The coefficient 8 represents the profit per unit of Product B.
- The objective is to maximize the total profit Z.

✍ *In this example, the manufacturing company wants to determine the optimal production quantities of Product A and Product B to maximize its profit. The objective function Z provides a mathematical representation of this goal, and the company will use linear programming techniques to find the values of x_1 and x_2 that maximize Z while satisfying the constraints, such as available resources and production capacities.*

Decision variables: Variables that represent the quantities to be determined or optimized in the LP problem. These are the unknowns to be solved for.

Example 3.2.b Let's consider an example related to production planning:

A company produces two types of products: Product A and Product B. The company wants to determine the optimal production quantities of each product to maximize their total profit.

The decision variables in this scenario are the quantities of Product A and Product B to be produced.

Let:

- x = Quantity of Product A to be produced
- y = Quantity of Product B to be produced

Objective Function: The objective is to maximize the total profit, which is a function of the production quantities of Product A (x) and Product B (y). Let's assume the profit per unit of Product A is $10, and the profit per unit of Product B is $15.

$$Objective\ function: Z\ =\ 10x\ +\ 15y$$

Constraints:

Production Capacity: The company has a limited production capacity. Let's say the maximum production capacity for Product A is 100 units, and for Product B is 80 units.

- *Constraint 1*: $x\ \leq\ 100$ (Production capacity for Product A)
- Constraint 2: $y\ \leq\ 80$ (Production capacity for Product B)

Resource Availability: The company has a limited availability of raw materials. Let's say it requires 2 units of raw material for each unit of Product A and 3 units for each unit of Product B. The total available raw material is limited to 200 units.

- *Constraint 3*: $2x + 3y \leq 200$ (Availability of raw materials)

Non-negativity Constraints: Since production quantities cannot be negative, we have the following non-negativity constraints:

- *Constraint 4*: $x \geq 0$
- *Constraint 5*: $y \geq 0$

✍ *Now, the objective is to find the values of x and y that maximize the objective function Z while satisfying all the given constraints. By solving this linear programming problem, the company can determine the optimal production quantities of Product A and Product B, leading to the maximum profit. The values of x and y should be integers in this example because the production quantities are typically whole numbers. In such cases, the problem becomes an Integer Linear Programming (ILP) problem.*

Constraints: Constraints are typically represented as linear equations or inequalities. Constraints are the restrictions or limitations placed on the decision variables. They define the feasible region where the optimal solution must lie. Each constraint is represented as a linear equation or inequality involving the decision variables and their coefficients. There can be multiple constraints, and they are typically denoted by "C_1", "C_2" and so on:

C_1: $a_{11} * x_1 + a_{12} * x_2 + \ldots + a_{1n} * x_n \leq b_1$

...

C_k: $a_{k1} * x_1 + a_{k2} * x_2 + \ldots + a_{kn} * x_n \geq b_2$

...

C_i: $a_{i1} * x_1 + a_{i2} * x_2 + \ldots + a_{in} * x_n = b_i$

…

Where:

$x_1, x_2, \ldots, x_n$ are the decision variables.

$a_{i1}, a_{i2}, \ldots, a_{in}$ are the coefficients of the decision variables in the i-th constraint.

b_i is the right-hand side constant of the i-th constraint.

Example 3.2.c Production Planning

A company manufactures two products, Product A and Product B, using two types of resources, Labour and Raw Materials. The company wants to determine the production quantities of each product to maximize profit, subject to the availability of labour and raw materials, production capacity, demand, and non-negativity of production quantities.

Decision Variables: Let: x = Quantity of Product A produced, y = Quantity of Product B produced.

Objective Function: The objective is to maximize the profit, which is the revenue from selling Product A and Product B minus the costs.

$$Maximize: Z = 50x + 80y$$

Constraints:

Labour constraint: The company has a total of 100 hours of labour available. Labour constraint equation: $2x + 3y \leq 100$

Raw Materials constraint: The company has 120 units of raw materials available. Raw Materials constraint equation: $4x + 2y \leq 120$

Production capacity constraint: The production capacity is limited to 80 units per day. Production capacity constraint equation: $x + y \leq 80$

Demand constraint: The maximum demand for Product A is 60 units, and for Product B is 50 units. Demand constraint equations: $x \leq 60$, $y \leq 50$

Non-negativity constraint: The production quantities cannot be negative. Non-negativity constraint: $x \geq 0$, $y \geq 0$

The production planning problem can be formulated as follows:

Maximize: $Z = 50x + 80y$

Subject to:

$$\begin{cases} 2x + 3y \leq 100 & (Labor\ constraint) \\ 4x + 2y \leq 120 & (Raw\ Materials\ constraint) \\ x + y \leq 80 & (Production\ capacity\ constraint) \\ x \leq 60 & (Demand\ constraint\ for\ Product\ A) \\ y \leq 50 & (Demand\ constraint\ for\ Product\ B) \\ x \geq 0, y \geq 0 & (Non-negativity\ constraints) \end{cases}$$

✍ *The company's objective is to determine the optimal production quantities of Product A and Product B (x and y) that maximize the profit while satisfying all the resource and demand constraints. Linear programming techniques can be used to solve this problem and find the optimal solution for production planning.*

Non-negativity constraints: Constraints that require the decision variables to be non-negative (greater than or equal to zero).

Integer Constraints (in Integer Linear Programming, ILP): In some cases, the decision variables may be restricted to integers (0 or positive integers). In this scenario, the LP problem becomes an Integer Linear Programming (ILP) problem.

Equality constraints: Constraints that represent exact equalities between decision variables and constants.

Example 3.2.d Production Planning: Suppose a company manufactures two products, Product A and Product B. The company has limited resources, such as labour hours and raw materials, to produce these products. The goal is to determine the optimal production quantities of Product A and Product B to maximize the total profit.

Objective function: Let's assume the profit per unit of Product A is $10 and the profit per unit of Product B is $15. The objective is to maximize the total profit, denoted as "Z," which is the sum of the profits from both products:

$$Z = 10A + 15B$$

where: A = Quantity of Product A to produce B = Quantity of Product B to produce

Constraints:

Labour hours constraint: The company has a total of 100 labour hours available for production.

$$Labor\ hours\ used\ for\ Product\ A + Labor\ hours\ used\ for\ Product\ B = 100$$

Raw materials constraint: The company has a total of 200 units of raw materials available.

$$Raw\ materials\ used\ for\ Product\ A\ +\ Raw\ materials\ used\ for\ Product\ B\ =\ 200$$

Non-negativity constraints: $A \geq 0$ (Product A cannot have a negative quantity) $B \geq 0$ (Product B cannot have a negative quantity)

✍ *The goal is to find the values of A and B that satisfy all the constraints and maximize the objective function Z. This problem can be solved using linear programming techniques to find the optimal production quantities of Product A and Product B that maximize the total profit while ensuring that the labour and raw material constraints are not violated. The equality constraints (labour hours and raw materials) ensure that the company utilizes all available resources efficiently without any slack or surplus.*

Inequality constraints: Constraints that represent inequalities between decision variables and constants.

Domain: The domain represents the allowable values for each decision variable. It may include non-negativity constraints, indicating that the decision variables cannot be negative:

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Boundary Conditions: For bounded variables, there may be lower and upper bounds:

$$L_1 \leq x_1 \leq U_1, \qquad L_2 \leq x_2 \leq U_2, \qquad \ldots L_n \leq x_n \leq U_n$$

Where:

$L_1, L_2, \ldots, L_n$ are the lower bounds for the decision variables.

$U_1, U_2, \ldots, U_n$ are the upper bounds for the decision variables.

Example 3.2.e Production Planning: A company manufactures two types of products, A and B. The production process for each product requires the use of two resources: labour hours and raw materials. The company has limited resources available for production: 500 labour hours and 800 kilograms of raw materials. The profit per unit of product A is $10, and for product B, it is $15.

Let's define the ***decision variables***: $x_1 =$ number of units of product A to produce $x_2 =$ number of units of product B to produce.

Objective Function: The objective is to maximize the total profit *Z*, which can be represented as the sum of the profits from producing product *A* $(10 * x_1)$ and product *B* $(15 * x_2)$:

$$Z = 10 * x_1 + 15 * x_2$$

Constraints:

Labour hours constraint: The total labour hours used for producing product A and product B cannot exceed the available labour hours (500 hours): $2 * x_1 + 3 * x_2 \leq 500$

Raw materials constraint: The total raw materials used for producing product A and product B cannot exceed the available raw materials (800 kilograms): $4 * x_1 + 2 * x_2 \leq 800$

Non-negativity constraints: $x_1 \geq 0\ x_2 \geq 0$

Boundary Conditions: The boundary conditions represent the lower and upper bounds on the decision variables (x_1 and x_2). Since the number of units produced cannot be negative, the boundary conditions are: $x_1 \geq 0\ x_2 \geq 0$

✍ *In this production planning problem, the boundary conditions ensure that the number of units produced for each product cannot be negative. The LP solver will consider these constraints*

along with the objective function and other resource constraints to find the optimal production plan that maximizes the total profit within the given resource limitations.

Feasible region: The set of all feasible solutions that satisfy all the constraints of the LP problem.

Example 3.2.f Let's consider an example of a linear programming problem with two decision variables (x and y) and multiple constraints, and we will find the feasible region:

Object function: $Maximize\ Z = 3x + 4y$

Subject to the following constraints: $\begin{cases} 2x + y \leq 8 \\ x + 2y \leq 6 \\ x \geq 0, y \geq 0 \end{cases}$

To find the feasible region, we need to identify the region that satisfies all the given constraints.

Step 1: Plot the constraints on a graph: Start by plotting the equations of the constraints on a coordinate plane. For each constraint, find the intercepts (x and y) by setting one variable to zero and solving for the other variable.

For constraint 1: $2x + y = 8$. When $x = 0,\ y = 8$. When $y = 2,\ x = 3$.

For constraint 2: $x + 2y = 6$. When $x = 0,\ y = 3$. When $y = 0,\ x = 6$.

Step 2: Shade the feasible region: Shade the region that satisfies all the constraints. In this case, the feasible region is the intersection of the shaded regions determined by constraints 1 and 2, limited by the non-negativity constraints ($x \geq 0, y \geq 0$).

Step 3: Identify the vertices of the feasible region: The vertices of the feasible region are the points where the boundary lines intersect. These points will be the potential solutions of the LP problem.

In this example, the vertices of the ***feasible region*** are points with coordinates (0, 3), (2, 2), and (4, 0).

Step 4: Evaluate the objective function at each vertex: Now, evaluate the objective function $Z = 3x + 4y$ at each vertex to determine which vertex provides the maximum value of Z.

$Z\ at\ (0,3) = 3(0) + 4(3) = 12,$

$Z\ at\ (2,2) = 3(2) + 4(2) = 14,$

$Z\ at\ (4,0) = 3(4) + 4(0) = 12.$

Step 5: Determine the optimal solution: The optimal solution is the vertex that yields the highest value of the objective function. In this case, the maximum value of Z is 14, which occurs at the vertex (2, 2).

◘ Therefore, the feasible region for this linear programming problem is the shaded region bounded by the lines $2x + y \leq 8$ and $x + 2y \leq 6$, limited by the non-negativity constraints $x \geq 0$ and $y \geq 0$. The optimal solution is $(x, y) = (2, 2)$ with the maximum objective function value $Z = 14$.

Feasible solutions: The solutions that satisfy all the constraints and belong to the feasible region.

Example 3.2.g Production Planning: A company produces two types of products: Product A and Product B. The company has a limited amount of resources, including labour hours and raw materials, available for production. The goal is to maximize the total profit while adhering to the resource constraints.

Objective function: $Maximize\ Z = 5A + 3B$

Constraints:
$$\begin{cases} 2A + 3B \leq 12 & \text{(Total available labor hours are 12)} \\ A + 2B \leq 8 & \text{(Total available raw materials are 8)} \\ A \geq 0, B \geq 0 & \text{(Both products' production quantities cannot be negative)} \end{cases}$$

We need to find the values of A and B that satisfy all the constraints and maximize the objective function Z.

Feasible solutions:

To find feasible solutions, we need to identify the points within the feasible region where all the constraints are satisfied.

Labour constraint: $2A + 3B \leq 12$ If we set A = 0, we can produce 0 units of Product A, and then $3B \leq 12$, which means $B \leq 4$. We can have various combinations of A and B that satisfy this constraint, such as *(0, 3), (0, 4).*

Raw material constraint: $A + 2B \leq 8$ If we set B = 0, we can produce 0 units of Product B, and then $A \leq 8$. Feasible combinations could be (8, 0), (5, 0), etc.

Non-negativity constraints: $A \geq 0$, $B \geq 0$ Both A and B should be greater than or equal to zero.

Feasible solutions within the feasible region include: (0, 3), (0, 4), (8, 0), (5, 0), etc.

◘ *For each of these feasible solutions, we can calculate the value of the objective function Z = 5A + 3B. The solution that gives the maximum value of Z will be the optimal solution for the production planning problem.*

✍ *Note: In this example, we are considering only integer values for A and B. In practice, linear programming problems often deal with fractional values for decision variables. However, for simplicity, we are using integers in this example.*

Linear equations: Mathematical expressions that represent the relationships between decision variables and constants, where each variable has an exponent of one.

Linear inequalities: Mathematical expressions that represent the restrictions on decision variables, where each variable has an exponent of one.

Slack variables: Introduced in LP problems to convert inequality constraints into equality constraints. They measure the "slack" or surplus in the constraints.

Example 3.2.h Let's consider a simple example of a linear programming problem with slack variables. Maximizing Profit with Slack Variables: *A company produces two products, Product A and Product B. The company wants to maximize its total profit by deciding how many units of each product to produce. The profit per unit of Product A is $5, and the profit per unit of Product B is $8. The company has limited resources: 120 hours of labour and 100 units of raw material available.*

Objective function: $Maximize\ Z = 5A + 8B$

Subject to the following constraints: $\begin{cases} 2A + 3B \leq 120 & \text{(Labor constraint)} \\ 4A + 2B \leq 100 & \text{(Raw material constraint)} \\ A \geq 0, B \geq 0 & \text{(Non-negativity constraints)} \end{cases}$

We can rewrite the constraints *using slack variables* to convert the inequalities into equalities:

Constraints: $\begin{cases} 2A + 3B + S_1 = 120 & \text{(Labor constraint)} \\ 4A + 2B + S_2 = 100 & \text{(Raw material constraint)} \\ A \geq 0, B \geq 0, S_1 \geq 0, S_2 \geq 0 & \text{(Non-negativity constraints)} \end{cases}$

Here, S_1 and S_2 are slack variables introduced to convert the inequalities into equalities. The slack variables represent the "slack" or surplus in the constraints. If S_1 or S_2 is equal to zero in the optimal solution, it means that the corresponding constraint is binding (fully utilized). If S_1 or S_2 is greater than zero, it indicates that there is some slack in the corresponding constraint.

✍ *Solving this linear programming problem will yield the optimal values of A and B, representing the number of units of Product A and Product B to produce, respectively, to maximize the company's total profit while fully utilizing the available resources. The slack variables will also provide insights into how much slack or surplus there is in each constraint.*

Surplus variables: Similar to slack variables, introduced to convert inequality constraints into equality constraints. They measure the excess or surplus in the constraints.

Artificial variables: Used in LP problems when solving systems with inequality constraints to create an initial feasible solution. They are later removed from the final solution.

Example 3.2.i Let's consider an example where artificial variables are used in a Linear Programming (LP) problem to find the initial feasible solution:

Maximize $Z = 3x + 2y$

Subject to the following constraints: $\begin{cases} x + y \leq 5 \\ 2x - y \geq 3 \\ x, y \geq 0 \end{cases}$

In this LP problem, we have two inequality constraints. To solve it using the Simplex method, we first need to convert the inequality constraints into equality constraints by introducing artificial variables. This is done to create an initial feasible solution from which we can start the Simplex method.

The first constraint ($x + y \leq 5$) is already in the form of an inequality. To convert it to an equality, we introduce a surplus variable, s_1, such that: $x + y + s_1 = 5$

The second constraint ($2x - y \geq 3$) is also an inequality. To convert it to an equality, we introduce another redundant variable, u_1 and an artificial variable, a_1, such that: $2x - y - u_1 + a_1 = 3$.

Now, the LP problem is reformulated with the additional artificial variables:

Subject function: $Maximize\ Z = 3x + 2y + 0 * s_1 + 0 * u_1 - M * a_1$

Subject to the following constraints: $\begin{cases} x + y + s_1 = 5 \\ 2x - y - u_1 + a_1 = 3 \\ x, y, s_1, u_1, a_1 \geq 0 \end{cases}$

◘ Next, we use the Simplex method to solve the LP problem. As the Simplex method proceeds, it will either make the artificial variables equal to zero (indicating that they are not needed in the final optimal solution) or identify that the LP problem is infeasible (no feasible solution exists).

◘ If the LP problem has a feasible solution, the artificial variables will eventually become zero during the Simplex method's iterations. Once all artificial variables are eliminated, the initial feasible solution is found, and the Simplex method continues with the original variables (x and y) to find the optimal solution.

✍ *Artificial variables play a crucial role in the Simplex method's initial steps to establish a starting feasible solution for the LP problem, especially when there are inequality constraints that need to be converted into equalities*.

Formulating LP problems: The process of defining the objective function, decision variables, and constraints to model a real-world problem as an LP problem.

Example 3.2.j Let's consider an example of a production planning problem that can be formulated as a Linear Programming (LP) problem.

A company manufactures two products, Product A and Product B. The production of each product requires labour hours and machine hours. The company wants to determine the optimal production quantities of each product to maximize their total profit, subject to constraints on labour and machine hours available.

Objective:

Maximize Total Profit: Let's assume that the profit per unit of Product A is $100 and the profit per unit of Product B is $150. The objective is to maximize the total profit, which can be represented as the objective function:

$$Maximize\ Z = 100A + 150B$$

Decision Variables:

Let A be the number of units of Product A to produce. Let B be the number of units of Product B to produce.

Constraints:

Labour hours constraint: The total labour hours available for production is limited to 800 hours. $3A + 2B \leq 800$

Machine hours constraint: The total machine hours available for production is limited to 600 hours. $4A + 3B \leq 600$

Non-negativity constraints: $A \geq 0\ B \geq 0$

Formulated LP Problem:

Maximize $Z = 100A + 150B$

Subject to:
$$\begin{cases} 3A + 2B \leq 800 \\ 4A + 3B \leq 600 \\ A \geq 0\ B \geq 0 \end{cases}$$

Interpretation:

In this LP problem, the decision variables A and B represent the quantities of Product A and Product B to produce, respectively. The objective function Z is to maximize the total profit, considering the profit per unit of each product. The constraints represent the limitations on labour and machine hours available for production.

✍ *By solving this LP problem using LP algorithms, like the Simplex method, the company can determine the optimal production quantities of Product A and Product B that will result in the maximum total profit while adhering to the labour and machine hours constraints.*

Mathematical modelling: The process of representing a real-world problem using mathematical equations and relationships.

Coefficients: The constant values that multiply the decision variables in the objective function and constraints, representing their importance or contribution to the overall optimization.

Resource allocation: The process of distributing limited resources among different activities or projects in a way that optimizes the desired objective.

Production planning: The process of determining the optimal production levels of different products or services given available resources and constraints.

✍ *Linear Programming is a powerful tool for optimizing various real-world problems, including resource allocation, production planning, transportation, and supply chain management, among others. By formulating the problem mathematically, LP allows decision-makers to make informed and optimal decisions.*

Questions you should be able to answer:

What is a Linear Programming (LP) problem, and what are its key characteristics?
How do you formulate an LP problem mathematically using decision variables, objective function, and constraints?
What are decision variables, and how are they used in LP problem formulation?
What is the objective function in an LP problem, and what role does it play in optimization?
What are the constraints in an LP problem, and how are they represented mathematically?
How do you classify constraints as equality or inequality constraints in LP?
What is the feasible region of an LP problem, and how is it related to feasible solutions?
What is the optimal solution of an LP problem, and how is it determined?
How do you interpret the values of decision variables and the objective function in the optimal solution of an LP problem?
What is graphical representation, and how is it used to solve LP problems graphically?
What is the Simplex method, and how does it work to solve LP problems?
What are the applications of LP in real-world scenarios and various industries?
How can LP be used for resource allocation, production planning, transportation problems, and other optimization tasks?
What are the limitations and assumptions of LP modelling in practical applications?
How do you handle unbounded and infeasible LP problems, and what are the implications of each case?
What are the differences between LP, Integer Programming (IP), and Nonlinear Programming (NLP) models?
How can LP be applied in multi-objective optimization and decision-making under uncertainty?
How can LP models be extended and adapted to address more complex real-world problems?

Proposed Exercises:

3.2.1 In each exercise, fill in the blanks with appropriate words or symbols to complete the mathematical representation of the LP problem. The objective is to reinforce the understanding of LP problem formulation and constraints:

a) A company produces two products, P1 and P2. The profit per unit of P1 is $6, and the profit per unit of P2 is $8. The company wants to maximize its _____.

Objective Function: Maximize ____ + ____ subject to constraints.

b) A farmer has 200 acres of land to plant two crops, C1 and C2. Crop C1 requires 2 acres of land, and crop C2 requires 3 acres of land per unit. Crop C1 has a profit of $500 per unit, and crop C2 has a profit of $700 per unit. The farmer wants to maximize the total _____.

Objective Function: Maximize (____ x ____ + ____ x ____) subject to constraints.

c) A manufacturing company wants to produce two products, A and B. The production of product A requires 4 units of labour and 2 units of raw materials, while the production of product B requires 6 units of labour and 3 units of raw materials. The company has 40 units of labour and 30 units of raw materials available. The profit per unit of product A is $10, and the profit per unit of product B is $12. The company wants to maximize its _____.

Objective Function: Maximize (___ x ___ + ___ x)

subject to (x ___ + ___ x) $\leq$ ___ , (x ___ + ___ x ___) $\leq$ ___ , ___ ≥ 0 , ___ ≥ 0.

d) A charity organization wants to distribute food packets to the homeless. Each food packet contains two types of food items, F1 and F2. Food packet A requires 3 units of F1 and 4 units of F2, while food packet B requires 5 units of F1 and 2 units of F2. The charity has 100 units of F1 and 80 units of F2 available. The charity wants to maximize the number of food packets distributed to the homeless.

Objective Function: Maximize (___ +)

subject to (+) $\leq$ ___ , (+ ___) $\leq$ ___ , ___ ≥ 0 , ___ ≥ 0.

3.2.2 In each exercise, pay attention to the given information, and translate the problem requirements into the mathematical representation of the LP problem.:

a) A company produces two types of products (P1 and P2). The company has 200 hours of labour and 150 units of raw materials available. The profit per unit of P1 is $10, and the profit per unit of P2 is $15. The production requirements and resource constraints are as follows:

Maximize: _____

Subject to:

Labour constraint: _____

Raw materials constraint: _____

Non-negativity constraint: _____

Fill in the missing words to complete the mathematical representation of the LP problem.

b) A store sells two products (A and B) with prices of $20 and $30, respectively. The store manager wants to analyse the sales mix to maximize the revenue. The store's monthly budget for

purchasing products is $1000. The number of units sold for products A and B is represented by X and Y, respectively:

b.1) Write the objective function to maximize the revenue. *b.2)* Write the constraint equation to represent the budget constraint. *b.3)* Formulate the LP problem to find the optimal sales mix.

c) A farmer has a total of 100 acres of land available for planting two crops (C1 and C2). Crop C1 requires 2 acres of land per unit, and crop C2 requires 3 acres of land per unit. The profit per unit of C1 is $300, and the profit per unit of C2 is $500. The farmer wants to maximize the total profit.

Maximize: _____

Subject to:

Land constraint: _____

Non-negativity constraint: _____

Fill in the missing words to complete the mathematical representation of the LP problem.

d) A company produces three products (X, Y, and Z) with selling prices of $50, $80, and $100, respectively. The company wants to maximize its daily revenue, subject to the following constraints:

- The production capacity for product X is 100 units per day.
- The production capacity for product Y is 150 units per day.
- The production capacity for product Z is 80 units per day.
- The company aims to produce at least 200 units in total daily.

d.1) Write the objective function to maximize the daily revenue. *d.2)* Write the constraint equations to represent the production capacity constraints. *d.3)* Write the constraint equation to ensure that the total daily production is at least 200 units.

3.2.3 For each exercise, sketch the feasible region, identify the corner points, and use these corner points to find the optimal solution (maximum or minimum) for the given objective function. Remember to check the feasibility and boundedness of the region before determining the optimal solution.

a) Maximize the objective function $Z = 3x + 4y$ subject to the following constraints:

$$\begin{cases} x + y \leq 6 \\ 2x + y \leq 8 \\ x \geq 0, y \geq 0 \end{cases}$$

a.1) Plot the feasible region on a graph. *a.2)* Identify the corner points of the feasible region. *a.3)* Calculate the objective function value at each corner point. *a.4)* Determine the optimal solution and the maximum value of Z.

b) Minimize the objective function $Z = 2x + 3y$ subject to the following constraints:

$$\begin{cases} \mathrm{x} + 2\mathrm{y} \geq 4 \\ 3\mathrm{x} + \mathrm{y} \geq 6 \\ \mathrm{x} \geq 0, \mathrm{y} \geq 0 \end{cases}$$

b.1) Plot the feasible region on a graph. *b.2)* Identify the corner points of the feasible region. *b.3)* Calculate the objective function value at each corner point. *b.4)* Determine the optimal solution and the maximum value of Z.

c) Maximize the objective function $Z = 5x + 2y$ subject to the following constraints:

$$\begin{cases} x + y \leq 10 \\ 3x + 2y \leq 18 \\ x \geq 0, y \geq 0 \end{cases}$$

c.1) Plot the feasible region on a graph. c.*2)* Identify the corner points of the feasible region. c.*3)* Calculate the objective function value at each corner point. c.*4)* Determine the optimal solution and the maximum value of Z.

d) Minimize the objective function $Z = 4x + 5y$ subject to the following constraints:

$$\begin{cases} 2x + y \geq 10 \\ x + 3y \geq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

d.1) Plot the feasible region on a graph. d.*2)* Identify the corner points of the feasible region. d.*3)* Calculate the objective function value at each corner point. d.*4)* Determine the optimal solution and the maximum value of Z.

3.2.4 For each exercise, define the decision variables, objective function, and constraints, and then graphically represent the feasible region by plotting the constraints on a two-dimensional coordinate system. Determine the optimal solution (maximum profit or revenue) and the values of decision variables at the corner points of the feasible region:

a) A company produces two products, Product A and Product B. The company's production process requires two resources: labour and raw materials. Product A requires 3 hours of labor and 4 units of raw materials, while Product B requires 5 hours of labor and 2 units of raw materials. The company has a total of 20 hours of labor and 16 units of raw materials available. The profit per unit of Product A is $50, and the profit per unit of Product B is $60. Formulate the LP model to maximize the company's profit and graphically represent the feasible region.

b) A farmer grows two crops, Crop X and Crop Y, on two different plots of land. Crop X requires 2 units of water per acre and 3 units of fertilizer per acre, while Crop Y requires 4 units of water per acre and 2 units of fertilizer per acre. The farmer has 12 units of water and 18 units of fertilizer available. Crop X yields a profit of $200 per acre, and Crop Y yields a profit of $300 per acre. Formulate the LP model to maximize the farmer's profit and graphically represent the feasible region.

c) A furniture manufacturer produces two types of tables, Table 1 and Table 2. Table 1 requires 4 hours of carpentry work and 2 hours of finishing work, while Table 2 requires 6 hours of carpentry work and 3 hours of finishing work. The manufacturer has 30 hours of carpentry work and 15 hours of finishing work available. The profit per unit of Table 1 is $100, and the profit per unit of Table 2 is $150. Formulate the LP model to maximize the manufacturer's profit and graphically represent the feasible region.

d) A company produces two types of smartphones, Smartphone X and Smartphone Y. Smartphone X requires 5 units of memory and 3 units of battery, while Smartphone Y requires 4 units of memory and 4 units of battery. The company has 30 units of memory and 24 units of battery available. Smartphone X sells for $400, and Smartphone Y sells for $450. Formulate the LP model to maximize the company's revenue and graphically represent the feasible region.

3.2.5 For each exercise, define the decision variables, objective function, and constraints mathematically. Make sure to specify whether the problem is a maximization or minimization problem:

a) A company produces two products, A and B. The profit per unit for A is $5, and for B is $8. The company has 60 units of labor available daily and 80 units of raw material. Product A requires 2 units of labor and 3 units of raw material per unit, while product B requires 4 units of labor and 2 units of raw material per unit. Formulate a linear programming problem to maximize the daily profit.

b) A farmer has 100 acres of land available for planting corn and wheat. Each acre of corn requires 2 units of labor and 3 units of fertilizer, while each acre of wheat requires 3 units of labor and 2 units of fertilizer. The profit per acre for corn is $200, and for wheat is $300. The farmer wants to maximize the total profit. Formulate the linear programming problem.

c) A furniture manufacturer produces two types of chairs, X and Y. The production of chair X requires 3 hours of labor and 4 units of wood, while chair Y requires 5 hours of labor and 6 units of wood. The company has 100 hours of labor and 120 units of wood available. The profit per unit for chair X is $30, and for chair Y is $40. The manufacturer wants to maximize the total profit. Formulate the linear programming problem.

d) A transportation company has three routes for delivering goods: Route 1, Route 2, and Route 3. The cost per mile for each route is $10, $8, and $12, respectively. The company has a budget of $8000 for transportation expenses. Route 1 is 500 miles long, Route 2 is 600 miles long, and Route 3 is 400 miles long. The company wants to minimize the total transportation cost while covering all three routes. Formulate the linear programming problem.

e) A dietitian is planning a balanced diet for a patient. The diet should include at least 2000 calories, 50 grams of protein, and 30 grams of fat. The dietitian has two food options: food A and food B. Each serving of food A provides 300 calories, 10 grams of protein, and 5 grams of fat. Each serving of food B provides 400 calories, 8 grams of protein, and 10 grams of fat. The cost per serving of food A is $2, and for food B is $3. The dietitian wants to minimize the total cost while meeting the nutritional requirements. Formulate the linear programming problem.

3.3 GEOMETRIC INTERPRETATION OF LP PROBLEMS

Keywords: *Geometric Interpretation, Linear Programming, LP Problems, Feasible Region, Objective Function, Constraints, Optimal Solution, Corner Points, Convex Polytope, Graphical Method, Linear Inequalities, Simplex Algorithm, Extreme Points, Feasible Solutions, Geometric Representation, Linear Optimization, Geometric Visualization, Vertex, Boundary Points, Linear Programming Model, Objective Function Coefficients, Resource Constraints, Production Constraints, Non-negativity Constraints, Geometric Feasibility.*

The geometric interpretation of Linear Programming (LP) problems provides valuable insights into understanding the feasible region, optimal solution, and the role of constraints and objective function in optimization. It involves graphing the constraints and objective function in the $n-dimensional$ space, where n is the number of decision variables.

Let's consider a two-variable LP problem for ease of visualization. Example LP Problem:

Object function: $Maximize\ Z = 3x + 4y$

Subject to the constraints: $\begin{cases} 2x + y \leq 6 \\ x + 2y \leq 8 \\ x \geq 0, y \geq 0 \end{cases}$

Geometric Interpretation:

Feasible Region: Each constraint in the LP problem represents a linear inequality that defines a boundary in the x-y plane. The feasible region is the intersection of the half-planes formed by these boundaries that satisfy all the constraints. In our example, it is the region bounded by the lines $2x + y = 6$, $x + 2y = 8$, and the non-negativity axes $x \geq 0$, $y \geq 0$. This region is polygonal and convex.

Feasible Solutions: The feasible solutions are the points (x, y) that lie within the feasible region and satisfy all the constraints. These solutions fulfill the resource limitations and other restrictions defined in the LP problem.

Objective Function Line: The objective function $Z = 3x + 4y$ represents a linear equation. We can plot this equation as a line in the $x - y\ plane$. For each pair (x, y), the objective function value *Z* can be computed. As we vary *x* and *y*, the objective function line sweeps through the space.

Optimal Solution: The optimal solution of the LP problem is the point (x, y) that maximizes or minimizes the objective function Z while lying in the feasible region. Geometrically, the optimal solution is the point where the objective function line is tangent to the feasible region, and it corresponds to the highest or lowest value of Z, depending on whether it is a maximization or minimization problem.

Iso-profit (Iso-objective) Lines: Iso-profit lines are parallel lines with constant values of the objective function Z. They run parallel to the objective function line and represent different levels of profit. These lines help visualize the trade-offs between different feasible solutions.

✍ *By graphically interpreting LP problems, decision-makers can gain a better understanding of the feasible solutions, the optimal solution, and the sensitivity of the objective function to changes in the decision variables. Additionally, the geometric representation provides insights into the feasible region's shape, convexity, and the impact of each constraint on the overall optimization process.*

Optimal solution in the geometric interpretation of LP problems: In the geometric interpretation of Linear Programming (LP) problems, the optimal solution is the point within the feasible region that maximizes or minimizes the objective function, subject to the given constraints. The feasible region is the region that satisfies all the constraints and non-negativity constraints of the LP problem. To find the optimal solution geometrically, follow these steps:

Step 1: Represent the constraints: Graph each constraint on a coordinate plane. Each constraint is typically represented as a line (in 2D) or a plane (in 3D) in the coordinate system.

Step 2: Identify the feasible region: The feasible region is the region that satisfies all the constraints and non-negativity constraints. It is the intersection of all the individual constraint regions.

Step 3: Plot the objective function: Plot the objective function on the same coordinate plane. The objective function is typically represented as a line (in 2D) or a plane (in 3D) with a slope determined by the coefficients of the decision variables in the objective function.

Step 4: Locate the optimal solution: The optimal solution is the point within the feasible region that optimizes (maximizes or minimizes) the objective function. In 2D, it is the vertex (corner point) of the feasible region that lies on the objective function line. In 3D, it is the point where the objective function plane is tangent to the feasible region.

Step 5: Read the values of decision variables: The coordinates of the optimal solution point represent the values of the decision variables that yield the maximum or minimum value of the objective function.

✍ *The geometric interpretation of LP problems is limited to 2D or 3D spaces. In higher dimensions, the feasible region becomes a polytope, and the optimal solution is a vertex of this polytope. However, in practice, LP solvers use mathematical algorithms (e.g., the Simplex method) to efficiently find the optimal solution, even in higher dimensions.*

The geometric interpretation provides an intuitive visualization of the LP problem, making it easier to understand and analyze the relationships between the objective function, constraints, and feasible region. However, for complex problems with many variables and constraints, the computational approach using LP solvers is more practical and efficient.

Example 3.3.a Let's consider a simple example of a geometric interpretation of an LP problem with two decision variables, x and y, and a linear objective function. Suppose we want to maximize the objective function $Z = 3x + 2y$ subject to the following constraints:

Constraint 1: $x + y \leq 4$ Constraint 2: $2x + y \leq 5$ Non-negativity constraints: $x \geq 0, y \geq 0$.

The feasible region of this LP problem is the area that satisfies all the constraints and the non-negativity constraints. We can graph the feasible region in a 2D coordinate plane.

Step 1: Plot the constraints: To graph the constraints, we first draw the lines corresponding to each constraint. For Constraint 1 ($x + y \leq 4$), we plot the line $x + y = 4$, and for Constraint 2 ($2x + y \leq 5$), we plot the line $2x + y = 5$.

Step 2: Identify the feasible region: The feasible region is the shaded area where both Constraint 1 and Constraint 2 are satisfied and the non-negativity constraints are met ($x \geq 0, y \geq 0$) (Fig. 3.2, the surface hatched in green and burgundy, bordered by a red line).

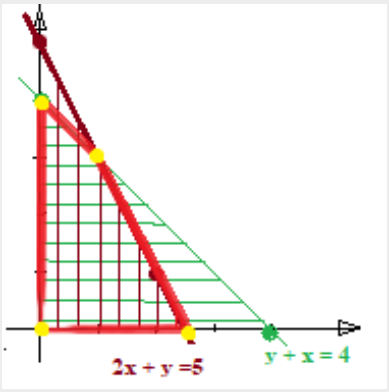


Figure 3.2 The illustration of the Feasible Region for Example 3.3.a

Step 3: Plot the objective function: Next, we plot the objective function $Z = 3x + 2y$ on the same coordinate plane. This represents the set of points where the objective function is equal to a constant value Z.

The vertices of the feasible region are the points where the boundary lines intersect. These points will be the potential solutions of the LP problem.

In this example, the vertices of the ***feasible region*** are: (0, 0), (0, 4), (1, 3), and (2.5, 0).

Step 4: Locate the optimal solution: The optimal solution is the point within the feasible region that maximizes the objective function $Z = 3x + 2y$. It is the vertex (*yellow vertex in Fig. 3.3.a*) of the feasible region that lies on the objective function line $Z = 3x + 2y$.

Evaluate the objective function at each vertex: Now, evaluate the objective function $Z = 3x + 2y$ at each vertex to determine which vertex provides the maximum value of Z.

$$Z\ at\ (0,0) = 3 * (0) + 2 * (0) = 0,$$

$$Z\ at\ (0,4) = 3 * (0) + 2 * (4) = 8,$$

$$Z\ at\ (1,3) = 3 * (1) + 2 * (3) = 9,$$

$$Z\ at\ (2.5,0) = 3 * (2.5) + 2 * (0) = 7.5.$$

Step 5: Determine the optimal solution: The optimal solution is the vertex that yields the highest value of the objective function. In this case, the maximum value of Z is 9, which occurs at the vertex (1, 3).

◘ *In this example, the optimal solution can be found at the intersection point of Constraint 1 and Constraint 2, where the objective function $Z = 3x + 2y$ is maximized. The coordinates of this point will provide the values of x and y that give the maximum value of Z.*

✍ *In more complex LP problems with more decision variables and constraints, the feasible region may be in higher dimensions, and the geometric interpretation becomes more challenging. In such cases, LP solvers use mathematical algorithms to find the optimal solution efficiently.*

The significance of shadow prices or dual prices in the context of LP problems: In the context of Linear Programming (LP) problems, shadow prices or dual prices have significant implications for understanding the sensitivity of the objective function to changes in the constraints.

Example 3.3.b Consider the following example of a transportation LP problem:

Objective: Minimize the cost of transporting goods from factories to warehouses.

Constraints: Factory 1: 3 units of goods are produced, and 5 units are demanded at Warehouse A.
Factory 2: 4 units of goods are produced, and 4 units are demanded at Warehouse A.
Factory 3: 6 units of goods are produced, and 8 units are demanded at Warehouse A.
Factory 1: 2 units of goods are produced, and 4 units are demanded at Warehouse B.
Factory 2: 2 units of goods are produced, and 3 units are demanded at Warehouse B.
Factory 3: 5 units of goods are produced, and 7 units are demanded at Warehouse B.

Objective Function: $Minimize\ Cost = 10x_{11} + 20x_{12} + 15x_{21} + 25x_{22} + 30x_{31} + 15x_{32}$

where x_{ij} represents the quantity of goods transported from Factory *i* to Warehouse *j*.

Solution:

The optimal solution suggests the following transportation plan: $x_{11} = 3$ (3 units of goods from Factory 1 to Warehouse A) $x_{21} = 1$ (1 unit of goods from Factory 2 to Warehouse A) $x_{32} = 2$ (2 units of goods from Factory 3 to Warehouse B)

Shadow Prices or Dual Prices: Shadow prices (also known as dual prices) are the rates of change in the objective function's optimal value with respect to a one-unit increase in the right-hand side (RHS) of the constraints. In other words, shadow prices indicate how much the objective function's optimal value would change if the availability or demand of a resource is increased or decreased by one unit while keeping the other constraints unchanged.

For the given example, let's assume there is an increase in the demand at Warehouse A by one unit (from 5 to 6 units). This increase affects the right-hand side of constraint 1, which is the demand constraint for Warehouse A:

New Constraint 1: Factory 1: 3 units of goods are produced, and 6 units are demanded at Warehouse A.

◘ *By solving the LP problem with the new constraint, we obtain a new optimal solution. The increase in demand will lead to a change in the optimal cost. The shadow price or dual price associated with constraint 1 represents the change in the objective function's optimal value (cost) per unit increase in demand at Warehouse A.*

For example, if the shadow price for constraint 1 is $2, it means that for each additional unit of goods demanded at Warehouse A, the total transportation cost will increase by $2. Conversely, if the shadow price for constraint 1 is -$2, it means that for each unit decrease in demand at Warehouse A, the total transportation cost will decrease by $2.

✍ *In summary, shadow prices or dual prices provide valuable information about the sensitivity of the objective function to changes in the constraints, helping decision-makers understand the impact of resource availability or demand fluctuations on the overall solution's optimality and cost. They allow for better decision-making in dynamic and uncertain environments, where conditions may change over time.*

The impact of changes in objective function coefficients on the optimal solution geometrically:

Example 3.3.c Let's consider a simple example of a 2-variable linear programming problem to analyze the impact of changes in objective function coefficients on the optimal solution geometrically:

Maximize Z = 3x + 2y

Subject to the constraints: $\begin{cases} x + y \leq 6 \\ 2x + y \leq 8 \\ x, y \geq 0 \end{cases}$

To analyze the impact of changes in the objective function coefficients (3 and 2) on the optimal solution geometrically, we'll create a graph and observe how the optimal solution changes as we vary these coefficients.

Plot the Feasible Region: To begin, graph the feasible region determined by the constraints. The feasible region is the intersection of the non-negative quadrants (x, y ≥ 0) with the area bounded by the lines defined by the constraints.

Determine the Original Optimal Solution: Find the optimal solution for the original objective function coefficients (3 and 2) by drawing the objective function line ($3x + 2y = Z$) on the graph and slide it parallel to itself until it just touches the feasible region. The point of tangency will be the optimal solution ($x*, y*$) that maximizes *Z*.

Observe the Impact of Coefficient Changes: Now, change the coefficients of the objective function to, for example, $Z = 5x + 2y$, and repeat the process of finding the optimal solution for this new objective function. Again, slide the objective function line parallel to itself until it touches the feasible region at a different point (x*, y*). Observe how the optimal solution changes.

Compare the Optimal Solutions: By comparing the two optimal solutions obtained for different objective function coefficients, you can analyze the impact of changes in the coefficients on the optimal solution geometrically. Specifically, observe how the position of the optimal solution changes on the graph and how it shifts along the objective function line as you modify the coefficients.

✍ *Keep in mind that in this example, we have a 2-dimensional graph for simplicity. In practical applications with more variables, the geometric visualization becomes challenging. However, the same principles apply, and changes in the objective function coefficients will still have a geometric impact on the optimal solution.*

The advantages and limitations of the geometric interpretation of LP problems:

Advantages of the geometric interpretation of LP problems:

Intuitive Visualization: The geometric interpretation allows for an intuitive visualization of the LP problem in two or three dimensions, making it easier to understand the feasible region, feasible solutions, and the optimal solution.

Geometric Insights: Geometric interpretations provide valuable insights into the structure of the LP problem. For example, the feasible region is a convex polytope, and the optimal solution lies at one of the extreme points (vertices) of the polytope.

Simple Problems: Geometric interpretation is particularly useful for LP problems with two or three decision variables as they can be easily visualized in a 2D or 3D space, facilitating analysis and optimization.

Graphical Solutions: In 2D LP problems, the optimal solution can be found by locating the intersection point of the objective function line and the constraint lines, making it a graphical solution method.

Limitations of the geometric interpretation of LP problems:

Limited to Low Dimensions: The geometric interpretation becomes less practical as the number of decision variables increases, as visualizing the feasible region and the optimal solution becomes challenging in higher dimensions.

Infeasibility and Unboundedness: It may be difficult to visualize infeasible or unbounded LP problems geometrically, especially in higher dimensions.

Non-linear Problems: Geometric interpretation is limited to linear LP problems. Non-linear LP problems do not have a simple geometric interpretation, and specialized techniques are required to solve them.

Complex Constraints: In cases where the constraints are complex, involving multiple inequalities and equations, the geometric representation may become cluttered and difficult to comprehend.

Computational Efficiency: While geometric interpretation helps in understanding the LP problem, it may not be the most efficient way to find the optimal solution, especially in high-dimensional problems. Other solution methods like the Simplex algorithm or interior-point methods are more efficient for large-scale LP problems.

✍ *In summary, the geometric interpretation of LP problems provides valuable insights and visualizations for understanding and solving LP problems, particularly in low dimensions. However, it has limitations in higher dimensions, with complex constraints or non-linear problems, where specialized algorithms and computational techniques are more suitable for finding optimal solutions efficiently.*

The geometric approach in various real-economic-world scenarios: Linear Programming (LP) problems can be applied in various real-economic-world scenarios using the geometric approach. The geometric approach involves visualizing the feasible region and the objective function graphically on a two-dimensional plane (for two decision variables) or a three-dimensional space (for three decision variables). Here are some examples of real-economic-world scenarios where the geometric approach of LP can be applied:

Production Planning: In manufacturing industries, the geometric approach can be used to optimize production planning. Decision variables may represent the quantities of different products to produce, and the objective function can be the total profit or cost. Constraints can represent resource limitations such as raw materials, labor hours, and machine capacities.

Resource Allocation: In businesses, the geometric approach of LP can help optimize resource allocation. Decision variables may represent the allocation of resources to different projects or departments, and the objective function can be the overall efficiency or profit. Constraints can represent budget constraints, time constraints, or resource availability.

Investment Portfolio Management: In finance, LP can be used to optimize investment portfolio management. Decision variables may represent the allocation of funds to different investment options, and the objective function can be the expected return or risk minimization. Constraints can represent investment limits or asset allocation requirements.

Supply Chain Management: In logistics and supply chain management, the geometric approach of LP can help optimize transportation and distribution decisions. Decision variables may represent the flow of goods between different locations, and the objective function can be the cost minimization. Constraints can represent capacity constraints, demand requirements, and shipping limitations.

Project Scheduling: In project management, LP can be used to optimize project scheduling and resource allocation. Decision variables may represent the start and end times of different project activities, and the objective function can be the project completion time or resource utilization. Constraints can represent project dependencies and resource availability.

Media Allocation: In advertising and marketing, LP can be applied to optimize media allocation decisions. Decision variables may represent the allocation of advertising budgets to different media channels, and the objective function can be the maximum reach or sales revenue. Constraints can represent budget constraints and target audience coverage.

✍ *In each of these scenarios, the geometric approach of LP involves plotting the feasible region defined by the constraints on a graph and then identifying the optimal solution by evaluating the objective function at the corner points of the feasible region. The graphical representation helps decision-makers visualize the trade-offs and make informed decisions to achieve the best possible outcomes in real-world economic scenarios.*

Example 3.3.c Let's consider a real-world scenario of resource allocation in a manufacturing company using the geometric approach for solving a Linear Programming (LP) problem.

Scenario: A manufacturing company produces two products, Product A and Product B. The company has limited resources, including labor hours and raw materials. The goal is to maximize the total profit from the production of these two products while considering the resource constraints.

Variables: Let's define the decision variables:

x: The number of units of Product A to produce

y: The number of units of Product B to produce

Objective Function: The objective is to maximize the total profit, which is the sum of the profits from Product A and Product B. Let's assume that the profit per unit of Product A is \$5 and the profit per unit of Product B is \$7.

Objective Function: $Z = 5x + 7y$

Constraints: The manufacturing company has the following resource constraints:

Labor hours constraint: The total number of labor hours available per day is limited to 40 hours. $2x + 3y \leq 40$

Raw materials constraint: The total amount of raw materials available is limited to 30 units. $4x + 2y \leq 30$

Non-negativity Constraints: Since the company cannot produce a negative number of products, the decision variables must be non-negative: $x \geq 0$ $y \geq 0$

Geometric Approach:

To solve this LP problem using the geometric approach, we graph the feasible region defined by the constraints. We plot the constraint equations as straight lines on the x-y plane and shade the region that satisfies all the constraints.

Graphically, the feasible region will be the intersection of the shaded areas representing the constraints. It will be a convex polygon bounded by the lines $2x + 3y = 40$, $4x + 2y = 30$, $x = 0$, and $y = 0$.

Objective Optimization: Next, we evaluate the objective function ($Z = 5x + 7y$) at each corner point of the feasible region to find the optimal solution that maximizes the profit.

Corner Points (Vertices of the feasible region):

$(x, y) = (0, 0)$

$(x, y) = (0, 15)$

$(x, y) = (10, 10)$

$(x, y) = (15, 0)$

We calculate the value of Z at each corner point:

$Z = 5 * (0) + 7 * (0) = \$0$

$Z = 5 * (0) + 7 * (15) = \$105$

$Z = 5 * (10) + 7 * (10) = \$120$

$Z = 5 * (15) + 7 * (0) = \$75$

Optimal Solution: The maximum profit of $120 is achieved at (x, y) = (10, 10), which means the company should produce 10 units of Product A and 10 units of Product B to maximize their profit while satisfying the resource constraints.

✍ *In this example, we have applied the geometric approach of LP to solve a resource allocation problem in a manufacturing company. The geometric representation helps visualize the feasible region and identify the optimal solution efficiently. The same geometric approach can be used in other real-world economic scenarios to optimize resource allocation, production planning, and various decision-making processes.*

Questions you should be able to answer after studying the topic of ” Geometric Interpretation of LP Problems”

What is a linear programming (LP) problem?

What are the components of an LP problem?

How can LP problems be geometrically represented in two dimensions?

How can LP problems be geometrically represented in three dimensions?

What are feasible solutions and how are they represented in the geometric interpretation of LP problems?

What is the feasible region in the context of LP problems?

How are objective functions represented geometrically in LP problems?

What is the optimal solution in the geometric interpretation of LP problems?

How do you identify the optimal solution geometrically on the feasible region?

What are corner points (vertices) of the feasible region, and how are they related to the optimal solution?

How do you find the optimal solution algebraically given the geometric representation of an LP problem?

How does the graphical method for LP differ in higher dimensions?

What is the significance of shadow prices or dual prices in the context of LP problems?

How can you analyze the impact of changes in objective function coefficients on the optimal solution geometrically?

How do you handle unbounded and infeasible LP problems using the geometric approach?

How does the graphical approach relate to the simplex method used to solve LP problems?

What are the advantages and limitations of the geometric interpretation of LP problems?

How can LP problems be applied in real-world scenarios using the geometric approach?

EXERCISES

3.3.1 For the geometrical interpretation and solving exercises, draw the feasible region accurately, and use graphical methods to identify the optimal solution:

a) Fill in the Missing Words Consider the following LP problem:

Maximize: $3x + 4y$

Subject to: $2x + y \leq 10,\ x + 3y \leq 15,\ x, y \geq 0$

Fill in the missing words:

a.1) The objective of the LP problem is to _____ the value of the objective function.

a.2) The region defined by the constraints is called the _____.

a.3) The feasible region is the set of all points that satisfy the _____.

a.4) The values of x and y that make the objective function as large as possible are the _____ of the LP problem.

b) Geometrical Interpretation and Solving of LP Problem. Consider the following LP problem:

Maximize: $4x + 3y$

Subject to: $x + y \leq 6,\ 2x + y \leq 8,\ x, y \geq 0$

b.1) Sketch the feasible region defined by the constraints on a coordinate plane.

b.2) Label the corner points of the feasible region.

b.3) Calculate the objective function value at each corner point.

b.4) Identify the optimal solution and its corresponding values of *x* and *y*.

c) Fill in the Missing Words Consider the following LP problem:

Minimize: $2x + 5y$

Subject to: $3x + y \geq 6,\ 2x + 3y \geq 9,\ x, y \geq 0$

Fill in the missing words:

c.1) The objective of the LP problem is to _____ the value of the objective function.

c.2) The region defined by the constraints is called the _____.

c.3) The feasible region is the set of all points that satisfy the _____

c.4) The values of x and y that make the objective function as small as possible are the _____ of the LP problem.

d) Geometrical Interpretation and Solving of LP Problem Consider the following LP problem:

Maximize: $5x + 2y$

Subject to: $3x + 4y \leq 12,\ x + 2y \leq 6,\ x, y \geq 0$

a) Sketch the feasible region defined by the constraints on a coordinate plane.

b) Label the corner points of the feasible region.

c) Calculate the objective function value at each corner point.

d) Identify the optimal solution and its corresponding values of *x* and *y*.

3.3.2 These exercises provide practice in formulating LP problems, interpreting feasible regions geometrically. Make sure to understand the economic interpretations of the LP problems and their solutions:

a) Fill in the Missing Words A company produces two types of products (P1 and P2) with profits of $5 and $8 per unit, respectively. The company has two constraints: labor and raw materials. The labor constraint is 100 hours, and the raw materials constraint is 200 units. The company wants to maximize its total profit.

Fill in the missing words in the LP problem formulation:

a.1) Maximize: $__P1 + __P2$

Subject to:

a.2) Labor constraint: $__P1 + __P2 \leq __$

a.3) Raw materials constraint: $__P1 + __P2 \leq __$

a.4) Non-negativity constraint: $P1 \geq 0,\ P2 \geq 0$

b) Geometrical Interpretation and Solving of LP Problem Consider the following LP problem:

Maximize: $4x + 3y$

Subject to: Constraint 1: $x + y \leq 5$; Constraint 2: $2x + y \leq 8$; Non-negativity constraint: $x, y \geq 0$.

b.1) Plot the feasible region defined by the constraints on a graph.

b.2) Identify the corner points of the feasible region.

b.3) Calculate the objective function value at each corner point.

b.4) Determine the optimal solution and the maximum value of the objective function.

c) Fill in the Missing Words A manufacturing company produces three products (P1, P2, and P3) with profits of $10, $15, and $12 per unit, respectively. The company has three constraints: labor, machine capacity, and raw materials. The labor constraint is 100 hours, the machine capacity constraint is 80 units, and the raw materials constraint is 150 units. The company wants to maximize its total profit.

Fill in the missing words in the LP problem formulation:

c.1) Maximize: $__P1 + __P2 + __P3$

Subject to:

c.2) Labor constraint: $__P1 + __P2 + __P3 \leq __$

c.3) Machine capacity constraint: $__P1 + __P2 + __P3 \leq __$

c.4) Raw materials constraint: $__P1 + __P2 + __P3 \leq __$

Non-negativity constraint: $P1 \geq 0,\ P2 \geq 0,\ P3 \geq 0$

d) Geometrical Interpretation and Solving of LP Problem. Consider the following LP problem:

Maximize: 6x + 5y

Subject to: Constraint 1: $2x + y \leq 8$; Constraint 2: $x + 2y \leq 6$; Non-negativity constraint: $x, y \geq 0$.

d.1) Plot the feasible region defined by the constraints on a graph.

d.2) Identify the corner points of the feasible region.

d.3) Calculate the objective function value at each corner point.

d.4) Determine the optimal solution and the maximum value of the objective function.

3.3.3 For each exercise, carefully draw the feasible region, label the corner points, and evaluate the objective function at each corner point to find the optimal solution. Use the graphical method to identify the maximum or minimum value of the objective function and the corresponding values of *x* and *y:*

a) Maximize: $4x + 3y$ Subject to: $x + y \leq 6;\ 2x + y \leq 8;\ x, y \geq 0$:

a.1) Sketch the feasible region defined by the constraints on a coordinate plane. a.*2)* Label the corner points of the feasible region. *a.3)* Calculate the objective function value at each corner point. *a.4)* Identify the optimal solution and its corresponding values of *x* and *y*.

b) Maximize: $3x + 2y$ Subject to: $2x + y \leq 5,\ x + 2y \leq 4,\ x, y \geq 0$

b.1) Sketch the feasible region defined by the constraints on a coordinate plane. *b.2)* Label the corner points of the feasible region. *b.3)* Calculate the objective function value at each corner point. *b.4)* Identify the optimal solution and its corresponding values of *x* and *y*.

c) Minimize: $2x + y$. Subject to: $x + y \geq 4,\ x + 2y \geq 6,\ x, y \geq 0$

c.1) Sketch the feasible region defined by the constraints on a coordinate plane. *c.2)* Label the corner points of the feasible region. *c.3)* Calculate the objective function value at each corner point. *c.4)* Identify the optimal solution and its corresponding values of *x* and *y*.

d) Maximize: $5x + 3y$. Subject to: $2x + y \leq 10,\ x + y \leq 6,\ x, y \geq 0$

d.1) Sketch the feasible region defined by the constraints on a coordinate plane. *d.2)* Label the corner points of the feasible region. *d.3)* Calculate the objective function value at each corner point. *d.4)* Identify the optimal solution and its corresponding values of *x* and *y*.

e) Maximize: $2x + 4y$. Subject to: $x + y \leq 5,\ x + 2y \leq 7,\ x, y \geq 0$

e.1) Sketch the feasible region defined by the constraints on a coordinate plane. *e.2)* Label the corner points of the feasible region. e.*3)* Calculate the objective function value at each corner point. *e.4)* Identify the optimal solution and its corresponding values of *x* and *y*.

3.3.4 For each exercise, draw the feasible region accurately, and identify the optimal solution by examining the corner points. The optimal solution will be the one that maximizes (or minimizes) the objective function while satisfying all the constraints. Geometric solutions provide valuable insights into the LP problems in economic mathematics:

a) Production Optimization A company manufactures two products (P1 and P2) using two resources (Labor and Raw Materials). The profit per unit of P1 is $8, and for P2, it is $6. The resource requirements and availability are as follows:

Resource Constraints:

Labor: $3P1 + 2P2 \leq 60$

Raw Materials: $2P1 + 4P2 \leq 80$

a.1) Sketch the feasible region defined by the resource constraints on a coordinate plane.
a.2) Calculate the objective function (profit) at each corner point of the feasible region.
a.3) Identify the optimal production quantities of P1 and P2 that maximize the total profit.

b) Investment Portfolio An investor wants to invest in two assets (A and B). The return on investment (ROI) per dollar for asset A is \$0.06, and for asset B, it is \$0.04. The investor has \$3000 to invest.

Investment Constraints:

Investment in Asset A: $A \leq \$1500$

Investment in Asset B: $B \leq \$2000$

b.1) Sketch the feasible region defined by the investment constraints on a coordinate plane.
b.2) Calculate the objective function (ROI) at each corner point of the feasible region.
b.3) Determine the investment quantities in Asset A and Asset B that maximize the total ROI.

c) Diet Planning A nutritionist wants to create a diet plan with two food items (F1 and F2). Each food item contains a certain amount of calories and proteins.

Nutrition Constraints:

Calories: $4F1 + 3F2 \geq 1200$

Proteins: $2F1 + 3F2 \geq 600$

c.1) Sketch the feasible region defined by the nutrition constraints on a coordinate plane.
c.2) Calculate the objective function (cost) at each corner point of the feasible region.
c.3) Find the combination of food items F1 and F2 that minimizes the total cost while meeting the nutritional requirements.

d) Resource Allocation A company wants to allocate its resources to two projects (P1 and P2). Each project requires a certain amount of time and money.

Resource Constraints:

Time: $2P1 + 3P2 \leq 80$

Money: $4P1 + 2P2 \leq 120$

d.1) Sketch the feasible region defined by the resource constraints on a coordinate plane.
d.2) Calculate the objective function (benefit) at each corner point of the feasible region.
d.3 Determine the allocation of resources to projects P1 and P2 that maximizes the total benefit.

3.4 LP SOLUTION TECHNIQUES

Keywords: *Linear Programming, Optimization, LP Models, LP Formulation, Objective Function, Constraints, Feasible Region, Simplex Method, Dual Simplex Method, Interior Point Methods, Sensitivity Analysis, Integer Linear Programming, Mixed Integer Linear Programming, Branch and Bound, Cutting Plane Methods, Dantzig-Wolfe Decomposition, Benders Decomposition, Column Generation, Lagrange Multipliers, KKT Conditions, Linear Programming Software, Linear Programming Applications, LP Sensitivity Analysis, LP Sensitivity Report, LP Sensitivity Range, LP Sensitivity Interpretation, LP Sensitivity Shadow Price, LP Sensitivity Reduced Cost, LP Sensitivity Allowable Increase/Decrease, LP Sensitivity Right Hand Side (RHS) Range.*

A linear programming problem requires finding the extremum point of a real function called the objective function (or, under other names, the goal function, or criterion function), represented by a linear expression, when the variables of this function must satisfy a set of constraints represented by a number of equalities or inequalities, which are also linear. Depending on the requirements of the problem to be solved, we discuss maximization problems when determining the maximum value of the objective function is sought, or minimization problems when finding the minimum value of this function is required.

Linear Programming (LP) solution techniques are algorithms used to find the optimal solution to LP problems efficiently. These techniques aim to maximize or minimize the objective function while satisfying the given constraints.

LP (Linear Programming) problems can be solved using various solution techniques, and the choice of technique depends on the characteristics of the LP problem and the specific requirements of the application

These LP solution techniques can handle a wide range of real-world economic problems efficiently, providing decision-makers with optimal solutions to resource allocation, production planning, investment management, and many other optimization challenges. The choice of the most suitable technique depends on the specific characteristics of the LP problem and the requirements of the decision-making process. Here are some commonly used techniques to find the optimal solution to LP problems:

Graphical Method: The graphical method is suitable for LP problems with two decision variables. It involves graphing the constraints and the objective function on a two-dimensional plane. The optimal solution is found at the corner point of the feasible region that maximizes or minimizes the objective function.

Simplex Method: The simplex method is the most widely used technique for solving LP problems with multiple decision variables. It is an iterative algorithm that starts at an initial feasible solution and moves along the edges of the feasible region to find the optimal solution. The simplex method efficiently converges to the optimal solution by improving the objective function value at each iteration.

Interior-Point Method: The interior-point method is another iterative algorithm used to solve LP problems. It works by finding the optimal solution within the interior of the feasible region, rather than on the boundary. Interior-point methods are known for their efficiency in solving large-scale LP problems.

Dual Simplex Method: The dual simplex method is a variant of the simplex method used when solving LP problems with a large number of constraints and few decision variables. It takes advantage of the dual variables to find the optimal solution efficiently.

Integer Linear Programming (ILP) Techniques: For LP problems where decision variables are required to take integer values, techniques like Branch and Bound, Branch and Cut, and Integer Cuts are used. These methods branch the problem into subproblems and solve them iteratively to find the optimal integer solution.

Dantzig-Wolfe Decomposition: The Dantzig-Wolfe decomposition method is used for large-scale LP problems with a specific structure. It decomposes the problem into smaller subproblems, which are then solved separately and combined to find the overall optimal solution.

Sensitivity Analysis: Sensitivity analysis is used to study how changes in the coefficients of the objective function or constraints impact the optimal solution. It helps identify the range of validity for the optimal solution and provides insights into the stability of the solution.

Network Simplex Method: The network simplex method is specialized for LP problems with a network structure, such as transportation and network flow problems. It efficiently finds the optimal solution in such cases.

Cutting Plane Method: The cutting plane method is used to strengthen the LP problem formulation by adding additional constraints iteratively until the optimal solution is found.

✍ *The choice of LP solution technique depends on the problem's size, complexity, and specific requirements. Different LP solvers and software tools use variations of these techniques to efficiently find the optimal solution to a wide range of real-world optimization problems*.

The different types of LP problems, and how do they differ in terms of objectives and constraints: Linear Programming (LP) problems can be categorized into different types based on their objectives and constraints. The main types of LP problems include:

Standard LP Problem: The standard LP problem involves maximizing or minimizing a linear objective function subject to a set of linear constraints. The objective is to optimize a single linear function while satisfying all the constraints. Both the objective function and constraints are linear equations or inequalities.

Integer Linear Programming (ILP) Problem: In an ILP problem, the decision variables are required to take integer values (0 or positive integers) instead of continuous values. This adds an additional level of complexity to the problem since it restricts the potential solutions to only integer values.

Mixed-Integer Linear Programming (MILP) Problem: MILP problems are a combination of continuous decision variables (allowing fractional values) and integer decision variables (limited to integer values). This type of problem is particularly useful when some variables must be discrete while others can be continuous.

Binary Linear Programming: In binary LP problems, the decision variables are restricted to only two possible values, typically 0 or 1. This type of LP problem is often used in binary decision-making scenarios, such as yes/no choices or selecting from multiple alternatives.

Quadratic Programming (QP) Problem: In a quadratic programming problem, the objective function includes quadratic terms, making it non-linear. However, the constraints remain linear. Solving QP problems is more complex than standard LP problems due to the presence of quadratic terms.

Nonlinear Programming (NLP) Problem: Nonlinear programming involves optimizing a non-linear objective function, which means both the objective function and constraints include non-linear terms. NLP problems are more challenging to solve compared to linear programming problems.

Network Flow Problems: Network flow problems involve optimizing the flow of goods, resources, or information through a network of interconnected nodes and edges. The objective is to maximize or minimize the flow subject to capacity constraints and flow balance constraints.

Transportation Problems: Transportation problems involve minimizing the cost of transporting goods from multiple sources to multiple destinations. The objective is to determine the optimal shipping quantities between sources and destinations while satisfying supply and demand constraints.

Assignment Problems: Assignment problems deal with assigning tasks or resources to individuals or machines in a way that optimizes a given objective. The objective could be minimizing the cost or maximizing the efficiency of the assignments.

✍ *Each type of LP problem requires a different modeling approach and solution technique due to the nature of the objective function and constraints involved. The choice of LP type depends on the specific requirements and complexity of the real-world problem being addressed.*

Formulation of LP problems in standard form and their conversion to canonical form:

Formulating LP Problems in Standard Form:

The canonical form of a Linear Programming (LP) problem is a specific representation that simplifies the LP problem and ensures consistency in its structure. The standard form of an LP problem is as follows:

$$Maximize\ (or\ Minimize)\ Z = c_1 * x_1 + c_2 * x_2 + \ldots + c_n * x_n$$

$$\text{Subject to: } \begin{cases} a_{11} * x_1 + a_{12} * x_2 + \ldots + a_{1n} * x_n \leq b_1 \\ a_{21} * x_1 + a_{22} * x_2 + \ldots + a_{2n} * x_n \leq b_2 \\ \ldots \\ a_{i1} * x_1 + a_{i2} * x_2 + \ldots + a_{in} * x_n \leq b_i \end{cases}$$

where:

$x_1, x_2, \ldots, x_n$	are the decision variables,
$c_1, c_2, \ldots, c_n$	are the coefficients of the objective function,
$a_{i1}, a_{i2}, \ldots, a_{in}$	are the coefficients of the decision variables in the ith constraint,
b_i	is the right-hand side constant of the ith constraint.

To convert an LP problem to canonical form, we follow these steps:

Maximize (or Minimize) Objective Function: Determine the objective of the problem (e.g., maximize profit, minimize cost) and express it in terms of decision variables.

Identify Decision Variables: Define the decision variables that need to be optimized to achieve the objective. Assign symbols like $x_1, x_2, \ldots, x_n$ to each variable.

Formulate Constraints: Identify the constraints that restrict the values of the decision variables. Express these constraints as linear inequalities ($\leq$, $\geq$, or =) involving the decision variables.

Non-Negativity Constraints: Specify that all decision variables must be non-negative ($x_1, x_2, \ldots, x_n \geq 0$), as negative values are not practical in most real-world situations.

Objective Function Coefficients: Write the coefficients of the decision variables in the objective function.

✍ *Once the LP problem is formulated in canonical form, it can be solved using various optimization techniques, such as the simplex method, to find the optimal values of the decision variables and the maximum (or minimum) value of the objective function.*

Formulating LP Problems in Standard Form:

The standard form of an LP problem is an equivalent form of the canonical form, which further simplifies the problem for computational purposes. The standard form of an LP problem has the following characteristics:

All Constraints as Equalities: All constraints are represented as equality (=) constraints.

All Non-Negativity Constraints: All decision variables must be non-negative (≥ 0).

To convert an LP problem from standard form to canonical form, follow these steps:

Step 1: Convert Inequalities to Equalities: For each inequality constraint ($\leq$ or $\geq$), introduce *slack variables (s)* for $\leq$ constraints and *surplus variables (u)* for $\geq$ constraints, making them equalities.

Step 2: Non-negativity: Constraints include non-negativity constraints for the slack and surplus variables ($s \geq 0$, $u \geq 0$).

Step 3: Eliminate Negative Variables (if applicable): If any decision variable has a negative coefficient in the objective function, replace it with a new non-negative variable by subtracting it from another non-negative variable.

Step 4: Eliminate Objective: Coefficients (if applicable): If the objective function contains any constants, introduce a new variable for each constant term to convert them into linear terms.

Step 5: Standard Form: The LP problem is now in standard form, with all constraints represented as equalities and all variables non-negative.

The standard form of an LP problem is as follows:

$$Maximize\ (or\ Minimize)\ Z = c_1 * x_1 + c_2 * x_2 + \ldots + c_n * x_n$$

$$\text{Subject to: } \begin{cases} a_{11} * x_1 + a_{12} * x_2 + \ldots + a_{1n} * x_n + s_1 = b_1 \\ a_{21} * x_1 + a_{22} * x_2 + \ldots + a_{2n} * x_n + s_2 = b_2 \\ \ldots \\ a_{i1} * x_1 + a_{i2} * x_2 + \ldots + a_{in} * x_n + s_i = b_i \end{cases}$$

where:

$x_1, x_2, \ldots, x_n$	are the decision variables.
$s_1, s_2, \ldots, s_i$	are the slack variables introduced to convert inequalities into equalities.
$c_1, c_2, \ldots, c_n$	are the coefficients of the objective function.
$a_{i1}, a_{i2}, \ldots, a_{in}$	are the coefficients of the decision variables in the ith constraint.
b_i	is the right-hand side constant of the ith constraint.

Example 3.4.a Let's formulate a linear programming problem in standard form and then convert it into canonical form using an example. Manufacturing Company's Production Planning: *A manufacturing company produces two products, Product A and Product B. The company wants to maximize its profit from these two products while considering resource constraints.*

Objective: $Maximize\ Z = 5x + 4y$

Constraints:

$$\begin{cases} \text{Labor hours constraint:} & 2x + 3y \leq 240 \\ \text{Raw materials constraint:} & 4x + 2y \leq 320 \\ \text{Non} - \text{negativity constraints:} & x \geq 0, y \geq 0 \end{cases}$$

Step 1: Formulate in Standard Form: The given problem is already in standard form since the objective function is to maximize, and all constraints are in the form of linear equations.

Standard Form:

$$Maximize\ Z = 5x + 4y$$

$$\text{Subject to: } \begin{cases} 2x + 3y \leq 240 \\ 4x + 2y \leq 320 \\ x \geq 0, y \geq 0 \end{cases}$$

Step 2: Convert to Canonical Form: To convert to canonical form, we introduce *slack variables* (s_1 and s_2) for the two constraints and rewrite them as equalities.

Canonical Form:

$$Maximize\ Z = 5x + 4y + 0 * s_1 + 0 * s_2$$

$$\text{Subject to: } \begin{cases} 2x + 3y + s_1 = 240 \\ 4x + 2y + s_2 = 320 \\ x \geq 0, y \geq 0, s_1 \geq 0, s_2 \geq 0 \end{cases}$$

✍ *The standard form is particularly useful for LP solvers, as it simplifies the LP problem to a form that is easier to solve computationally. Converting the LP problem to canonical form does not change its feasible region or the optimal solution. It only represents the LP problem in a standardized way for computational efficiency.*

Formulating LP problems in Matrix and Vector form: Each of the standard or canonical forms can be transcribed in matrix or vector form.

The matrix transcription of a linear programming problem in canonical form is as follows:

$\max(min)\ Z(x) = c^T x$

$Ax = b$

$x \geq 0$

where:

$x \in R^n$, the vector of variables,

$A = [a_{ij}]$, the matrix of system coefficients (with *m* rows and *n* colums)

$b \in R^n$, the vector of free terms of the constraints

$c \in R^n$, the constant vector of objective function coefficients

The Vector Form is represented by the following relations:

$\max(min)\ Z(x) = c^T x$

$a_1 x_1 + a_2 x_{2+} \ \ldots + \ a_n x_n = 0$

$x \geq 0$

where, the vectors $a_1, a_2, \ldots, a_n$ are the column vectors of the matrix A.

Example 3.4.b Let's reformulate the given problem and express it in its standard and canonical matrix forms. Problem - Production Planning: *A manufacturing company produces two types of products, Product X and Product Y. The company wants to determine the optimal production quantities of these products to maximize their total revenue while considering resource constraints and demand requirements.*

Objective: Maximize Total Revenue (*Z*), which is the sum of revenues generated from selling Product X and Product Y.

Decision Variables: Let x_1 be the quantity of Product X produced, and x_2 be the quantity of Product Y produced.

Constraints:

Resource Constraint: The company has a limited number of labor hours and raw materials available for production. The total labor hours required to produce one unit of Product X is 2, and for Product Y, it is 1. The total raw materials required to produce one unit of Product X is 3 units, and for Product Y, it is 2 units. The company has 20 labor hours and 30 units of raw materials available for production. The resource constraint can be expressed as: $2x_1 + x_2 \leq 20$ (Labor hours constraint) $3x_1 + 2x_2 \leq 30$ (Raw materials constraint)

Demand Constraint: The company has market demand constraints for both products. The demand for Product X is at least 5 units, and for Product Y, it is at least 4 units. The demand constraint can be expressed as: $x_1 \geq 5$ (Demand for Product X) $x_2 \geq 4$ (Demand for Product Y)

Non-negativity Constraints: $x_1, x_2 \geq 0$ (Since production quantities cannot be negative)

Formulation of the Linear Problem in *Standard Form*:

$$Maximize\ Z = 3x_1 + 2x_2$$

$$\text{Subject to: } \begin{cases} 2x_1 + x_2 \leq 20 \\ 3x_1 + 2x_2 \leq 30 \\ x_1 \geq 5 \\ x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

Canonical Form: For each inequality constraint (≤ or ≥), introduce slack variables (s) for ≤ constraints and surplus variables (u) for ≥ constraints, making them equalities. Constraints include non-negativity constraints for the slack and surplus variables ($s \geq 0$, $u \geq 0$).

The Linear Problem in Standard Form is:

$$Maximize\ Z = 3x_1 + 2x_2$$

$$\text{Subject to: } \begin{cases} 2x_1 + x_2 + s_1 = 20 \\ 3x_1 + 2x_2 + s_2 = 30 \\ x_1 - u_1 = 5 \\ x_2 - u_2 = 4 \\ x_1, x_2, s_1, s_2, u_1, u_2 \geq 0 \end{cases}$$

Matrix Form: Represent the coefficients of the decision variables and surplus variables in a matrix and vector.

The Matrix transcription of a Linear Programming problem in canonical form is as follows:

$$\max\ Z(x) = c^T x$$
$$Ax = b$$
$$x \geq 0$$

where:

$x = [x_1, x_2, s_1, s_2, u_1, u_2]^T$, the vector of variables,

$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$, the matrix of system coefficients (with *4* rows and *6* colums),

$b = [20, 30, 4, 5]^T$, the vector of free terms of the constraints,

$c^T = [3, 2, 0, 0, 0, 0]$, the constant vector of objective function coefficients.

Vector Form: Represent the decision variables and surplus variables in vector form.

The Vector transcription of a Linear Programming problem in canonical form is as follows:

$\max(min) F(x) = c^T x$

$a_1 x_1 + a_2 x_2 + a_3 s_1 + a_4 s_2 + a_5 u_1 + a_6 u_2 = 0$

$x \geq 0$

where, the vectors $a_1, a_2, \dots, a_6$ are the column vectors of the matrix A.

✍ *These different representations are equivalent ways to describe the same linear programming problem, and each form can be used in various LP algorithms for finding the optimal solution.*

Types of solutions of a Linear Problem (LP): Considering the system of constraints and non-negativity conditions of a given linear programming problem, multiple types of solutions can be defined and used to establish the resolution methods for the problem.

An admissible (feasible) solution: An admissible (feasible) solution of the given linear programming problem is a vector in R^n whose components simultaneously satisfy the constraints, and non-negativity conditions.

A basic solution: A basic solution is an admissible solution that has at most m strictly positive components (where m is the number of equations in the system of constraints). Since an admissible solution must satisfy the non-negativity conditions, the remaining components of the basic solution are naturally zeros.

A non-degenerate solution: A basic solution that has exactly m strictly positive components is called non-degenerate.

A degenerate solution: If the basic solution has less than m strictly positive components, it is called degenerate.

An optimal solution: An optimal solution is an admissible solution that corresponds to the maximum value (for maximization problems) or minimum value (for minimization problems) of the objective function among the set of admissible solutions. A linear programming problem can have either a unique optimal solution or an infinity of optimal solutions. It can be demonstrated that any point belonging to the segment determined by two optimal solutions is also an optimal solution.

✍ *Therefore, solving a linear programming problem means finding the answer to the question: among the vectors in R^n whose components simultaneously satisfy the* constraints *conditions and which one yields the highest (or lowest) value of the objective function?*

Depending on the specifics of the problem to be solved, different coefficients have different meanings. For instance, the coefficients c_i of the objective function represent unit benefits, unit prices, or unit production costs. The coefficients a_{ij} (referred to as technical coefficients) of the constraint system represent specific consumption rates, machine capacities, etc. The free terms b_i generally represent available resources or technological limits.

The Simplex Method for solving LP problems: The Simplex Method is a widely used algorithm for solving linear programming (LP) problems. It is an iterative approach that efficiently finds the optimal solution by moving along the edges of the feasible region defined by the constraints until reaching the optimal vertex (corner point). The method was developed by George Dantzig in 1947 and has since become one of the most important algorithms in optimization.

Here's a step-by-step explanation of how the Simplex method works:

Step 1: Canonical Form: The LP problem is first converted to canonical form, which involves expressing all inequalities as equalities and introducing slack or surplus variables for inequalities in the constraint equations.

The canonical form of an LP problem is as follows:

$$Maximize\ (or\ Minimize)\ Z = c_1 * x_1 + c_2 * x_2 + \ldots + c_n * x_n$$

$$\text{Subject to: } \begin{cases} a_{11} * x_1 + a_{12} * x_2 + \ldots + a_{1n} * x_n + s_1 = b_1 \\ a_{21} * x_1 + a_{22} * x_2 + \ldots + a_{2n} * x_n + s_2 = b_2 \\ \ldots \\ a_{m1} * x_1 + a_{m2} * x_2 + \ldots + a_{mn} * x_n + u_k = b_m \end{cases}$$

where:

$x_1, x_2, \ldots, x_n$	are the decision variables.
$s_1, s_2, \ldots, s_l$	are the slack variables introduced to convert inequalities into equalities.
$u_1, u_2, \ldots, u_k$	are the surplus variables introduced to convert inequalities into equalities.
$c_1, c_2, \ldots, c_n$	are the coefficients of the objective function.
$a_{i1}, a_{i2}, \ldots, a_{in}$	are the coefficients of the decision variables in the ith constraint.
b_i	is the right-hand side constant of the ith constraint.

Step 2: Initial Basic Feasible Solution: An initial basic feasible solution is obtained by setting some variables to zero while leaving the rest at their non-negative constraints. This solution satisfies the non-negativity constraints and some of the equality constraints.

Step 3: Optimality Test: At each iteration, the algorithm checks if the current solution is optimal. It does so by calculating the reduced costs (the coefficients of the objective function) for non-basic variables. If all the reduced costs are non-negative, the current solution is optimal, and the algorithm terminates. Otherwise, the algorithm proceeds to the next step.

Step 4: Pivot Operation: The pivot operation is performed to improve the solution and move towards the optimal vertex. It involves selecting an entering variable (a variable with a negative reduced cost) and a leaving variable (a variable that leaves the basis). The algorithm then adjusts the values of the variables to create a new basic feasible solution.

Step 5: Iteration: Steps 3 and 4 are repeated until an optimal solution is found. The algorithm moves along the edges of the feasible region, pivoting from one basic feasible solution to another, each time improving the value of the objective function until it reaches the optimal solution.

Step 6: Termination: The Simplex method terminates when an optimal solution is reached, and the values of the decision variables correspond to the optimal vertex (corner point) of the feasible region, where the objective function is maximized or minimized.

The simplex method applied geometrically: The Simplex method, when applied geometrically, is a graphical approach to solving linear programming problems with two decision variables. It involves the following steps:

Step 1: Graph the Feasible Region: Start by graphing the constraints as lines on a two-dimensional Cartesian coordinate system. Each constraint is treated as an equation, and its corresponding line is plotted. Identify the feasible region, which is the region bounded by or within the lines representing the constraints. Ensure that the feasible region satisfies the non-negativity constraints $(x, y \geq 0)$.

Step 2: Identify the Objective Function Line: Graph the objective function as a line in the same coordinate system. The objective function line represents the equation $Z = cx + dy$, where c and d are the coefficients of the objective function.

Step 3: Determine the Optimal Solution: The objective function line *(Z)* will have a specific slope determined by the coefficients c and d. Move the objective function line in the direction that maximizes the objective function *Z*, while still intersecting the feasible region. The optimal solution is the point of intersection between the objective function line and the feasible region. If the objective function line is parallel to one of the constraint lines and does not intersect the feasible region, the problem is unbounded, and the solution is infinite.

Step 4: Iterative Process (if necessary): If the initial feasible region does not contain the optimal solution, use the Simplex method iteratively to improve the solution. At each iteration, move the objective function line *(Z)* towards the direction of optimization while maintaining intersection with the feasible region. After each iteration, check if the new point of intersection is a better solution (maximizing Z). Keep repeating this process until no further improvement is possible, and the optimal solution is reached.

Step 5: Interpret the Results: The coordinates of the point of intersection represent the optimal values of the decision variables x and y that maximize the objective function *Z*. The value of *Z* at the optimal solution provides the maximum or minimum value of the objective function.

Example 3.4.c To illustrate the Simplex Method, consider the following two-dimensional linear programming problem:

$$Maximize\ Z = x_1 + 2x_2$$

$$\text{Subject to: } \begin{cases} x_1 + x_2 \leq 7 \\ -x_1 + 2x_2 \leq 8 \\ x_1 \leq 5 \\ x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

The Simplex Method follows the movement on the frontier of the domain of admissible solutions, along the edges of the simplex, following the projection of the gradient of the function (which is constant) on these edges.

In fig. 1.1 the simplex method of solving the problem was illustrated graphically. Here you can see the range of admissible solutions (the convex polygon OABCD) and the lines containing the points for which the objective function has a constant value.

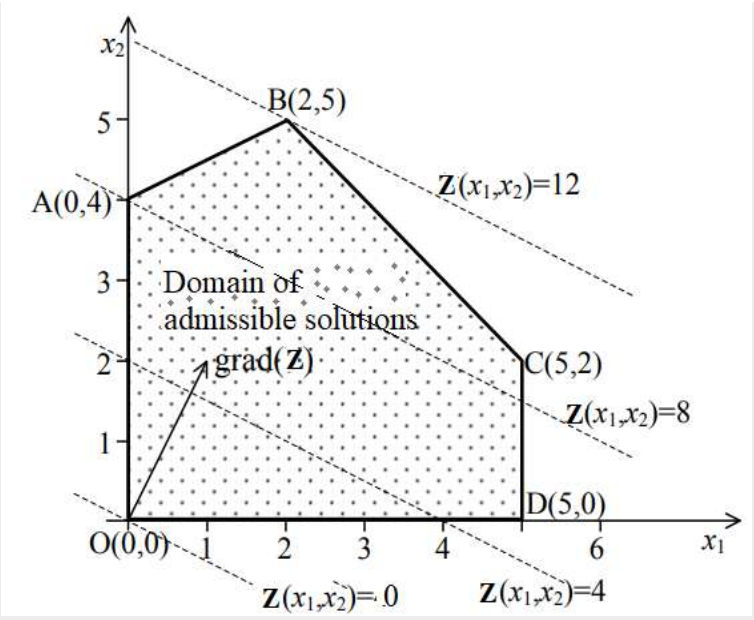


Figure 3.3 The illustration of the Feasible Region for Example 3.4.c

If one starts from the origin as an initial basic solution (where the objective function has zero value) it is increased by moving along the two edges OA and OB that meet in this vertex. Since the growth of the function is faster along the edge OA (the projection of the gradient of the function on it is greater) the next basic solution will be the peak $A(0,4)$ characterized by the value $Z(0,4) = 8$. From this extreme point, side AB will be followed to the top B, which represents the optimal solution.

The optimal solution of the problem will be $x_1 = 2$, $x_2 = 5$ and the maximum value of the objective function $Z(2,5) = 12$.

✍ *It's essential to note that the graphical approach using the Simplex Method is limited to problems with two decision variables due to its visual nature. For problems with more than two variables, the algebraic version of the Simplex Method is typically used. However, the geometric approach provides an intuitive understanding of how the optimization process works by visualizing the feasible region and the objective function line.*

The matrix applied Simplex Method: Applying the simplex method graphically is impossible for large-scale problems with many restrictions. For this reason, an analytical, iterative method must be obtained, in the form of a well-structured algorithm, with specified steps and stopping conditions. For this, it is considered that, at a given moment, a basic solution has been obtained. Without restricting the generality, it can be considered that this solution has as basic variables (those that can be strictly positive) the first m components, the others being null (if this is not the case, you can get

here by renumbering the variables of the problem). Under these conditions, the following notations are made:

- vector of basic variables
  $x^B = [x_1 \dots . x_i \dots x_m]^T$
- the vector of secondary variables
  $x^R = [x_{m+1} \dots x_n]^T$
- the vector of coefficients of the basic variables
  $c^B = [c_1 \dots . c_i \dots c_m]^T$
- the vector of the coefficients of the secondary variables
  $c^R = [c_{m+1} \dots c_n]^T$
- the matrix of the coefficients of the restrictions corresponding to the basic variables (matrix that constitutes a basis in the space R^m)

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{im} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

- the matrix of the coefficients of the restrictions corresponding to the secondary variables

$$R = \begin{bmatrix} a_{1\,m+1} & a_{1\,m+2} & \cdots & a_{1\,n} \\ \vdots & & & \\ a_{i\,m+1} & a_{i\,m+2} & \cdots & a_{i\,n} \\ \vdots & & & \\ a_{m\,m+1} & a_{m\,m+2} & \cdots & a_{m\,n} \end{bmatrix}$$

- the vector of free terms
  $b = [b_1 \dots . b_i \dots b_m]^T$

  With these notations, the linear programming problem can be put in the equivalent form:
  $\max Z(x) = (c^B)^T x^B + (c^R)^T x^R$
  $B * x^B + R * x^R = b$
  $x^B \geq 0, x^R \geq 0$

Since the matrix B is non-singular, from the system of restrictions, by multiplying to the left with the inverse of the matrix B, we can obtain:

$$x^B = B^{-1}b - B^{-1}R\,x^R$$

This relationship allows the determination of a first basic solution if it is considered that all the secondary variables are zero, respectively $x^R = 0$. For this it is noted that $\bar{x}^B$ defined as:

$$\bar{x}^B = B^{-1}b$$

it is the basic solution if it has no negative component. If $\bar{x}^B$ thus determined has negative components, the solution is not admissible, so it cannot be a basic solution either. In this case, another basis B can be considered (another non-singular quadratic matrix composed of m columns of matrix A is chosen) until a basic solution is obtained.

The value of the objective function for the basic solution thus determined will be:

$$Z(\bar{x}_1^B, \dots, \bar{x}_m^B, 0, \dots, 0) \overset{not}{=} \bar{Z} = (c^B)^T \bar{x}^B = (c^B)^T B^{-1} b$$

Next, the matrix with m lines and $(n-m)$ columns will be denoted by G:

$$G = B^{-1}R = \begin{bmatrix} g_{1\,m+1} & g_{1\,m+2} & \cdots & g_{1\,n} \\ \vdots & & & \\ g_{i\,m+1} & g_{i\,m+2} & \cdots & g_{i\,n} \\ \vdots & & & \\ g_{m\,m+1} & g_{m\,m+2} & \cdots & g_{m\,n} \end{bmatrix}$$

Since G has the same size as R, the same indices were used. Starting from the basic solution obtained, its iterative improvement is sought so as to obtain, at each step, an increase in the value of the objective function. For this purpose, it is considered that one of the variables in x^R becomes non-zero, so it enters between the basic variables. Let this component x_j with j having a value between $m+1$ and n. The values of the components of the solution thus modified result from:

$$x^B = B^{-1}b - B^{-1}Rx^R = \bar{x}^B - G * \left[0 \dots\ 0\ x_j\ 0 \dots 0\right]^T$$

or, customizing for a specific component:

$$x_i^B = \bar{x}_i^B - g_{ij}x_j$$

As can be seen from this relationship, as x_j increases, all components of x^B corresponding to positive g_{ij} values will decrease. Since the resulting solution must still be a basic solution (have at most m positive components and the rest zero) the growth of x_i will have to stop when the first of its components becomes zero, which corresponds to the minimum of the ratio $\frac{\bar{x}_s^B}{g_{ij}}$:

$$x_j = \underbrace{min}_{k=\overline{1,m}} \left\{ \frac{\bar{x}_k^B}{g_{kj}} \middle|\ g_{kj} > 0 \right\}$$

The goal function value for the new solution will be:

$$Z(x) = \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{m} c_i x_i^B + c_j x_j = \sum_{i=1}^{m} c_i x_i - x_j\left(\sum_{i=1}^{m} c_i g_{ij} - c_j\right)$$

or, with the notations:

$$Z(x) \overset{not}{=} \bar{Z}$$

$$Z_j \overset{not}{=} \sum_{i=1}^{m} c_i g_{ij}$$

how to get:

$$Z = \bar{Z} - x_j\left(Z_j - c_j\right)$$

So, for the objective function to increase ($Z > \bar{Z}$), since x_j is positive, the difference $Z_j - c_j$ must be negative. If the condition is met:

$$Z_j - c_j \geq 0, \forall j = \overline{m+1, n}$$

the value of the objective function can no longer increase by passing a variable from x^R to x^B, so the basic solution obtained is the optimal solution.

If there is a variable for which $Z_j - c_j < 0$ and, in addition, $g_{ij} < 0 \; \forall \; i = \overline{1,n}$ then the objective function does not admit a finite minimum.

In this case, the value of this variable can be however high because, along with it, the other variables in x^B increase, and the value of the objective function increases to infinity.

It follows that by analyzing the sign of the differences $Z_j - c_j$ and the elements of G it is possible to establish:

- whether the current basic solution is an optimal solution or not,
- if the given problem does not have an optimal solution.

If none of these conclusions result, then the current solution can be improved. The choice of the variable that enters the base is not critical as long as the condition that $Z_j - c_j$ is negative is respected. An incorrect choice can only lead to increasing the number of steps after which the problem is solved and the optimal solution is obtained.

The variable that passes into the vector of secondary variables is obtained as the first variable x_s that cancels:

$$\frac{\bar{x}_s^B}{g_{sl}} = \underbrace{min}_{k=\overline{1,m}} \left\{ \frac{\bar{x}_k^B}{g_{kl}} \middle| \; g_{kl} > 0 \right\}$$

The Primal Simplex Algorithm consists of the following steps:

Step 1: An initial basic solution is assumed to be known. B is determined and the matrix is calculated: $G = B^{-1}R$.

Step 2: The differences are determined: $Z_j - c_j, \forall j \in J \overset{not}{=} \{m+1, m+2, \dots, n\}$ and their sign is analyzed

- If $Z_j - c_j \geq 0, \forall j \in J,$ the basic solution is optimal,
- If there are negative $Z_j - c_j$ differences, denote by $J_1 = \{j \in J \mid Z_j - c_j < 0\}$ the set of indices of the negative differences.

Step 3: We analyze the sign of the components of the column vectors g_j of G for indices $j \in J_1$:

- If there is at least one vector $g_j \leq 0, \; j \in J_1$ (which has no positive component) it follows that the problem has no bounded solution,
- If for all indices $j \in J_1$ the vectors g_j have positive components, the vector a_e that will enter the basis is determined from the condition:

$$|Z_e - c_e| = \underbrace{max}_{j=\overline{m+1,n}} \{|Z_j - c_j| \mid j \in J_1\}$$

the vector a_l will come out of the basis, which satisfies the condition:

$$\frac{\bar{x}_l^B}{g_{le}} = \underbrace{min}_{k=\overline{1,m}} \left\{ \frac{\bar{x}_k^B}{g_{ke}} \middle| \; g_{ke} > 0 \right\}$$

- The new base is formed by replacing a_l with a_e, the solution corresponding to this base is determined, and we return to step 1.

The initial solution will be improved iteratively, until the optimal solution is obtained or until it is determined that the problem does not have a finite maximum.

To apply the algorithm, a table called simplex table is completed and has the following form:

		c_1	c_2	c_m	c_{m+1}	c_k	c_n	
c	Baza	a_1	a_2	a_m	a_{m+1}	a_k	a_n	b
c_1	a_1	1	0	0	g_{1m+1}	g_{1k}	g_{1n}	$\tilde{b}_1$
c_2	a_2	0	1	0	g_{2m+1}	g_{2k}	g_{2n}	$\tilde{b}_2$
c_m	a_m	0	0	1	g_{mm+1}	g_{mk}	g_{mn}	$\tilde{b}_m$
Z_j		Z_1	Z_2	Z_m	Z_{m+1}	Z_k	Z_n	Z
Z_j-c_j		0	0	0	Z_{m+1}-c_{m+1}	Z_k-c_k	Z_n-c_n	

Figure 3.4 Illustration of simplex table forms

The Z_j value is obtained as the scalar product between the vector C and the column a_i of the simplex table. For the base variables, $Z_i = c_i$ will be obtained (the column being a versor of the base), $Z_i - c_i = 0$. The Z value of the objective function is equal to the dot product of c and b.

To move to the next simplex table, the pivoting method is applied, the pivot being the element located at the intersection of the vector line leaving the base with the vector column entering the base. To apply this method proceed as follows:

- the pivot line is divided at it;
- except for the pivot, its column is completed with 0;
- considering that the pivot is a_{ij}, any element that is not found neither on the line n or on the pivot column is determined as:

$$\tilde{a}_{ke} = a_{ke} - \frac{a_{ie}a_{kj}}{a_{ij}}$$

$$\tilde{b}_k = b_k - \frac{b_i a_{kj}}{a_{ij}}$$

$\tilde{a}, \tilde{b}$ being the values from the modified table, a, b those from the previous table.

Example 3.4.d (Maximum LP) Let's consider the following LP problem and use the Simplex method to find the optimal solution:

$$Maximize\ Z\ =\ 2x_1 + 6x_2 + 2x_3$$

Subject to: $\begin{cases} 2x_1 + 5x_2 + 3x_3 \ \leq\ 8 \\ 4x_1 + 3x_2 + 8x_3 \ \leq 16 \\ x_1, x_2, x_3 \ \geq\ 0 \end{cases}$

Converting the Standard Form of the LP problem to the Canonical Form:

$$Maximize\ Z\ =\ 2x_1 + 6x_2 + 2x_3 + 0 * s_1 + 0 * s_2$$

Subject to: $\begin{cases} 2x_1 + 5x_2 + 3x_3 + s_1 = 8 \\ 4x_1 + 3x_2 + 8x_3 + s_2 = 16 \\ x_1, x_2, x_3, s_1, s_2 \ \geq\ 0 \end{cases}$

The matrix A and b is constructed from the system of equations:

$A = \begin{bmatrix} 2 & 5 & 3 & 1 & 0 \\ 4 & 3 & 8 & 0 & 1 \end{bmatrix}$, $\quad x = [x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2]^T$ $\quad$ and $\quad b = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

The maximization function becomes: $Maximize\ Z\ =\ 2x_1 + 6x_2 + 2x_3 + 0 * s_1 + 0 * s_2$

The matrix C is constructed: $C = [2 \quad 6 \quad 2 \quad 0 \quad 0]$

To identify the optimal solutions, the following simplex table is drawn:

pivot — Enter the base — Out of the base

Initial table	C_B	C_j	2	6	2	0	0	MinRatio
Base (B)		b	x_1	x_2, ↓	x_3	s_1	s_2	b / x_e
← s_1	0	8	2	**5**	3	1	0	$8/5 \Rightarrow min$
s_2	0	16	4	3	8	0	1	16/3
Z = 0		Z_j	0	0	0	0	0	
$Z_j - C_j$			-2	−6	-2	0	0	
sign			< 0	< 0	< 0	$= 0$	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B1 = s_1; \\ B2 = s_2 \end{cases}$

C_B column items

Each cell of this column is equal to the coefficient, which corresponds to the base variable in the corresponding row: $\begin{cases} C_{B,1} = 0 \\ C_{B,2} = 0 \end{cases}$.

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products: $Z = C_{B,1} * b_1 + C_{B,2} * b_2 = 0 * 8 + 0 * 6 = 0$

Evaluated Control Variables ($Z_j - C_j$)

We calculate the estimates for each controlled variable, by element-wise multiplying the value from the variable column, by the value from the C_B column, summing up the results of the products, and subtracting the coefficient of the objective function from their sum, with this variable.:

$$x_1: \left(\left(C_{B,1} * x_{1,1}\right) + \left(C_{B,2} * x_{1,2}\right)\right) - C_{x_1} = 0 * 2 + 0 * 4 - 2 = -2$$

$$x_2: \left(\left(C_{B,1} * x_{2,1}\right) + \left(C_{B,2} * x_{2,2}\right)\right) - C_{x_2} = 0 * 5 + 0 * 3 - 6 = -6$$

$$x_3: \left(\left(C_{B,1} * x_{3,1}\right) + \left(C_{B,2} * x_{3,2}\right)\right) - C_{x_3} = 0 * 3 + 0 * 8 - 2 = -2$$

$$s_1: \left(\left(C_{B,1} * s_{1,1}\right) + \left(C_{B,2} * s_{1,2}\right)\right) - C_{s_1} = 0 * 1 + 0 * 0 - 0 = 0$$

$$s_2: \left(\left(C_{B,1} * s_{2,1}\right) + \left(C_{B,2} * s_{2,2}\right)\right) - C_{s_2} = 0 * 0 + 0 * 1 - 0 = 0$$

MinRatio column items

Since there are negative values among the estimates of the controlled variables, the current table does not yet have an optimal solution. Therefore, in the basis we introduce the variable with the smallest negative estimate.

The number of variables in the basis is always constant, so it is necessary to choose which variable to derive from the basis, for which we calculate MinRatio. The elements of the MinRatio column are calculated by dividing the values from column b by the value from the column corresponding to the variable that is entered in the basis (only for positiv values):

$$MR1 = \frac{b_1}{x_{2,1}} = \frac{8}{5} = 1.6$$

$$MR2 = \frac{b_2}{x_{2,2}} = \frac{16}{3} = 5.33$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

			pivot		Enter the base		Out of the base	
Iteration 1	C_B	C_j	2	6	2	0	0	MinRatio b / x_e
Base (B)		b	x_1	x_2	x_3	s_1	s_2	
x_2	6	1.6	0.4	1	0.6	0.2	0	
s_2	0	11.2	2,8	0	6.2	-0.6	1	
Z = 9.6	-	Z_j	0.8	2	1.2	-0.6	0	

		$C_j - Z_j$	0.4	0	1.6	1.2	0	
			> 0	$= 0$	> 0	> 0	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B1 = x_2; \\ \mathrm{B2} = \mathrm{s}_2 \end{cases}$

C_B column items

Each cell of this column is equal to the coefficient, which corresponds to the base variable in the corresponding row: $\begin{cases} C_{B,1} = 2 \\ C_{B,2} = 0 \end{cases}$.

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = 6 * 1.6 + 0 * 11.2 = 9.6.$

Evaluated Control Variables ($z_j - c_j$)

We calculate the estimates for each controlled variable, by element-wise multiplying the value from the variable column, by the value from the C_B column, summing up the results of the products, and subtracting the coefficient of the objective function from their sum, with this variable.:

$x_1: \left(\left(\mathrm{C}_{B,1} * \mathrm{x}_{1,1}\right) + \left(\mathrm{C}_{B,2} * \mathrm{x}_{1,2}\right)\right) - \mathrm{C}_{x_1} = 6 * \frac{2}{5} + 0 * \frac{14}{5} - 2 = \frac{2}{5} = 0.4$

$x_2: \left(\left(\mathrm{C}_{B,1} * \mathrm{x}_{2,1}\right) + \left(\mathrm{C}_{B,2} * \mathrm{x}_{2,2}\right)\right) - \mathrm{C}_{x_2} = 6 * 1 + 0 * 0 - 6 = 0$

$x_3: \left(\left(\mathrm{C}_{B,1} * \mathrm{x}_{3,1}\right) + \left(\mathrm{C}_{B,2} * \mathrm{x}_{3,2}\right)\right) - \mathrm{C}_{x_3} = 6 * \frac{3}{5} + 0 * \frac{31}{5} - 2 = \frac{8}{5} = 1.6$

$s_1: \left(\left(\mathrm{C}_{B,1} * \mathrm{s}_{1,1}\right) + \left(\mathrm{C}_{B,2} * \mathrm{s}_{1,2}\right)\right) - \mathrm{C}_{s_1} = 6 * \frac{1}{5} + 0 * 0 - 0 = \frac{6}{5} = 1.2$

$s_2: \left(\left(\mathrm{C}_{B,1} * \mathrm{s}_{2,1}\right) + \left(\mathrm{C}_{B,2} * \mathrm{s}_{2,2}\right)\right) - \mathrm{C}_{s_2} = 6 * 0 + 0 * 1 - 0 = 0$

Since there are no negative values among the estimates of the controlled variables, the current table has an optimal solution.

The value of the objective function:

$Z^* = 9.6$;

The variables that are present in the basis are equal to the corresponding cells of the column b, all other variables are equal to zero:

$x_1 = 0, x_2 = 1.6, x_3 = 0.$

A linear programming problem has an infinity of solutions when it has multiple optimal solutions, and the optimal value of the objective function is unbounded. This situation typically arises in the context of maximization problems when the objective function can be increased indefinitely without violating any of the constraints.

In the simplex method, if the initial tableau has more than one basic feasible solution that achieves the same optimal objective function value, it indicates the existence of an infinity of solutions. This occurs when the objective function has a slope of zero in the direction of optimization, meaning it runs parallel to one of the constraint lines within the feasible region.

In such cases, the simplex method will continue to pivot, moving along an edge of the feasible region towards the direction of optimization, while maintaining the same objective function value. Since the objective function has no strict maximum value, the problem is unbounded, and an infinity of solutions exists along the parallel constraint lines.

The general optimal solution is written as a convex linear combination of the obtained optimal independent solutions. Let x^1 and x^2 are two distinct optimal solutions. The general solution is of the form:

$$x_G = \alpha_1 x^1 + \alpha_2 x^2$$

Example 3.4.e (Maximum LP) Let's consider the following LP problem and use the Simplex method to find the optimal solution:

$$Maximize\ Z = -x_1 + x_2$$

$$\text{Subject to: } \begin{cases} -2x_1 + x_2 \leq 2 \\ x_1 - x_2 \leq 2 \\ x_1 + x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases}$$

Converting the Standard Form of the LP problem to the Canonical Form:

$$Maximize\ Z = -x_1 + x_2 + 0 * s_1 + 0 * s_2 + 0 * s_3$$

$$\text{Subject to: } \begin{cases} -2x_1 + x_2 + s_1 = 2 \\ x_1 - x_2 + s_2 = 2 \\ x_1 + x_2 + s_3 = 5 \\ x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{cases}$$

The matrix A and b is constructed from the system of equations:

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix},\ x = [x_1 \quad x_2 \quad s_1 \quad s_2 \quad s_3]^T \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

The maximization function becomes: $Maximize\ Z = -x_1 + x_2 + 0 * s_1 + 0 * s_2 + 0 * s_3$

The matrix C is constructed: $C = [-1 \quad 1 \quad 0 \quad 0 \quad 0]$

To identify the optimal solutions, the following simplex table is drawn:

pivot — Enter the base — Out of the base

Initial table	C_B	C_j	-1	1	0	0	0	MinRatio b / x_e
Base (B)		b	x_1	x_2, ↓	s_1	s_2	s_3	
← s_1	0	2	-2	**1**	1	0	0	2/1 = 2
s_2	0	2	1	-1	0	1	0	

s_3	0	5	1	1	0	0	1	2/1=2
Z = 0		Z_j	0	0	0	0	0	
$Z_j - C_j$			1	$\boxed{-1}$	0	0	0	
sign			> 0	< 0	$= 0$	$= 0$	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1 \\ B_2 = s_2 \\ B_3 = s_3 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 + C_{B,3} * b_3 = 0 * 2 + 0 * 2 + 0 * 5 = 0$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

Step 1: We iteratively search for the first optimal solution (x^1)*:*

			pivot		Enter the base		Out of the base	
Iteration 1	C_B	C_j	-1	1	0	0	0	MinRatio b / x_e
Base (B)		b	x_1 ↓	x_2	s_1	s_2	s_3	
x_2	1	2	-2	1	1	0	0	
s_2	0	4	-1	0	1	1	0	
← s_3	0	3	$\boxed{3}$	0	-1	0	1	$\boxed{3/3 = 1}$
Z = 2		Z_j	-2	1	1	0	0	
$Z_j - C_j$			$\boxed{-1}$	0	1	0	0	
sign			< 0	$= 0$	> 0	$= 0$	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = x_2; \\ B_2 = s_2 \\ B_3 = x_1 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 + C_{B,3} * b_3 = 1 * 2 + 0 * 4 + (-1) * 3 = -1$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

			pivot		Enter the base		Out of the base	
Initial table	C_B	C_j	-1	1	0	0	0	MinRatio b / x_e
Base (B)		b	$x1$	x_2	s_1	s_2	s_3	
x_2	1	4	0	1	1/3	0	2/3	
s_2	0	5	0	0	2/3	1	1/3	
x_1	-1	1	1	0	-1/3	0	1/3	
Z = 3		Z_j	-1	1	2/3	5	1/3	
$Z_j - C_j$			0	0	2/3	5	1/3	
sign			$= 0$	$= 0$	> 0	> 0	> 0	

Since there are no negative values among the estimates of the controlled variables, the current table has an optimal solution.

The value of the objective function:

$Z^{*1}(x^1) = 3$;

The variables that are present in the basis are equal to the corresponding cells of the column b, all other variables are equal to zero:

$x^1 = (x_1^1, x_2^1) = (1, 4)$.

Step 2: We iteratively search for the second optimal solution (x^2)*:*

To identify the optimal solutions, the following simplex table is drawn:

			pivot		Enter the base		Out of the base	
Initial table	C_B	C_j	-1	1	0	0	0	MinRatio b / x_e
Base (B)		b	x_1	$x_2, \downarrow$	s_1	s_2	s_3	
s_1	0	2	-2	1	1	0	0	$2/1 = 2$
s_2	0	2	1	-1	0	1	0	
$\leftarrow s_3$	0	5	1	1	0	0	1	$2/1 = 2$
Z = 0		Z_j	0	0	0	0	0	
$Z_j - C_j$			1	-1	0	0	0	

sign	> 0	< 0	$= 0$	$= 0$	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1 \\ B_2 = s_2 \\ B_3 = s_3 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 + C_{B,3} * b_3 = 0 * 2 + 0 * 2 + 0 * 5 = 0$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

			pivot		Enter the base		Out of the base	
Iteration 1	C_B	C_j	-1	1	0	0	0	MinRatio b / x_e
Base (B)		b	$x1$	x_2	s_1	s_2	s_3	
s_1	0	-3	-3	0	1	0	-1	
s_2	0	7	2	0	0	1	1	
x_2	1	5	1	1	0	0	1	
Z = 5		Z_j	1	1	0	0	1	
$Z_j - C_j$			2	0	0	0	1	
sign			> 0	$= 0$	$= 0$	$= 0$	> 0	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1; \\ B_2 = s_2 \\ B_3 = x_2 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 + C_{B,3} * b_3 = 0 * (-3) + 0 * 7 + 1 * 5 = 5$$

Since there are no negative values among the estimates of the controlled variables, the current table has an optimal solution.

The value of the objective function:

$Z^{*2}(x^2) = 5;$

The variables that are present in the basis are equal to the corresponding cells of the column b, all other variables are equal to zero:

$x^2 = (x_1^2, x_2^2) = (0, 5)$.

Step 3: Writing the general solution*:* The general solution is of the form:

$$x_G = \alpha_1 x^1 + \alpha_2 x^2,\ \alpha_1 + \alpha_2 = 1$$

So, $x_G = \alpha_1 * (1, 4) + \alpha_2 * (0, 5),\ \forall \alpha_1, \alpha_2 \in R^*:\ \alpha_1 + \alpha_2 = 1$

Step 3: Writing the particular solution (if necessary)*:* So, for $\alpha_1 = 0.3\ and\ \alpha_2 = 0.7$ which satisfy the condition $\alpha_1 + \alpha_2 = 1$, we obtain the following particular solution:

$$x_p = 0.3 * (1, 4) + 0.7 * (0, 5) = (0.3, 1.2) + (0, 3.5) = (0.3, 4.7)$$

The value of the objective function:

$$Z_p^* = -1 *\ 0.3 + 1 * 4.7 = -0.3 + 4.7 = 4.4$$

The Simplex Method is an iterative algorithm used to solve linear programming problems by optimizing the objective function, either maximizing or minimizing it, subject to linear constraints. When applying the Simplex method to a minimization problem, the goal is to find the values of the decision variables that minimize the objective function while satisfying all the constraints.

The steps of the *Simplex Method for Minimization* Problems:

Step 1: Convert the Linear Programming Problem into Canonical Form: Rewrite the objective function and constraints in standard form by introducing slack or surplus variables as necessary. The standard form requires *all constraints to be in the form of* $\leq$ and all decision variables to be non-negative.

Step 2: Write the Initial Simplex Tableau:

- Create the initial simplex tableau using matrices to represent the coefficients of the canonical form constraints and the objective function.
- The objective row represents the coefficients of the negated objective function (multiply by -1).
- The constraint rows represent the coefficients of the constraints.
- The right-hand side column represents the constants on the right side of the constraints.

Step 3: Perform the Simplex Iterations

- Choose a pivot column: Identify the most negative coefficient in the objective row, known as the pivot column.
- Choose a pivot row: Identify the constraint with the minimum non-negative ratio between its right-hand side constant and the corresponding coefficient in the pivot column. This is known as the pivot row.
- Perform the pivot operation: Use the pivot row to eliminate all other coefficients in the pivot column, except for the pivot element.
- Update the simplex tableau: Adjust the objective row and constraint rows using the pivot row's information.
- Repeat the process until all coefficients in the objective row are non-negativ.

Step 4: Identify the Optimal Solution:

- Once all coefficients in the objective row are non-negativ, the simplex method is complete.

- The basic variables (corresponding to pivot columns) are non-zero, while the non-basic variables (corresponding to columns with all zeros except for a 1 in the pivot row) are set to zero.
- The value in the last column represents the optimal value of the negated objective function (multiply by -1).
- The decision variables corresponding to the basic variables in the final tableau provide the optimal solution.

Example 3.4.f (Simplex Method for Minimization Problem): Let's consider the following LP problem and use the Simplex method to find the optimal solution:

$$Minimize \;\; Z = 2x_1 + x_2$$

$$\text{Subject to: } \begin{cases} x_1 + x_2 \geq 4 \\ 2x_1 + x_2 \geq 5 \\ x_1, x_2 \geq 0 \end{cases}$$

Preliminary stage:

The preliminary stage begins with the need to get rid of negative values (if any) in the right part of the restrictions. For what the corresponding restrictions are multiplied by -1. After this manipulation, the sign of inequality is reversed.

Next, you need to get rid of inequalities, for which we introduce compensating variables in the left-hand side of the inequalities. If an inequality of the form ≤, then the compensating variable has the sign +, if the inequality of the form ≥, then the compensating variable has the sign -. Compensating variables are included in the objective function of the problem with a zero coefficient.

Now in the constraint system it is necessary to find a sufficient number of basis variables. Each constraint must have one basis variable. The basic is a variable that has a coefficient of 1 with it and is found only in one constraint. If there are no basis variables in some restriction, then we add them artificially, and artificial variables enter the objective function with the coefficient $-M$ if the objective function tends to max and $+M$, if the objective function tends to min.

Converting the Standard Form of the LP problem to the Canonical Form:

$$Minimize \;\; Z = 2x_1 + x_2 + 0 * u_1 + 0 * u_2 + Ma_1 + Ma_2$$

$$\text{Subject to: } \begin{cases} x_1 + x_2 - u_1 + a_1 = 4 \\ 2x_1 + x_2 - u_2 + a_2 = 5 \\ x_1, x_2, u_1, u_{2,} a_1, a_2 \geq 0 \end{cases}$$

The matrix A and b is constructed from the system of equations:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 & 0 & 1 \end{bmatrix},\; x = [x_1 \quad x_2 \quad u_1 \quad u_2 \quad a_1 \quad a_2]^T \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

The minimiztion function becomes:

$$Minimize\, Z = 2x_1 + x_2 + 0 * u_1 + 0 * u_2 + Ma_1 + Ma_2$$

The matrix C is constructed: $C = [2 \quad 1 \quad 0 \quad 0 \quad M \quad M]$.

To identify the optimal solutions, the following simplex table is drawn:

pivot element — Enter the base — Out of the base

Initial table	C_B	C_j	2	1	0	0	M	M	MinRatio b / x_e
Base (B)		b	x_1 ↓	x_2	u_1	u_2	s_1	s_2	
s_1	M	4	1	1	-1	0	1	0	4/1=4
← s_2	M	5	2	1	0	-1	0	1	5/2 = 2.5
Z = 9M		Z_j	3M	2M	-M	-M	M	M	
$Z_j - C_j$			$3M - 2$	2M-1	-M	-M	0	0	
sign			> 0	> 0	< 0	< 0	$= 0$	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1 \\ B_2 = s_2 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = M * 4 + M * 5 = 9M$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

pivot — Enter the base — Out of the base

Initial table	C_B	C_j	2	1	0	0	M	M	MinRatio b / x_e
Base (B)		b	x_1	x_2 ↓	u_1	u_2	s_1	s_2	
← s_1	M	1.5	0	0.5	-1	0.5	1	-0.5	1.5/0.5 = 3
x_1	2	2.5	1	0.5	0	-0.5	0	0.5	2.5/0.5=5
$Z = 1.5M + 5$		Z_j	2	0.5M+1	-M	0.5M-1	M	-0.5M+1	
$Z_j - C_j$			0	$0.5M$	-M	0.5M-1	0	-1.5M+1	
sign			$= 0$	> 0	< 0	> 0	$= 0$	< 0	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1 \\ \mathrm{B}_2 = \mathrm{x}_1 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = M * 1.5 + 2 * 2.5 = 1.5M + 5$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

			Pivot element		Enter the base		Out of the base		
Initial table	C_B	C_j	2	1	0	0	M	M	MinRatio b / x_e
Base (B)		b	x_1	x_2	u_1	u_2	s_1	s_2	
x_2	1	3	0	1	-2	0.5	2	1	
x_1	2	1	1	0	-1	-1	-1	1	
$Z = 5$		Z_j	2	1	-4	-1.5	0	3	
$Z_j - C_j$			0	0	-4	-1.5	-M	3-M	
sign			$= 0$	$= 0$	< 0	< 0	< 0	< 0	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = x_2 \\ \mathrm{B}_2 = \mathrm{x}_1 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = 1 * 3 + 2 * 1 = 5$$

Since there are no pozitive values among the estimates of the controlled variables, the current table has an optimal solution.

The value of the objective function:

$Z^* = 5$;

The variables that are present in the basis are equal to the corresponding cells of the column b, all other variables are equal to zero:

$x = (x_1, x_2) = (3, 1)$.

✍ *The Matrix Form of the Simplex Method is an efficient way to solve linear programming problems with many variables and constraints. It allows for easy manipulation and updating of the tableau, making it a powerful tool for optimization.*

Dual Simplex Method: The Dual Simplex Method is an extension of the Simplex Method used to solve linear programming problems that involve unbounded solutions or infeasible solutions. It is particularly useful for problems with infeasible initial solutions or degenerate cases.

The main idea of the Dual Simplex Method is to pivot between feasible solutions along the dual feasible path until an optimal solution is reached. The method starts with an infeasible basic feasible solution and iteratively moves along the dual feasible path towards the optimal solution while maintaining feasibility. Here are the basic steps of the Dual Simplex Method (the maximization case):

Step 1: Convert the Linear Programming Problem into Canonical Form: Rewrite the objective function and constraints in standard form by introducing slack or surplus variables as necessary. The standard form requires all constraints to be in the form of $\leq$, and all decision variables to be non-negative.

Step 2: Write the Initial Simplex Tableau: Create the initial simplex tableau using matrices to represent the coefficients of the canonical form constraints and the objective function.

- The objective row represents the coefficients of the negated objective function (multiply by -1).
- The constraint rows represent the coefficients of the constraints.
- The right-hand side column represents the constants on the right side of the constraints.

Step 3: Perform the Dual Simplex Iterations: Choose a pivot column: Identify the most negative coefficient in the objective row, known as the pivot column.

- Choose a pivot row: Identify the constraint with the minimum non-negative ratio between its right-hand side constant and the corresponding coefficient in the pivot column. This is known as the pivot row.
- Perform the pivot operation: Use the pivot row to eliminate all other coefficients in the pivot column, except for the pivot element.
- Update the simplex tableau: Adjust the objective row and constraint rows using the pivot row's information.
- Repeat the process until all coefficients in the objective row are non-negative and the right-hand side constants remain non-negative.

Step 4: Identify the Optimal Solution or Infeasibility: If all coefficients in the objective row are non-negative and the right-hand side constants remain non-negative, the simplex method is complete, and an optimal solution is reached.

- If there are negative coefficients in the objective row or negative right-hand side constants during the iterations, the problem may be infeasible, and additional iterations are needed.

Step 5: Handle Infeasible Solutions: When the problem is infeasible, the Dual Simplex Method uses the feasibility conditions to modify the tableau and move towards a feasible solution. Continue the iterations, pivoting along the dual feasible path, until reaching a feasible solution or determining that the problem is unbounded.

Example 3.4.f (Simplex Method for Minimization Problem): Let's consider the following LP problem and use the Simplex method to find the optimal solution:

$$Minimize \;\; Z = 2x_1 + 3x_2$$

Subject to: $\begin{cases} 2x_1 - x_2 - x_3 \geq 3 \\ x_1 - x_2 + x_3 \geq 2 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

In order to apply the dual simplex method, convert *Minimize Z* to *Maximize Z* and all ≥ constraint to ≤ constraint by multiply -1:

$Maximize \quad Z = -2x_1 - 3x_2$

Subject to: $\begin{cases} -2x_1 + x_2 + x_3 \leq -3 \\ -x_1 + x_2 - x_3 \leq -2 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

Converting the Standard Form of the LP problem to the Canonical Form:

$Maximize \quad Z = -2x_1 - 3x_2 + 0 * s_1 + 0 * s_2$

Subject to: $\begin{cases} -2x_1 + x_2 + x_3 + s_1 = -3 \\ -x_1 + x_2 - x_3 + s_2 = -2 \\ x_1, x_2, x_3, s_1, s_2 \geq 0 \end{cases}$

The matrix A and b is constructed from the system of equations:

$A = \begin{bmatrix} -2 & 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \end{bmatrix}$, $x = [x_1 \quad x_2 \quad x_3 \quad s_1 \quad s_2]^T$ and $b = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

The maximiztion function becomes:

$$Minimize\ Z = -2\boldsymbol{a_1} - 3\boldsymbol{a_2} + 0 * \boldsymbol{a_3} + 0 * \boldsymbol{s_1} + 0\boldsymbol{s_2}$$

The matrix C is constructed: $C = [-2 \quad -3 \quad 0 \quad 0 \quad 0]$.

To identify the optimal solutions, the following simplex table is drawn:

pivot element — Enter the base — Out of the base

Initial table	C_B	C_j	-2	-3	0	0	0	MinRatio b / x_e
Base (B)		b	x_1↓	x_2	x_3	s_1	s_2	
← s_1	0	-3	−2	1	1	1	0	(−3)/(−2) = 1.5
s_2	0	-2	-1	1	-1	0	1	(-2)/(-1)=2
Z = 0		Z_j	0	0	0	0	0	
$Z_j - C_j$			2	3	0	0	0	
			> 0	> 0	0	= 0	= 0	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = s_1 \\ B_2 = s_2 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = 0 * (-3) + 0 * (-2) = 0$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

pivot element — Enter the base — Out of the base

Initial table	C_B	C_j	-2	-3	0	0	0	MinRatio b / x_e
Base (B)		b	x_1	x_2	x_3 ↓	s_1	s_2	
x_1	-2	1.5	1	-0.5	-0.5	-0.5	0	
← s_2	0	-0.5	0	0.5	−1.5	-0.5	1	1.5/0.5 = 3
Z = -3		Z_j	-2	1	1	1	0	
$Z_j - C_j$			0	4	1	1	0	
sign			$= 0$	> 0	> 0	> 0	$= 0$	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = x_1 \\ B_2 = s_2 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = -2 * (1.5) + 0 * (-0.5) = -3$$

We deduce from the basis the variable with the least positive value of MiniRatio.

At the intersection of the line that corresponds to the variable that is derived from the basis, and the column that corresponds to the variable that is entered into the basis, is the resolving element (pivot element).

This element will allow us to calculate the elements of the table of the next iteration.

pivot element — Enter the base — Out of the base

Initial table	C_B	C_j	-2	-3	0	0	0	MinRatio b / x_e
Base (B)		b	x_1	x_2	x_3	s_1	s_2	
x_1	-2	5/3	1	-2/3	0	-1/3	-1/3	
x_3	0	1/3	0	-1/3	1	1/3	-2/3	
Z = -10/3		Z_j	-2	4/3	0	2/3	2/3	
$Z_j - C_j$			0	13/3	0	2/3	2/3	

sign	$= 0$	> 0	$= 0$	> 0	> 0	

Elements of the column basis (B)

Transfer to the table the basic elements that we identified in the preliminary stage: $\begin{cases} B_1 = x_1 \\ B_2 = x_3 \end{cases}$

Objective function value (Z)

We calculate the value of the objective function by elementwise multiplying the column Cb by the column b, adding the results of the products:

$$Z = C_{B,1} * b_1 + C_{B,2} * b_2 = -2 * \left(\frac{5}{3}\right) + 0 * \left(\frac{1}{3}\right) = -\frac{10}{3}$$

Since there are no negative values among the estimates of the controlled variables, the current table has an optimal solution.

The value of the objective function:

$Z^*_{max} = -\frac{10}{3} \Rightarrow Z^*_{min} = \frac{10}{3}$.

The variables that are present in the basis are equal to the corresponding cells of the column b, all other variables are equal to zero:

$x_1 = \frac{5}{3}, x_2 = 0, x_3 = \frac{1}{3}$.

The Dual Problem:

The dual problem in linear programming (LP) is a mathematical formulation that is associated with the primal LP problem. It involves maximizing or minimizing a function, subject to its own set of constraints, derived from the original LP problem. The dual problem plays a crucial role in LP, particularly in solving LP problems and understanding their properties.

Here are the key components of the dual problem:

1. **Primal Problem (Primal LP):**
   - The original LP problem is referred to as the primal problem. It involves maximizing or minimizing an objective function, subject to a set of constraints.
2. **Dual Problem (Dual LP):**
   - The dual problem is a related LP problem associated with the primal problem. It involves finding the best possible lower bound (minimization) or upper bound (maximization) on the objective function of the primal problem while satisfying its own set of constraints.
3. **Objective Function of the Dual Problem:**
   - The objective function of the dual problem is determined by the coefficients of the constraints in the primal problem. For each constraint in the primal problem, there is a corresponding dual variable in the dual problem. The objective function of the dual problem is a weighted sum of these dual variables.
4. **Constraints of the Dual Problem:**
   - The constraints in the dual problem are derived from the coefficients of the objective function in the primal problem. The dual constraints are typically inequalities, and each constraint is associated with a variable in the primal problem.
5. **Relationship Between Primal and Dual Problems:**

- Strong duality theory establishes a close relationship between the optimal solutions of the primal and dual problems. If the primal problem has an optimal solution, then the dual problem also has an optimal solution, and the optimal objective values of both problems are equal.

6. **Complementary Slackness:**
   - Complementary slackness conditions are used to determine the relationship between the primal and dual solutions. These conditions state that if a constraint in the primal problem is binding (holds with equality), the corresponding dual variable in the dual problem must be nonzero.
7. **Solving the Dual Problem:**
   - Solving the dual problem can provide information about the quality of the primal solution. The dual problem helps in finding the optimal values of the dual variables, which, in turn, can be used to assess the sensitivity of the primal solution to changes in the problem parameters.

✍ *In summary, the dual problem in linear programming provides insights into the relationship between the primal and dual solutions and is used for sensitivity analysis, assessing the quality of the primal solution, and providing bounds on the objective function. The duality theory is a fundamental concept in linear programming.*

To construct the dual problem for a given primal linear programming (LP) problem, you can follow a set of rules and guidelines. Here are the steps to construct the dual problem:

Primal Problem (Original Problem): Start with the primal LP problem, which is usually stated in the following standard form:

Maximize (or Minimize): $f_{pimal} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ \cdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k \\ \cdots \\ a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n \geq b_l \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{cases}$$

where:

- f_{pimal} is the objective function of the primal problem to be maximized (or minimized),
- $c_1, c_2, \ldots c_n$ are the coefficients of the objective function,
- $x_1, x_2, \ldots x_n$ are the decision variables,
- a_{ij} are the coefficients of the constraints
- $b_1, b_2, \ldots, b_m$ are the right-hand sides of the constraints.

! The constraints can be of the form ≤ (less than or equal to) or ≥ (greater than or equal to).

Rules for Constructing the Dual Problem: The rules for constructing the dual problem are as follows:

1. For each constraint in the primal problem, introduce a corresponding dual variable in the dual problem. These dual variables are denoted by $y_1, y_2, \dots, y_m$,
2. The objective function of the dual problem is to minimize (if the primal problem is a maximization problem) or maximize (if the primal problem is a minimization problem) the sum of the products of the dual variables and the right-hand side values of the primal constraints.
3. The constraints of the dual problem are based on the coefficients of the primal objective function:
   - If the primal objective function coefficient is c_i, then the dual problem will have a constraint of the form $\sum_{i=1}^{m} a_{ik} y_j \geq c_k$, (or $\sum_{i=1}^{m} a_{ik} y_j \leq c_k$, or $\sum_{i=1}^{m} a_{ik} y_j = c_k, j = 1,2, \dots n$)
4. For each decision variable in the primal problem, introduce a constraint in the dual problem. These constraints will depend on the type of primal constraint $(\leq, \geq, =)$.
5. If a primal constraint is in concordance with the type of problem $\left(\leq for\ the\ Max\ f_{primal}; \geq for\ the\ MIN\ f_{primal}\right)$, the corresponding dual constraint will be ≥ 0.
6. If a primal constraint is in unconcordance with the type of problem $\left(\geq for\ the\ Max\ f_{primal}; \leq for\ the\ MIN\ f_{primal}\right)$, the corresponding dual constraint will be ≤ 0.
7. If a primal constraint is of the form $\sum_{i=1}^{m} a_{ik} y_j = c_k$ is unrestricted, the corresponding dual constraint will have no sign restrictions.
8. The Dual Problem has as many variables (or restrictions) as the Primal Problem has restrictions (or variables).

✍ *These are the basic rules for constructing the dual problem from a given primal problem in linear programming. The resulting dual problem can then be solved to obtain dual optimal values and dual variables, which can be used for various purposes, including sensitivity analysis and assessing the quality of the primal solution.*

Example3.4.g: Write the dual problem for the given primal problem:

[A]

$$\underbrace{\begin{bmatrix} MAX\ f(X) = 8x_1 + 3x_2 + x_3 + 5x_4 + 7x_5 \\ 3x_1 - 2x_2 + x_3 + 4x_4 - x_5 \leq 6 \\ 2x_1 + 2x_3 + 5x_4 + x_5 = 9 \\ x_1 + 6x_2 + x_3 + 2x_4 \geq 3 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3\ unsign\ var. \\ x_4\ unsign\ var. \\ x_5 \leq 0 \end{bmatrix}}_{PIMAL\ PROBLEM} \Leftrightarrow \underbrace{\begin{bmatrix} \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \end{bmatrix}}_{RELATIONS} \Leftrightarrow \underbrace{\begin{bmatrix} MIN\ g(Y) = 6y_1 + 9y_2 + 3y_3 \\ y_1 \geq 0 \\ y_2\ unsign\ var. \\ y_3 \leq 0 \\ 3y_1 + 2y_2 + y_3 \leq 8 \\ -2y_1 + 6y_3 \leq 3 \\ y_1 + 2y_2 + 6y_3 = 1 \\ 4y_1 + 5y_2 + 2y_3 = 5 \\ -y_1 + y_2 \geq 0 \end{bmatrix}}_{DUAL\ PROBLEM}$$

[B]

$$\underbrace{\begin{bmatrix} MIN\ f(X) = 8x_1 + 3x_2 + x_3 + 5x_4 + 7x_5 \\ 3x_1 - 2x_2 + x_3 + 4x_4 - x_5 \leq 6 \\ 2x_1 + 2x_3 + 5x_4 + x_5 = 9 \\ x_1 + 6x_2 + x_3 + 2x_4 \geq 3 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3\ unsign\ var. \\ x_4\ unsign\ var. \\ x_5 \leq 0 \end{bmatrix}}_{PIMAL\ PROBLEM} \Leftrightarrow \underbrace{\begin{bmatrix} \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \end{bmatrix}}_{RELATIONS} \Leftrightarrow \underbrace{\begin{bmatrix} MAX\ g(Y) = 6y_1 + 9y_2 + 3y_3 \\ y_1 \leq 0 \\ y_2\ unsign\ var. \\ y_3 \geq 0 \\ 3y_1 + 2y_2 + y_3 \geq 8 \\ -2y_1 + 6y_3 \geq 3 \\ y_1 + 2y_2 + 6y_3 = 1 \\ 4y_1 + 5y_2 + 2y_3 = 5 \\ -y_1 + y_2 \leq 0 \end{bmatrix}}_{DUAL\ PROBLEM}$$

The Fundamental Theorem of Duality in linear programming establishes a strong relationship between the optimal solutions of the primal and dual problems. This theorem is a key result that plays a crucial role in understanding the duality theory in linear programming.

The Fundamental Theorem of Duality states:

Strong Duality Theorem:
If the primal linear programming problem (Primal LP) has an optimal solution, then the dual linear programming problem (Dual LP) also has an optimal solution, and the optimal objective values of both problems are equal.

Mathematically:
If Z^*_{pimal} is the optimal objective value of the primal problem, and Z^*_{dual} is the optimal objective value of the dual problem, then: $Z^*_{pimal} = Z^*_{dual}$.

Key implications of the Strong Duality Theorem:

1. **Existence of Optimal Solutions:**
   - If the primal problem has an optimal solution, it implies that the dual problem also has an optimal solution. The existence of an optimal solution is guaranteed for both the primal and dual problems.
2. **Equality of Optimal Objective Values:**
   - The optimal values of the objective functions in the primal and dual problems are equal. This equality provides a critical insight into the relationship between the primal and dual optimal solutions.
3. **Duality Gap:**
   - The difference between the optimal values of the primal and dual problems is known as the duality gap. The Strong Duality Theorem asserts that there is no duality gap when both problems have optimal solutions.
4. **Complementary Slackness:**
   - Complementary slackness conditions provide a way to characterize the relationship between the primal and dual solutions. These conditions state that if a primal constraint

is binding (holds with equality), the corresponding dual variable is nonzero, and vice versa.

✍ *The Strong Duality Theorem is a fundamental result that underlines the power and symmetry of linear programming duality. It enables analysts and practitioners to leverage information from both the primal and dual perspectives, providing insights into the structure and nature of the underlying optimization problems. The theorem is a cornerstone in the theory and application of linear programming*.

The dual problem in linear programming has important economic interpretations that provide insights into resource allocation, shadow prices, and the value of resources.

Here are some key economic interpretations of the dual problem:

1. **Shadow Prices or Dual Prices:**
   - The dual variables in the dual problem are often referred to as shadow prices or dual prices. These prices represent the rate of change in the optimal value of the objective function of the primal problem with respect to a one-unit increase in the right-hand side of the corresponding constraint. In economic terms, the shadow price indicates the marginal value of an additional unit of a resource.
2. **Resource Valuation:**
   - Each dual variable in the dual problem corresponds to a resource or constraint in the primal problem. The value of a dual variable reflects the economic value or opportunity cost associated with relaxing or tightening the corresponding constraint. A higher dual variable implies a higher economic value or opportunity cost for that resource.
3. **Optimal Resource Allocation:**
   - The optimal dual solution provides information about the optimal allocation of resources in the primal problem. The dual variables indicate which constraints are binding (have a nonzero dual value) and provide a measure of the scarcity or importance of each resource in the optimal solution.
4. **Complementary Slackness:**
   - Complementary slackness conditions link the primal and dual solutions. The conditions state that if a primal constraint is binding (holds with equality), the corresponding dual variable is nonzero, and vice versa. This implies that resources with a binding constraint in the primal problem have a positive shadow price in the dual problem.
5. **Economic Interpretation of Dual Objective Function:**
   - The objective function of the dual problem is often interpreted as the minimum total cost of acquiring the available resources to meet the demands specified in the primal problem. This interpretation highlights the economic trade-offs and costs associated with obtaining the necessary resources.
6. **Feasibility and Infeasibility:**

- The feasibility of the dual problem reflects the feasibility of the primal problem. If the dual problem is feasible, it indicates that the primal problem has a feasible solution, and vice versa. Infeasibility in the dual problem implies infeasibility in the primal problem.

✍ *Understanding the economic interpretation of the dual problem is crucial for decision-makers to make informed decisions about resource allocation, pricing, and production planning. The dual information complements the insights gained from the primal problem and provides a comprehensive view of the economic implications of linear programming models*.

Questions you should be able to answer after studying the topic of ” LP Solution Techniques”

What is Linear Programming (LP), and what are its main objectives?

What are the key components of a linear programming problem?

What are the different types of LP problems, and how do they differ in terms of objectives and constraints?

How do you formulate LP problems in standard form and convert them into canonical form?

What is the Simplex method, and how does it work to find the optimal solution for LP problems?

How do you set up the initial feasible solution for the Simplex method?

What is the role of pivot operations in the Simplex method, and how do they improve the solution?

How do you interpret the Simplex method's iterations and the resulting solution?

What are the potential outcomes of the Simplex method, and how do you identify unbounded and infeasible LP problems?

How can you handle degeneracy and cycling issues in the Simplex method?

What is the duality theory in LP, and how is it related to the primal LP problem?

How do you formulate the dual LP problem and interpret its significance?

What is the relationship between the primal and dual solutions in LP problems?

How do you interpret the shadow price or dual value in the context of LP problems?

What are sensitivity analysis and post-optimality analysis in LP, and how do they help in decision-making?

How can you use LP to solve real-world optimization problems in various industries or sectors?

What are the limitations of LP, and when are other optimization techniques more suitable?

How do you apply LP solution techniques using software or programming tools?

How do LP solutions contribute to efficient resource allocation and decision-making processes?

EXERCISES

3.4.1 In these exercises, try to use LP terminology accurately and apply the appropriate LP solution techniques to find the optimal solutions. Additionally, understand the significance of shadow prices and sensitivity analysis in LP

a) Fill in the missing words in the following sentences with appropriate terms related to Linear Programming (LP):

a.1) In Linear Programming, the objective is to _____ (maximize/minimize) a linear _____ (function/equation) subject to _____ (constraints/variables).

a.2) The _____ (coefficients/variables) of the objective function represent the _____ (costs/profits) or _____ (revenues/costs) of the decision variables.

a.3) LP problems with only _____ (less than/greater than) constraints are known as _____ (open/closed) problems.

b) LP Solution Simplex Technique - Graphical Method Consider the following LP problem:

Maximize: $3x + 2y$

Subject to:

$2x + y \leq 8$

$x + 2y \leq 6$

$x, y \geq 0$

b.1) Plot the feasible region defined by the constraints on a graph.

b.2) Find the corner points of the feasible region.

b.3) Evaluate the objective function at each corner point to determine the optimal solution.

b.4) Identify the optimal values of *x* and *y* that maximize the objective function.

c) LP Solution Techniques - Simplex Method Consider the following LP problem:

Maximize: 5x + 3y

Subject to:

$2x + y \leq 10$

$x + 3y \leq 18$

$x, y \geq 0$

c.1) Convert the inequalities into equations by introducing slack variables.

c.2) Set up the initial tableau for the Simplex Method.

c.3) Apply the Simplex Method to find the admisibil solution.

d) Fill in the Missing Words - LP Sensitivity Analysis Fill in the missing words in the following sentences with appropriate terms related to LP Sensitivity Analysis:

d.1) Sensitivity Analysis in LP helps us examine how changes in the _____ (objective function/coefficient matrix) or _____ (constraints/objective function) affect the optimal solution.

d.2) The _____ (shadow price/dual variable) of a constraint indicates the rate of change of the objective function value with respect to a one-unit increase in the right-hand side of the constraint.

d.3) If the _____ (shadow price/dual variable) of a constraint is zero, a small change in the constraint's right-hand side will not affect the _____ (feasibility/optimal solution).

3.4.2 For each exercise, follow the steps of the Simplex Method carefully, including selecting the pivot element, performing row operations, and updating the tableau. Pay attention to signs and inequalities while setting up the tableau and identifying the optimal solution. Also, check for feasibility and degeneracy in the LP problems as appropriate

a) Basic Solution Consider the following LP problem in standard form:

Maximize: $2x + 3y$

Subject to: $x + y \leq 4,\ 2x + y \leq 5,\ x, y \geq 0$

a.1) Find the basic solution corresponding to the initial basic feasible solution ($x = 0, y = 0$). *a.2)* Verify if the basic solution is feasible. *a.3)* Calculate the objective function value for the basic solution.

b) Simplex Method - Maximization Use the Simplex Method to solve the following LP problem:

Maximize: $4x + 3y$

Subject to: $2x + y \leq 8,\ x + 2y \leq 6,\ x, y \geq 0$

b.1) Set up the initial Simplex tableau. *b.2)* Perform the first iteration of the Simplex Method. *b.3)* Determine whether an optimal solution has been reached. If not, continue the iterations until an optimal solution is found.

c) Simplex Method - Minimization Use the Simplex Method to solve the following LP problem:

Minimize: $5x + 4y$

Subject to: $2x + 3y \geq 6,\ 4x + y \geq 4,\ x, y \geq 0$

c.1) Set up the initial Simplex tableau. *c.2)* Perform the first iteration of the Simplex Method. *c.3)* Determine whether an optimal solution has been reached. If not, continue the iterations until an optimal solution is found.

d) Degeneracy and Cycling Consider the following LP problem:

Maximize: 3x + 2y

Subject to: x + y ≤ 4 2x + y ≤ 5 x, y ≥ 0

d.1) Find the initial basic feasible solution. *d.2)* Use the Simplex Method to find the optimal solution. *d.3)* Determine if the problem is degenerate and identify any cycling that occurs during the iterations.

3.4.3 For each exercise, you can use either manual calculations or software/tools to apply the Simplex Method. The exercises aim to reinforce your understanding of LP solution techniques and their application in economic mathematics. Additionally, sensitivity analysis helps you understand how changes in objective function coefficients affect the optimal solution.

a) Basic Solution Consider the following LP problem:

Maximize: $Z = 3x + 4y$

Subject to: $2x + y \leq 8,\ x + 2y \leq 6,\ x, y \geq 0$

a.1) Determine the feasible region defined by the constraints. *a.2)* Find the basic solution corresponding to each corner point of the feasible region. *a.3)* Evaluate the objective function Z at each basic solution. d) Identify the optimal solution and the optimal value of Z.

b) Simplex Method - Maximization

Maximize: $Z = 5x + 3y$

Subject to: $2x + y \leq 10,\ 4x + 3y \leq 24,\ x, y \geq 0$

b.1) Set up the initial Simplex tableau. *b.2)* Apply the Simplex Method to find the optimal solution and the optimal value of Z. *B.3)* Interpret the results in the context of the problem.

c) Simplex Method - Minimization

Minimize: $Z = 2x + 4y$

Subject to: $2x + y \geq 8,\ x + 3y \geq 6,\ x, y \geq 0$

c.1) Set up the initial Simplex tableau. *c.2)* Apply the Simplex Method to find the optimal solution and the optimal value of Z. *c.3)* Interpret the results in the context of the problem.

d) Sensitivity Analysis Consider the following LP problem:

Maximize: $Z = 4x + 6y$

Subject to: $2x + 3y \leq 12,\ x + 2y \leq 8,\ x, y \geq 0$

d.1) Find the optimal solution and the optimal value of Z using the Simplex Method. d.*2)* Perform sensitivity analysis to determine the range of variation in the objective function coefficients (c1 = 4 and c2 = 6) that maintains the current optimal solution. *d.3)* Interpret the results of sensitivity analysis in terms of changes in the objective function coefficients.

3.4.4 In each exercise, follow the steps of the Simplex method, including identifying the entering and leaving variables, computing the pivot row, and updating the tableau in each iteration. Ensure that you fully understand the concept of basic solutions, feasibility, and the steps of the Simplex method to solve the given LP problems:

a) Basic Solution Consider the following LP problem in standard form:

Maximize: $2x1 + 3x2$

Subject to:

$x1 + 2x2 \leq 8$

$2x1 + x2 \leq 10$

$x1, x2 \geq 0$

a.1) Determine the basic solution corresponding to the initial feasible solution (x1 = 0, x2 = 0). *a.2)* Verify if the basic solution is feasible. *a.3)* Calculate the objective function value for this basic solution.

b) Simplex Method - Maximization Use the Simplex method to solve the following LP problem:

Maximize: $4x1 + 3x2$

Subject to:

$2x1 + x2 \leq 6$

$x1 + 2x2 \leq 8$

$x1, x2 \geq 0$

b.1) Write the initial simplex tableau for the given LP problem. *b.2)* Perform the Simplex iterations to find the optimal solution. *b.3)* Identify the optimal solution and the maximum value of the objective function.

c) Simplex Method - Minimization Use the Simplex method to solve the following LP problem:

Minimize: 5x1 + 4x2

Subject to:

$x1 + x2 \geq 4$

$2x1 + 3x2 \geq 6$

$x1, x2 \geq 0$

c.1) Write the initial simplex tableau for the given LP problem. *c.2)* Perform the Simplex iterations to find the optimal solution. *c.3)* Identify the optimal solution and the minimum value of the objective function.

d) Degeneracy in Simplex Method Use the Simplex method to solve the following LP problem:

Maximize: 3x1 + 2x2

Subject to:

$x1 + x2 \leq 4$

$2x1 + 2x2 \leq 5$

$x1, x2 \geq 0$

d.1) Write the initial simplex tableau for the given LP problem. *d.2)* Perform the Simplex iterations until reaching the optimal solution. *d.3)* Identify if the final solution is degenerate, and explain the implications of degeneracy in the Simplex method.

3.4.5 For each exercise, you use the Simplex Method and the Dual Simplex Method, as appropriate, to solve the LP problem. Show all the steps involved in finding the basic solution or the optimal solution using the chosen method. Also, explain the concept of basic solution, basic feasible solution, and degeneracy in the context of Linear Programming

a) Basic Solution Consider the following LP problem in standard form:

Maximize: $3x + 2y$

Subject to: $x + 2y \leq 6,\ 3x + 2y \leq 12,\ x, y \geq 0$

a.1) Find the basic solution for this LP problem. *b.2)* Determine whether the basic solution is feasible.

b) Dual Simplex Method Consider the following LP problem in standard form:

Maximize: $2x + y$

Subject to: $3x + y \leq 6,\ x + 2y \leq 4,\ x, y \geq 0$

b.1) Set up the initial tableau for the LP problem. *b.2)* Apply the Dual Simplex Method to find the optimal solution step-by-step.

c) Basic Feasible Solution Consider the following LP problem in standard form:

Minimize: $2x + y$

Subject to: $3x + y \geq 6,\ x + 2y \leq 4,\ x, y \geq 0$

c.1) Find the basic feasible solution for this LP problem. *c.2)* Determine whether the basic feasible solution is optimal.

d) Degeneracy and Dual Simplex Method Consider the following LP problem in standard form:

Maximize: $4x + y$

Subject to: $2x + y \leq 6,\ x + 2y \leq 4,\ x, y \geq 0$

d.1) Set up the initial tableau for the LP problem. *d.2)* Apply the Dual Simplex Method to find the optimal solution step-by-step. *d.3)* Determine whether the LP problem is degenerate and explain its impact on the Dual Simplex Method.

3.5 LP APPLICATIONS

Keywords: *Linear Programming Applications, Economic Optimization, Resource Allocation, Production Planning, Supply Chain Management, Financial Planning, Portfolio*

Optimization, Marketing Budget, Allocation, Sales Territory Planning, Agricultural Planning, Public Sector Resource Management, Transportation Planning, Traffic Optimization, Energy Planning, Labor Scheduling, Workforce Planning, Cost Minimization, Profit Maximization, Investment Decision-Making, Budget Allocation in Economics.

Linear programming (LP) has a wide range of applications in the field of economics. It is a powerful optimization technique used to make informed decisions and allocate resources efficiently in various economic scenarios. Here are some common applications of LP in the economy:

Resource Allocation: LP is frequently used to allocate limited resources such as labor, capital, raw materials, and production capacity optimally. Companies can use LP models to determine the best mix of resources to maximize profits or minimize costs while meeting demand.

Production Planning: LP is used to optimize production plans by determining the optimal quantities of different products to manufacture based on available resources and market demand. It helps companies achieve cost-efficient production and maximize revenue.

Supply Chain Management: LP is employed to optimize supply chain logistics, including inventory management, transportation, and distribution. It aids in minimizing transportation costs, reducing inventory holding costs, and ensuring timely delivery of goods to customers.

Financial Planning: In the financial sector, LP models are used for portfolio optimization, where investment decisions are made to maximize returns while minimizing risk. It helps investors allocate their funds across different assets efficiently.

Marketing and Sales: LP is used for marketing budget allocation and sales territory planning. It helps companies allocate their marketing resources effectively to maximize sales and profitability.

Agricultural Planning: LP can be used to optimize agricultural production by determining the best crop mix and allocation of resources to maximize agricultural output and profits.

Resource Management in Public Sector: Governments use LP to allocate public resources efficiently, such as budget allocation, workforce planning, and distribution of public services.

Transportation Planning: LP is used in urban planning and transportation systems to optimize traffic flow, public transportation routes, and vehicle fleet management.

Energy Planning: LP is employed in the energy sector to optimize energy production and distribution, considering various factors like costs, demand, and environmental constraints.

Labor Scheduling: LP is used to optimize workforce scheduling, especially in service industries like healthcare and retail, to match labor supply with demand and minimize labor costs.

Economic Optimization: Economic optimization refers to the process of maximizing or minimizing a specific economic objective subject to certain constraints. It is a fundamental concept in economics, aimed at making the best use of available resources to achieve the most desirable outcomes.

In economic optimization, a mathematical model is formulated to represent the economic problem. This model typically includes an objective function that represents the quantity to be

maximized or minimized, as well as a set of constraints that represent the limitations or restrictions on the decision variables.

The objective function can represent various economic goals, such as maximizing profit, minimizing costs, optimizing utility, maximizing revenue, or minimizing risk. The decision variables are the economic parameters that can be controlled or adjusted, and the constraints represent the limitations on these variables, such as budget constraints, resource availability, or production capacity.

Linear programming is a common technique used for economic optimization problems, but there are also other optimization methods like nonlinear programming, dynamic programming, and integer programming, depending on the complexity and nonlinearity of the problem.

✍ *Economic optimization is widely applied in various economic sectors, including business management, finance, marketing, supply chain management, resource allocation, and public policy. It helps decision-makers make data-driven choices to achieve efficient resource allocation, cost-effectiveness, and optimal outcomes in economic activities.*

Cost Minimization: Cost minimization is an economic optimization concept that aims to find the most cost-efficient way of producing a given level of output or achieving a specific goal. It is a fundamental principle in business management and economics, as it allows companies to optimize their production processes and resource allocation to minimize expenses while maintaining the desired level of output.

In cost minimization, businesses seek to identify the combination of inputs (such as labor, capital, raw materials) that will result in the lowest production costs while still meeting the production requirements. This is typically represented mathematically as an optimization problem with an objective function that represents the total cost of production and a set of constraints that reflect the production requirements.

The objective function in cost minimization represents the total cost of production, which includes both variable costs (costs that change with the level of output) and fixed costs (costs that remain constant regardless of the level of output). The decision variables are the quantities of inputs used in the production process, and the constraints represent the production requirements, such as the minimum level of output to be achieved.

Linear programming and other optimization techniques are often used to solve cost minimization problems in practical applications. By finding the optimal combination of inputs and production levels, companies can reduce waste, improve efficiency, and enhance their competitiveness in the market.

Example: Cost Minimization in Production: Imagine a manufacturing company that produces two types of products, Product A and Product B. The production process involves labor (L) and raw materials (R). The company wants to minimize its total production costs while producing a certain quantity of each product to meet market demand.

Step 1: <u>Formulating the Real Economic Problem</u>

Objective:

Minimize the total production cost.

Decision Variables:

Let x represent the quantity of *Product A* produced. Let y represent the quantity of *Product B* produced.

Constraints:

Labor constraint: $2x + y \leq 100$ (Total labor hours available).

Raw material constraint: $3x + 2y \leq 150$ (Total raw material units available).

Non-negativity constraint: $x \geq 0$, $y \geq 0$ (Quantities of products cannot be negative).

Step 2: Mathematical Formulation

Objective Function:

$Minimize\ Z = C(x, y) = 5x + 4y\ (Total\ production\ cost)$

Subject to the following constraints:

$$\begin{cases} 2x + y \leq 100 \\ 3x + 2y \leq 150 \\ x \geq 0, y \geq 0 \end{cases}$$

The objective function Z represents the total production cost, where C(x, y) is the cost function with coefficients 5 and 4 representing the cost per unit of Product A and Product B, respectively.

The constraints represent the labor and raw material limitations, as well as the non-negativity constraints. The company cannot produce negative quantities of products, and the production quantities must not exceed the available labor and raw material resources.

Step 3: Solving the Cost Minimization Problem

Using linear programming techniques like the Simplex Method, the company can find the optimal values of *x* and *y* (quantities of Product A and Product B) that minimize the total production cost while satisfying all the constraints. The optimal solution will indicate the most cost-effective way to produce the desired quantity of each product.

✍ *By solving this cost minimization problem, the manufacturing company can make informed decisions to optimize its production process, allocate resources efficiently, and achieve cost-effective production. This will help the company remain competitive in the market by offering products at competitive prices while maximizing profitability.*

Cost minimization is especially crucial in competitive industries where firms aim to produce goods and services at the lowest cost to offer competitive prices to consumers. It is also relevant in public policy, where governments seek to minimize the costs of public services and infrastructure projects while ensuring quality and effectiveness. Overall, cost minimization is a key aspect of economic decision-making, as it allows businesses and policymakers to make informed choices that lead to cost-effective and sustainable outcomes.

Transportation planning: Transportation planning is the process of designing and implementing strategies to efficiently move people and goods from one location to another. It is a critical aspect of urban and regional development that aims to improve the transportation system's performance, reduce congestion, enhance accessibility, and promote sustainable mobility options. Transportation planning involves a combination of analysis, forecasting, and decision-making to address various transportation-related challenges and meet the needs of communities and economies.

Key Components of Transportation Planning:

Traffic Analysis: Transportation planners analyze traffic patterns, flow, and congestion to identify areas of concern and potential bottlenecks. This involves collecting data on traffic volumes, travel times, and road conditions. The transport problem

Demand Forecasting: Planners use demographic, economic, and social data to forecast future transportation needs. They predict travel demand to estimate the number of people and goods requiring transportation services.

Route Planning: Determining the most efficient routes for different modes of transportation, such as cars, buses, trains, and bicycles, to optimize travel times and reduce congestion.

Public Transit Planning: Designing and expanding public transportation systems, including buses, trains, and light rail, to provide accessible and convenient options for commuters.

Multi-Modal Integration: Coordinating various modes of transportation to create a seamless and interconnected network that encourages people to use a combination of transit options for their journeys.

Infrastructure Development: Identifying the need for new roads, bridges, highways, and public transit facilities and planning their construction and maintenance.

Environmental Impact Assessment: Evaluating the environmental consequences of transportation projects to ensure sustainable and eco-friendly solutions.

Land Use and Zoning: Coordinating transportation planning with land use and zoning policies to promote urban development that supports efficient transportation systems.

Safety Analysis: Assessing the safety of transportation infrastructure and implementing measures to improve road safety and reduce accidents.

Accessibility and Equity: Ensuring that transportation planning considers the needs of all members of the community, including those with disabilities and vulnerable populations, to promote social equity and inclusivity.

✍ *Transportation planning aims to enhance mobility, reduce congestion, minimize environmental impacts, and improve the overall quality of life for residents and businesses in a region. It plays a crucial role in shaping urban development and ensuring that transportation systems support economic growth and sustainable development*.

The Transport Problem: The Transport Problem is a classic optimization problem in operations research and linear programming that deals with the efficient allocation of goods from multiple sources to multiple destinations, considering the transportation costs. It is used to find the optimal way to transport goods from suppliers to customers while minimizing the total transportation costs.

In the Transport Problem, the following elements are involved:

- *Sources (Suppliers)*: The locations where goods are produced or available for shipment.
- *Destinations (Customers*): The locations where goods need to be delivered.
- *Supply*: The quantity of goods available at each source.
- *Demand:* The quantity of goods required at each destination.
- *Transportation Cost*: The cost of transporting one unit of goods from each source to each destination.

The objective of the Transport Problem is to find the allocation of goods from sources to destinations that minimizes the total transportation cost while satisfying the supply and demand

constraints. The goal is to optimize the transportation plan by determining how much to transport from each source to each destination.

Mathematically, the Transport Problem can be formulated as a linear programming problem. Let:

x_{ij} be the quantity of goods transported from source *i* to destination *j*,

c_{ij} be the transportation cost per unit from source *i* to destination *j*,

a_i be the supply at source *i*,

b_j be the demand at destination *j*.

The objective function is to minimize the total transportation cost:

$Minimize\ Z\ =\ \sum(c_{ij}x_{ij}),\ for\ all\ i\ and\ j.$

Subject to the following constraints:

Supply Constraints: $\Sigma\ x_{ij}\ =\ a_i$ for all $i = \overline{1,m}$ (Total goods transported from each source should equal the supply available).

Demand Constraints: $\Sigma\ x_{ij}\ =\ b_i$ for all $j = \overline{1,n}$ (Total goods received at each destination should meet the demand).

In mathimatical terms, the above problem can be expressed as finding a set of x_{ij}'s, to meet supply and demand requirements at a minimum distribution cost. The corresponding linear model is:

$Minimize\ Z\ =\ \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}x_{ij})$

Subject to the following constraints:

$$\begin{cases}\sum_{j=1}^{n} x_{ij} = a_i\,, & i = \overline{1,m} \\ \sum_{i=1}^{m} x_{ij} = b_j\,, & j = \overline{1,n} \\ x_{ij}, & i = \overline{1,m},\quad j = \overline{1,n}\end{cases}$$

✍ *The Transport Problem can be solved using various algorithms and optimization techniques, such as the North-West Corner Method, Vogel's Approximation Method, and the MODI method. These methods help find the optimal solution that minimizes the total transportation cost and allocates the goods efficiently between the sources and destinations.*

The North-West Corner Method

The North-West Corner Method is a simple and intuitive heuristic used to find an initial feasible solution for the Transportation Problem. It starts from the top-left corner of the transportation matrix and allocates units to the destination cells until either the supply from the source or the demand at the destination is exhausted.

The method follows a step-by-step process:

Step 1: Start at the Top-Left Corner Begin at the top-left corner cell of the transportation matrix.

Step 2: Allocate Units Allocate units from the source to the destination until either the supply of the source or the demand of the destination is met. In each step, choose the minimum value between the remaining supply of the source and the remaining demand of the destination.

Step 3: Move to the Next Cell Move to the next cell to the right if the demand of the current destination is met, or move to the next cell downwards if the supply of the current source is exhausted. Repeat Step 2 for the new cell.

Step 4: Repeat Until Complete Continue the allocation process until all the supply and demand values are exhausted.

Advantages of the North-West Corner Method:

Simplicity: The North-West Corner Method is straightforward and easy to understand. It involves a systematic approach of allocating units starting from the top-left corner, making it accessible to individuals with basic knowledge of linear programming.

Feasible Solution: The method guarantees a feasible solution (initial basic feasible solution) to the Transportation Problem. It quickly provides an allocation that satisfies the supply and demand constraints.

Quick Initialization: The method requires minimal computation and is useful for large-scale problems. It efficiently initializes the transportation matrix by allocating units in a simple manner.

Intuitive Solution: The initial solution obtained using the North-West Corner Method has an intuitive interpretation, as it follows a simple pattern of allocation from the top-left corner.

Limitations of the North-West Corner Method:

Sub-Optimal Solution: The initial solution obtained using the North-West Corner Method may not be optimal. It can lead to higher transportation costs compared to other advanced methods like the Modified Distribution (MODI) or the Stepping Stone Method.

Degeneracy: In some cases, the method may result in degenerate solutions where one or more cells in the initial solution have a value of zero. Degeneracy can lead to challenges in using some optimization algorithms.

Inefficiency: The North-West Corner Method does not consider the actual transportation costs associated with allocating units. It may lead to inefficient allocations when compared to more sophisticated optimization techniques.

Lack of Iteration: The method does not involve iterative improvement, which means it may not identify the optimal solution in a single step. Additional iterations are needed to refine the initial solution.

N*ot Suitable for Unbalanced Problems*: The North-West Corner Method may not work well for unbalanced transportation problems, where the total supply does not equal the total demand. It may lead to unused capacity or unmet demand.

✍ *While the North-West Corner Method is a useful heuristic for quickly obtaining an initial feasible solution, it is often combined with other optimization techniques to improve the solution's efficiency and achieve the optimal allocation in the Transportation Problem.*

Example 3.5.b Let's illustrate the North-West Corner Method using a simple example of the Transport Problem: A company has three factories (F1, F2, and F3) producing goods and four warehouses (W1, W2, W3, and W4) that need to receive these goods.

The transportation costs (in dollars per unit) from each factory to each warehouse are as follows:

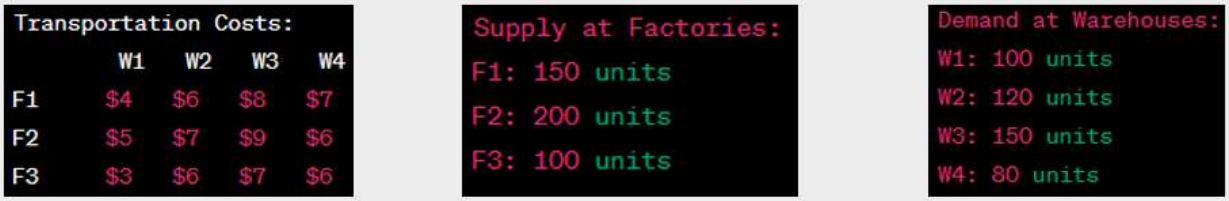


Figure 3.5 The illustration of the transportation costs, supply available at each factory and the demand at each warehouse

Step 1: Initialization: Start with the top-left corner cell (F1, W1) and allocate the maximum possible units (minimum of the supply at F1 and the demand at W1). In this case, we allocate 100 units from F1 to W1.

Allocations					
Iteration 1	W1	W2	W3	W4	Supply
F1	*100*	-	-	-	**150 (-100)**
F2	-	-	-	-	200
F3	-	-	-	-	100
Demand	**100 (-100)**	120	150	80	

Step 2: Update Supply and Demand: Adjust the remaining supply and demand based on the allocation made in Step 1. And move to the Next Cell: Move to the next cell to the right (F1, W2) and allocate the maximum possible units. In this case, we allocate 50 units from F1 to W2.

Allocations					
Iteration 2	W1	W2	W3	W4	Supply
F1	**100**	*50*	-	-	**50 (-50)**
F2	-	-	-	-	200
F3	-	-	-	-	100
Demand	0	**120 (-50)**	150	80	

Step 3: Update Supply and Demand: Adjust the remaining supply and demand based on the allocation made in Step 2. And continue Allocation: Continue the allocation process until all supply and demand values are zero. The final allocations will be:

Allocations					
Iteration 3	W1	W2	W3	W4	Supply
F1	**100**	**50**	-	-	0
F2	-	**70**	-	-	**200 (-70)**
F3	-	-	-	-	100
Demand	0	**70 (-70)**	150	80	

Allocations					
Iteration 4	W1	W2	W3	W4	Supply
F1	**100**	**50**	-	-	0
F2	-	**70**	**130**	-	**130 (-130)**
F3	-	-	-	-	100
Demand	0	0	**150 (-130)**	80	

Allocations					
Iteration 5	W1	W2	W3	W4	Supply
F1	**100**	**50**	-	-	0
F2	-	**70**	**130**	-	0
F3	-	-	**20**	-	**100 (-20)**
Demand	0	0	**20 (-20)**	80	

Allocations					
Iteration 6	W1	W2	W3	W4	Supply
F1	**100**	**50**	-	-	0
F2	-	**70**	**130**	-	0
F3	-	-	**20**	**80**	0
Demand	0	0	0	0	

Step 4: The total transport cost using the North-West Corner Method:

$$Z = \sum_{i=1}^{3}\sum_{j=1}^{4}(c_{ij}x_{ij}) = (c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14}) + (c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23} + c_{24}x_{24}) + (c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34})$$

$$Z = (4 * 100 + 6 * 50 + 8 * 0 + 7 * 0) + (5 * 0 + 7 * 70 + 9 * 130 + 6 * 0) + (3 * 0 + 6 * 0 + 7 * 20 + 6 * 80) = 700 + 1660 + 620 = \$2980$$

The total transport cost using the North-West Corner Method for this example is $2980.

This cost represents the sum of transportation costs incurred in transporting goods from factories to warehouses according to the initial allocation obtained through the North-West Corner Method.

The North-West Corner Method provides a starting feasible solution for the Transportation Problem. However, it may not be the optimal solution. To find the optimal solution, more sophisticated algorithms like the Modified Distribution (MODI) Method or the Stepping Stone Method are employed to iterate and improve the initial solution until no further improvements can be made.

The Stepping Stone Method

The Stepping Stone Method is an iterative procedure used in linear programming and transportation problems, particularly in finding the optimal solution for the Transportation Problem. It is often employed in combination with other optimization techniques like the Modified Distribution (MODI) Method to improve the initial feasible solution and find the optimal allocation of goods between sources and destinations.

The Stepping Stone Method involves the following steps:

Step 1: Start with a Basic Feasible Solution: Begin with a basic feasible solution, which is an initial allocation of goods that satisfies the supply and demand constraints but may not be the optimal solution.

Step 2: Identify Empty Cells (Non-Basic Variables): Identify the empty cells in the transportation matrix, which are the cells not assigned any units in the current solution.

Step 3: Find Closed Loops: Starting from each empty cell, find a closed loop that involves only non-empty cells (basic variables). A closed loop is a path that starts and ends at the same empty cell, traveling through basic cells in a way that maintains the balance of supply and demand.

Step 4: Determine Improvement: For each closed loop, calculate the improvement (or reduction) in the objective function (total transportation cost) if one unit is transferred from a source cell (basic) to a destination cell (non-basic).

Step 5: Choose the Best Closed Loop: Select the closed loop that offers the maximum improvement in the objective function. This closed loop represents the best reallocation of one unit between cells to decrease the total transportation cost.

Step 6: Move the Unit and Update the Solution: Move one unit along the selected closed loop, reallocating it from the source cell to the destination cell, and update the solution accordingly. This means adjusting the units in the cells involved in the closed loop.

Step 7: Repeat the Process: Continue the process by identifying new closed loops, determining improvements, and moving units along the selected closed loop until there are no more negative improvements. This indicates that the current solution is optimal, and no further improvements can be made.

Advantages of the Stepping Stone Method:

Flexibility: The Stepping Stone Method can be applied to various optimization problems, especially the Transportation Problem, and can handle large and complex transportation networks.

Iterative Improvement: It offers an iterative approach to improving the initial feasible solution. By repeatedly reallocating units, it converges towards the optimal solution, ensuring efficiency in finding the best allocation.

Optimal Solution: When used in combination with other optimization techniques like the Modified Distribution (MODI) Method, the Stepping Stone Method guarantees an optimal solution for the Transportation Problem, assuming there are no degenerate solutions.

Local Improvements: The method identifies local improvements by exploring closed loops in the transportation matrix, which allows for targeted and systematic adjustments to the current solution.

Suitable for Manual Calculation: The Stepping Stone Method is suitable for manual calculation, making it accessible to analysts and practitioners who might not have access to specialized optimization software.

Limitations of the Stepping Stone Method:

Complexity: As the problem size increases, finding all possible closed loops and determining improvements for each can become computationally expensive and time-consuming.

Degeneracy: The Stepping Stone Method may encounter difficulties when dealing with degenerate solutions, where more than one cell in the initial feasible solution has a value of zero. Degeneracy can lead to cycling (looping) in the method, making it challenging to find the optimal solution.

Multiple Optimal Solutions: In some cases, the Transportation Problem may have multiple optimal solutions, leading to the possibility of obtaining different optimal solutions through the Stepping Stone Method.

Subjective Selection of Closed Loops: If there are multiple closed loops with the same improvement, the method requires subjective selection of the loop to use for reallocation. Different choices might lead to different optimal solutions.

Large Solution Space: The Stepping Stone Method explores only a small subset of the entire solution space, and there is no guarantee that the global optimal solution will be found.

Manual Calculation Limitation: While the method is suitable for manual calculation in smaller problems, for larger and more complex problems, using optimization software that incorporates advanced algorithms may be more efficient.

✍ *Overall, the Stepping Stone Method is a powerful iterative approach for finding the optimal solution for the Transportation Problem. While it has its limitations, it remains a valuable tool in operations research and transportation planning, providing a systematic way to improve initial feasible solutions and reduce transportation costs.*

Example 3.5.c Let's illustrate the Stepping Stone Method with a simple example of the Transportation Problem: A company has two factories (F1 and F2) producing goods and three warehouses (W1, W2, and W3) that need to receive these goods. The transportation costs (in dollars per unit) from each factory to each warehouse are as follows:

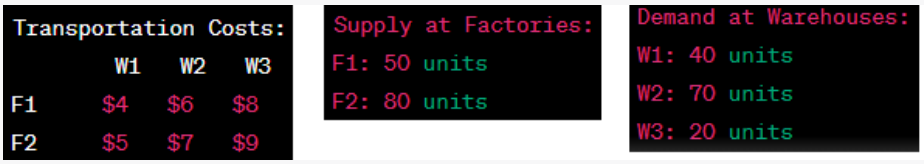


Figure 3.6 The illustration of the transportation costs, supply available at each factory and the demand at each warehouse

The corresponding linear model is:

$Minimize\ Z = \sum_{i=1}^{2}\sum_{j=1}^{3}(c_{ij}x_{ij})$

Subject to the following constraints:

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i, & i = \overline{1,2} \\ \sum_{i=1}^{m} x_{ij} = b_j, & j = \overline{1,3} \\ x_{ij}, & i = \overline{1,2},\quad j = \overline{1,3} \end{cases}$$

Step 1: Start with a Basic Feasible Solution: Begin with an initial feasible solution obtained using the North-West Corner Method or any other heuristic. For this example, we'll use the allocations obtained from the North-West Corner Method:

Allocations				
Iteration 1	W1	W2	W3	Supply
F1	-	-	-	50
F2	-	-	-	80
Demand	40	70	20	

Allocations				
Iteration 2	W1	W2	W3	Supply
F1	***40***	-	-	10
F2	-	-	-	80
Demand	0	70	20	

Allocations				
Iteration 3	W1	W2	W3	Supply
F1	***40***	**10**	-	0
F2	-	-	-	80
Demand	0	60	20	

Allocations				
Iteration 4	W1	W2	W3	Supply
F1	***40***	**10**	-	0
F2	-	**60**	-	20
Demand	0	0	20	

Allocations				
Iteration 5	W1	W2	W3	Supply
F1	***40***	**10**	-	0
F2	-	**60**	**20**	0
Demand	0	0	0	

The total transport cost using the North-West Corner Method:

$$Z = \sum_{i=1}^{2}\sum_{j=1}^{3}(c_{ij}x_{ij}) = (c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13}) + (c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23})$$

$$Z = (4 * 40 + 6 * 10 + 8 * 0) + (5 * 0 + 7 * 60 + 9 * 20) = 220 + 600 = \$820$$

So, the initial basic feasible solution: $Z = \$820$

Step 2: Identify Empty Cells (Non-Basic Variables): Identify the empty cells in the transportation matrix ⇒ (F2, W1). The cell $(F2, W1)$, non basic with cost 5, is empty so allocate one unit to it. Now draw a closed path from AD. The result of allocating one unit along with the necessary adjustments in the adjacent cells is indicated in *Transpotation Matrix 2*.

Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

Allocations				
Transportation Matrix 1	W1	W2	W3	Supply
F1	***40*** [4]	**10** [6]	***0*** [8]	50
F2	***0*** [5]	***60*** [7]	***20*** [9]	80
Demand	40	70	20	
net cost change				
$Z = 40 * 4 + 10 * 6 + 60 * 7 + 20 * 9 = \$820$				

Allocations				
Transportation Matrix 2	W1	W2	W3	Supply
F1	- ***39*** [4]	+ ***11*** [6]	***0*** [8]	50
F2	+ ***1*** [5]	- ***59*** [7]	***20*** [9]	80
Demand	40	70	20	-
net cost change	+5-4+6-7=0		*Neither increase nor decrease*	
$Z = 1 * 5 + 39 * 4 + 11 * 6 + 59 * 7 + 9 * 20 = \$820$				

Allocations				
Transportation Matrix 2	W1	W2	W3	Supply
F1	+ ***40*** [4]	- ***10*** [6]	+ ***1*** [8]	50
F2	- ***0*** [5]	+ ***60*** [7]	- ***19*** [9]	80
Demand	40	70	20	-
net cost change	+8-6+7-9=0		*Neither increase nor decrease*	
net cost change	+8-9+7-5+4-6=-1		*Cost decrease*	
$Z = 1 * 8 + 19 * 9 + 60 * 7 + 40 * 4 + 10 * 6 = \$819$				

We must evaluate all such unoccupied cells in this manner by finding closed paths and calculating the net cost change as shown below.

Since all the values of unoccupied cells are greater than or equal to zero, the solution obtained is **optimal**.

Minimum transportation cost is: $Z = \$819$

✍ *The Stepping Stone Method is an essential part of finding the optimal solution for the Transportation Problem in linear programming. When combined with other methods like MODI, it allows for iterative improvement of the initial feasible solution, ultimately leading to the optimal allocation of goods and the minimum total transportation cost.*

The Modified Distribution (MODI) Method

The Modified Distribution (MODI) Method is an iterative optimization technique used to check optimality of the basic feasible solution for the Transportation Problem and finding the optimal solution. It improves upon the initial feasible solution obtained from methods like the North-West Corner Method or the Vogel's Approximation Method.

The MODI Method involves the following steps:

Step 1: Identify the Initial Feasible Solution: Start with an initial feasible solution obtained from the North-West Corner Method or other heuristics.

Step 2: Calculate Opportunity Costs: For each empty cell (i.e., a cell not allocated in the initial solution), calculate the opportunity cost. The opportunity cost represents how much the transportation cost of that cell would have to decrease per unit to make it worthwhile to allocate goods to that cell instead of the current allocation.

Step 3: Identify the Optimal: Cell Identify the cell with the highest opportunity cost. This cell represents the best candidate for reallocation to potentially reduce the total transportation cost.

Step 4: Apply the Stepping Stone Method: Once the optimal cell is identified, apply the Stepping Stone Method to find a new feasible solution by reallocating units through a closed loop that starts and ends at the optimal cell.

Step 5: Recalculate Opportunity: Costs After obtaining the new feasible solution, recalculate the opportunity costs for all empty cells.

Step 6: Repeat the Process: Continue the process by identifying the cell with the highest opportunity cost, applying the Stepping Stone Method, and recalculating opportunity costs until no negative opportunity costs exist. This indicates that the current solution is optimal and no further improvements can be made.

Advantages of the Modified Distribution (MODI) Method:

Guaranteed Optimality: The MODI Method guarantees finding the optimal solution for the Transportation Problem, provided there are no degenerate solutions (multiple cells with zero allocations in the initial feasible solution). It systematically improves the initial solution until the optimal solution is achieved.

Iterative Improvement: MODI is an iterative method that continuously improves the current solution by reallocating units between cells. It examines the opportunity costs of unallocated cells and

identifies the best reallocation paths through the Stepping Stone Method. This iterative nature allows for fine-tuning the solution for better cost efficiency.

Reduces Transportation Costs: By reallocating goods based on opportunity costs, MODI aims to minimize the total transportation cost. It provides a systematic approach to finding the most cost-efficient allocation of goods between sources and destinations, leading to potential cost savings for transportation operations.

Efficient Handling of Large Problems: MODI can efficiently handle large and complex Transportation Problems with multiple sources, destinations, and transportation costs. Its step-by-step approach allows for incremental improvements without recalculating the entire problem from scratch.

Flexibility: MODI can be used in conjunction with other optimization methods, such as the North-West Corner Method or Vogel's Approximation Method, to generate an initial feasible solution. It can then take that solution and refine it through the Stepping Stone Method to obtain the optimal solution.

Limitations of the Modified Distribution (MODI) Method:

Sensitivity to Initial Solution: The efficiency and success of the MODI Method can be sensitive to the quality of the initial feasible solution. If the initial solution is far from optimal, it may require a higher number of iterations to converge to the optimal solution.

Computationally Intensive: For large-scale problems, the MODI Method can be computationally intensive and time-consuming. The number of iterations required to reach the optimal solution may increase significantly with problem size.

Degeneracy Issues: The presence of degenerate solutions (multiple cells with zero allocations) can complicate the process and make it difficult to converge to the optimal solution. In some cases, special handling may be required to deal with degeneracy.

Local Optima: While MODI aims to find the global optimal solution, there is a possibility of getting stuck in a local optimum, particularly if the initial solution is suboptimal. Additional strategies may be needed to ensure exploration of different solution spaces.

Example 3.5.d Let's illustrate the Stepping Stone Method with a simple example of the Transportation Problem: A company has two factories (F1 and F2) producing goods and three warehouses (W1, W2, and W3) that need to receive these goods. The transportation costs (in dollars per unit) from each factory to each warehouse are as follows:

Transportation Costs:	W1	W2	W3
F1	$4	$6	$8
F2	$5	$7	$9

Supply at Factories:
F1: 50 units
F2: 80 units

Demand at Warehouses:
W1: 40 units
W2: 70 units
W3: 20 units

The corresponding linear model is:

$Minimize\ Z\ = \sum_{i=1}^{2}\sum_{j=1}^{3}(c_{ij}x_{ij})$

Subject to the following constraints:

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i\,, & i = \overline{1,2} \\ \sum_{i=1}^{m} x_{ij} = b_j\,, & j = \overline{1,3} \\ x_{ij}, & i = \overline{1,2},\quad j = \overline{1,3} \end{cases}$$

Step 1: Start with a Basic Feasible Solution:

Allocations				
Transportation Matrix 2	W1	W2	W3	Supply
F1	***40*** [4]	***10*** [6]	***1*** [8]	50
F2	[5]	***60*** [7]	***19*** [9]	80
Demand	40	70	20	-
$Z = 40 * 4 + 10 * 6 + 1 * 8 + 60 * 7 + 19 * 9 = \$819$				

Step 2: Introduce dual variables (u & v) corresponding to supply and demand constraints: Dual variables u_i and v_j are such that, $u_i + v_j = c_{ij}$(where $i = \overline{1,2}$ & $j = \overline{1,3}$) and correspond to filled cells.

$u_i \setminus v_j$	v_1	v_2	v_3
u_1	**4**	**6**	**8**
u_2		**7**	**9**

As per the note above, we can derive equation as follows:

$$\begin{cases} u_1 + v_1 = 4 \\ u_1 + v_2 = 6 \\ u_1 + v_3 = 8 \\ u_2 + v_2 = 7 \\ u_2 + v_3 = 9 \end{cases} \overset{\text{we solve the system}}{\Rightarrow} \begin{cases} u_1 = 8 - t \\ u_2 = 9 - t \\ v_1 = -4 + t \\ v_2 = -2 + t \\ v_3 = t,\ \ t \in R \end{cases} \overset{If\ t=5}{\Rightarrow} \begin{cases} u_1 = 3 \\ u_2 = 4 \\ v_1 = 1 \\ v_2 = 3 \\ v_3 = 5 \end{cases}$$

Step 3: Carry out Vacant Cell Evaluation: Vacant cell evaluation is carried out by taking difference for each unoccupied call. Evaluation of cell: $= c_{ij} - (u_i + v_j)$, where c_{ij} is value of unoccupied cell. Remember that $(u_i + v_j)$, is known as *Implicit Cost*. *Implicit Cost* is the summation of dual variables of row and columns for each unoccupied cell.

Vacant cells are to be filled using above formula as follows:

$u_i \setminus v_j$	1	3	5
3			
4	**5-(4+1) = 0**		

In this method after performing vacant cell evaluation, if the values of the cells are/is :

(i) Positive (+) → then the current basic feasible solution (BFS) is optimal and stop the procedure. If we alter the present solution, the solution will become poorer.

(ii) Zero (0) → then also the current basic feasible solution (BFS) is optimal, but alternate allocation are possible with same transportation cost.

(iii) Negative (-) → then the obtained solution is not optimal and further improvement is possible.

As all the values in the non-allocated cells are now obtained zero (0) by using vacant cell evaluation, the present solution is optimal.

✍ *Despite these limitations, the Modified Distribution (MODI) Method remains a powerful and widely used technique for finding the optimal solution in Transportation Problems. Its ability to iteratively refine the initial solution makes it an essential tool for transportation planning and logistics optimization.*

The MODI Method is widely used in transportation planning and logistics to find the most cost-efficient allocation of goods from sources to destinations, minimizing overall transportation costs while meeting supply and demand constraints. It guarantees an optimal solution for the Transportation Problem when there are no degenerate solutions (a situation where more than one cell in the initial feasible solution has a value of zero).

△ *The Transport Problem has practical applications in various industries, such as supply chain management, logistics, and distribution, where it helps companies optimize their transportation routes and minimize transportation costs while meeting customer demands*.

Questions you should be able to answer after studying the topic of ”LP Applications”

What is LP (Linear Programming), and what are its main components?

What are the key steps involved in formulating an LP problem?

What are the applications of LP in different sectors or industries, such as economics, finance, supply chain management, etc.?

How can LP be used for resource allocation and production planning?

What role does LP play in financial planning and portfolio optimization?

How is LP applied in marketing and sales strategies?

What are some examples of LP applications in agriculture and energy planning?

How is LP used in transportation planning and traffic optimization?

What are the benefits of using LP in public sector resource management?

How does LP help in labor scheduling and workforce planning?

Can you explain the concept of duality in LP and its significance?

What are some limitations or challenges associated with using LP in real-world scenarios?

How can LP contribute to decision-making processes and optimization in various economic and business contexts?

What are some real-world examples of LP applications that have resulted in significant improvements or cost savings?

How do LP models and solutions compare to other optimization techniques in terms of efficiency and effectiveness?

EXERCISES

3.5.1 In each exercise, fill in the missing words with appropriate terms such as "profit," "impact," "return," "cost," "transportation cost," "objective function value," etc., based on the context of the LP application. These exercises aim to reinforce your understanding of LP applications in various economic mathematics scenarios:

a) A company wants to maximize its profit by producing and selling two products, A and B. The profit per unit of product A is $10, and the profit per unit of product B is $15. The company has limited production capacity, and it can produce up to 100 units of product A and up to 150 units of product B. The production time for product A is 2 hours per unit, and for product B is 3 hours per unit. The total production time available is 300 hours. The LP problem aims to _____ the production quantities of products A and B to maximize the total profit.

b) A farmer has a total of 200 acres of land available for planting two crops, corn, and wheat. The profit per acre from corn is $300, and from wheat is $200. The farmer's objective is to _____ the crop allocation to maximize the total profit. The number of acres allocated to corn and wheat should not exceed the total available land.

c) A retail store wants to determine the optimal mix of two products, X and Y, to include in a promotion bundle. The profit per unit of product X is $8, and the profit per unit of product Y is $10. The store has a budget of $500 to spend on the promotion. The LP problem aims to _____ the quantities of products X and Y in the bundle to maximize the total profit within the budget constraint.

d) A transportation company wants to determine the optimal route for its delivery trucks to minimize transportation costs. The LP problem aims to _____ the delivery quantities to different destinations to minimize the total transportation cost, considering distance and vehicle capacity constraints.

e) A manufacturing company wants to plan its production schedule for the next month. The company has multiple production lines, and each product requires a specific amount of machine time on each line. The objective of the LP problem is to _____ the production quantities of different products to meet the demand while minimizing the total machine time used.

f) Resource Allocation A non-profit organization wants to allocate its available resources efficiently to provide services to the community. The organization has a limited budget and needs to allocate it across different programs. Each program has specific costs and impact levels on the community. The organization aims to find the allocation that maximizes its community _____ while staying within the budget constraints.

g) Portfolio Optimization An investor wants to create an optimal investment portfolio with different assets (A1, A2, A3, etc.). Each asset has an expected return and risk associated with it. The investor wants to find the optimal allocation of funds to each asset to maximize the portfolio's expected _____ while minimizing the portfolio's risk.

h) Transportation Problem A logistics company needs to transport goods from multiple sources to multiple destinations. Each source has a specific supply of goods, and each destination has a specific demand. The company wants to find the optimal transportation plan that minimizes the total _____ while satisfying the supply and demand constraints.

i) Blending Problem A chemical company wants to produce a final product by blending different raw materials. Each raw material has specific characteristics and costs. The company needs to find the optimal blend of raw materials that meets the product's specifications while minimizing the total _____.

j) Project Scheduling A construction company wants to schedule different construction projects to maximize its total _____ while considering project durations, resource requirements, and project dependencies. The company needs to find the optimal project schedule that maximizes its overall profitability.

3.5.2 For each exercise, show the step-by-step solution process using the North-West Corner Method and the Modified Distribution Method. Perform the optimality test to ensure the obtained solution is optimal. Additionally, explore the concept of sensitivity analysis and its application in the context of the Transportation Problem:

a) Transportation Problem Formulation: A company needs to transport goods from three factories (F1, F2, F3) to three distribution centers (D1, D2, D3). The transportation costs per unit are given in the following table:

	D1	D2	D3	Supply
F1	3	5	7	***100***
F2	2	6	4	***150***
F3	5	8	3	***200***
Demand	***120***	***100***	***180***	

a.1) Formulate the transportation problem to minimize the total transportation cost. *a.2)* Set up the initial tableau for the North-West Corner Method.

b) North-West Corner Method: Use the North-West Corner Method to find an initial basic feasible solution for the transportation problem in Exercise 1.

c) Modified Distribution Method: Use the Modified Distribution Method (also known as the Vogel's Approximation Method) to improve the initial basic feasible solution obtained in Exercise 2.

d) Optimality Test: Using the initial basic feasible solution obtained in Exercise C, perform the optimality test to check if the solution is optimal, using the Stepping Stone Method..

e) Sensitivity Analysis: Suppose the transportation costs from F2 to D1 change from 2 to 3. Recalculate the optimal solution using the Modified Distribution Method and analyze how the changes in transportation costs affect the total transportation cost and the allocation of goods.

3.5.3 Minimize the total transportation cost by formulating the transportation problem. Use the North-West Corner Method to set up the initial tableau and find the initial basic feasible solution. Improve the initial basic feasible solution using the Modified Distribution Method. Perform the optimality test on the obtained solution using the Stepping Stone Method to check for optimality:

a) Transport Problem - Goods Distribution A company has three factories (F1, F2, F3) and three distribution centers (D1, D2, D3). The transportation costs per unit are given in the following table:

	D1	D2	D3	Supply
F1	4	6	7	100
F2	5	3	4	150
F3	6	8	5	200
Demand	120	100	180	

b) Transport Problem - Goods Allocation A company has four factories (F1, F2, F3, F4) and four distribution centers (D1, D2, D3, D4). The transportation costs per unit are given in the following table:

	D1	**D2**	**D3**	**D4**	**Supply**
F1	4	6	3	8	100
F2	7	5	2	9	150
F3	6	8	4	5	200
F4	3	2	7	6	120
Demand	90	70	110	100	

c) Transport Problem - Supply and Demand A company has two factories (F1, F2) and three distribution centers (D1, D2, D3). The transportation costs per unit are given in the following table:

	D1	D2	D3	Supply
F1	2	4	7	70
F2	5	3	2	100
Demand	50	80	40	

d) Transport Problem - Multiple Suppliers A company needs to transport goods from three suppliers (S1, S2, S3) to two distribution centers (D1, D2). The transportation costs per unit are given in the following table:

	D1	D2	Supply
S1	3	5	50
S2	6	4	70
S3	2	7	90
Demand	80	130	

3.5.4 For each exercise, show the step-by-step solution process using the North-West Corner Method and the Modified Distribution Method. Perform the optimality test to ensure the obtained solution is optimal. Additionally, explore the concept of sensitivity analysis and its application in the context of the Transportation Problem

a) Transportation Problem Formulation: A company needs to transport goods from three supply sources (S1, S2, S3) to three demand destinations (D1, D2, D3). The transportation costs per unit are given in the following table:

	D1	D2	D3	Supply
S1	2	3	7	100

	D1	D2	D3	Supply
S2	6	5	8	150
S3	9	4	6	200
Demand	120	100	180	

a.1) Formulate the transportation problem to minimize the total transportation cost. *a.2)* Set up the initial tableau for the North-West Corner Method.

b) North-West Corner Method: Use the North-West Corner Method to find an initial basic feasible solution for the transportation problem in *exercise a*.

c) Modified Costs Method: Use the Modified Costs Method to improve the initial basic feasible solution obtained in *exercise b*.

d) Optimality Test: Using the initial basic feasible solution obtained in *exercise c*, perform the optimality test to check if the solution is optimal, using the Stepping Stone Method.

e) Sensitivity Analysis: Suppose the transportation costs from S2 to D3 change from 8 to 6. Recalculate the optimal solution using the Modified Costs Method and analyze how the changes in transportation costs affect the total transportation cost and the allocation of goods.

3.5.5 Minimize the total transportation cost by formulating the transportation problem. Use the North-West Corner Method to set up the initial tableau and find the initial basic feasible solution. Improve the initial basic feasible solution using the Modified Costs Method. Perform the optimality test on the obtained solution using the Stepping Stone Method to check for optimality:

a) Consider the following transportation problem:

	D1	D2	D3	Supply
S1	2	3	7	100
S2	6	5	8	150
S3	9	4	6	200
Demand	120	100	180	

b) Consider the following transportation problem with modified transportation costs:

	D1	D2	D3	Supply
S1	3	5	2	150
S2	4	6	1	200
S3	2	4	3	100
Demand	120	180	150	

c) Consider the following transportation problem with unbalanced supply and demand:

	D1	D2	D3	D4	Supply
S1	4	1	5	7	120
S2	6	3	8	2	180
S3	2	5	6	3	90
Demand	130	100	150	110	

d) Consider the following transportation problem with mixed constraints:

	D1	D2	D3	Supply
S1	5	2	6	100
S2	4	5	8	150
S3	3	6	4	200
Demand	120	100	180	

3.6 LP EXTENSIONS AND VARIATIONS. SOFTWARE TOOLS FOR LP

Keywords: *LP Extensions, Linear Programming Variations, Integer Linear Programming (ILP), Mixed-Integer Linear Programming (MILP), Binary Linear Programming, Nonlinear Programming, Quadratic Programming, Convex Optimization, Combinatorial Optimization, LP Software Tools, LP Solvers, LP Modeling Software, Optimization Software, LP Algorithms, LP Formulations, LP Sensitivity Analysis, LP Duality, LP Relaxation, LP Integer Cuts, LP Branch and Bound, LP Branch and Cut, LP Interior-Point Methods, LP Simplex Method, LP Sensitivity Report, LP Feasibility, LP Infeasibility, LP Unboundedness, LP Objective Function, LP Constraints, LP Bounds, LP Optimality, LP Modeling Language, LP Modeling Environment, LP Solver Libraries.*

LP Extensions and Variations and Software Tools for LP in economics refers to the application of advanced techniques and tools to enhance the capabilities of Linear Programming (LP) and address complex economic problems. LP is a powerful mathematical optimization method used to find the best outcome in a linear objective function subject to linear constraints. In economics, LP has widespread applications in areas such as resource allocation, production planning, transportation optimization, and supply chain management. Key topics covered in this context include:

Integer Linear Programming (ILP): Extends LP to handle decision variables that must take on integer values, allowing for more realistic modeling in economic scenarios with discrete choices.

Mixed-Integer Linear Programming (MILP): Combines integer and continuous decision variables, making it suitable for modeling complex economic problems that involve both discrete and continuous choices.

Nonlinear Programming (NLP): Expands LP to handle non-linear objective functions and constraints, enabling the optimization of non-linear economic models.

Quadratic Programming (QP): Focuses on optimizing quadratic objective functions, widely applicable in economic models involving quadratic relationships between variables.

Convex Optimization: Deals with optimizing convex functions, which are relevant in various economic contexts due to their desirable mathematical properties.

Multi-Objective Linear Programming (MOLP): Addresses optimization problems with multiple conflicting objectives, allowing for trade-offs and Pareto-optimal solutions.

Stochastic Linear Programming: Accounts for uncertainty in economic models by considering probabilistic parameters and optimizing over various scenarios.

Robust Optimization: Addresses optimization under uncertainty by considering a set of possible scenarios and ensuring the solution's stability across all scenarios.

Sensitivity Analysis: Evaluates the impact of changes in input parameters on the optimal solution, providing valuable insights into the economic model's robustness.

Duality Theory: Explores the relationship between the primal and dual formulations of LP problems, shedding light on the economic interpretation of LP solutions.

Optimization Software Tools: Various specialized software tools, such as CPLEX, Gurobi, and Excel Solver, are utilized to efficiently solve large-scale and complex LP problems in economics.

Network Optimization: Focuses on optimizing flows through networks, relevant in economic applications like transportation and communication systems.

Supply Chain Optimization: Utilizes LP techniques to optimize supply chain processes, distribution, and inventory management, leading to cost reduction and efficiency improvements.

Production Planning: Applies LP in optimizing production schedules, capacity allocation, and resource utilization in manufacturing and industrial settings.

Extensions of Linear Programming (LP): One extension of Linear Programming (LP) in economics is the application of Mixed-Integer Linear Programming (MILP) to address problems with both continuous and discrete decision variables. MILP allows for more realistic modeling of economic scenarios that involve discrete choices, such as selecting among different investment projects, product options, or production quantities.

Example 3.6.a: Capital Budgeting Decision: Consider a company that wants to optimize its capital budgeting decisions to select the best investment projects within a limited budget. The company has a set of potential investment projects with different costs, expected returns, and project lifetimes. The objective is to maximize the total expected return while adhering to the budget constraint.

Variables: Let x_ii be a binary decision variable that takes the value 1 if project *i* is selected, and *0* otherwise.

Objective Function: $Maximize\ Z\ =\ \Sigma\ (Return_i\ *\ x_i)$

Constraints:

Budget Constraint: $\Sigma\ (Cost_i\ *\ x_i)\ \leq\ Budget$

Discrete Decision: $x_i\ \in\ \{0, 1\}$ for all projects *i* (indicating whether the project is selected or not).

Here, the objective function aims to maximize the total expected return from the selected investment projects. The budget constraint ensures that the total cost of selected projects does not exceed the available budget. The discrete decision variables (x_i) ensure that only one investment project is selected or not selected.

To solve this MILP problem, specialized optimization solvers like CPLEX or Gurobi can be used. These solvers efficiently handle both continuous and discrete decision variables, exploring the solution space to find the optimal combination of projects that maximizes the expected return while respecting the budget constraint.

✍ *MILP extensions in economics are valuable for decision-making processes involving choices with limited resources, budget constraints, and discrete alternatives, making them applicable in various fields, including financial planning, project selection, portfolio optimization, and resource allocation.*

Variations of Linear Programming (LP): One variation of Linear Programming (LP) that is commonly used in economics is Multi-Period LP or Dynamic Linear Programming. This variation

extends the traditional LP model to consider multiple time periods, allowing for more comprehensive and realistic economic planning and decision-making over time.

Example 3.6.b: Investment Planning in a Manufacturing Company: Suppose a manufacturing company is planning its production and investment strategy for the next three years. The company produces two types of products, A and B, and aims to maximize its profits while considering production capacity constraints and demand forecasts for each product.

Objective: Maximize the total profit over three years.

Decision Variables:

X_1, X_2, and X_3: the quantities of product *A* to produce in years 1, 2, and 3, respectively,

Y_1, Y, and Y_3: the quantities of product *B* to produce in years 1, 2, and 3, respectively,

I_1, I_2, and I_3: the investments in production capacity for years 1, 2, and 3, respectively.

Constraints:

Production Capacity: The production of each product cannot exceed the available production capacity in each year. For each year ($t = 1, 2, 3$):

$$\begin{cases} X_t + Y_t \leq Capacity_A \\ 2 * X_t + Y_t \leq Capacity_B \end{cases}$$

Demand Constraints: The demand for each product must be met in each year. For each year ($t = 1, 2, 3$):

$$\begin{cases} X_t \geq Demand_A \\ Y_t \geq Demand_B \end{cases}$$

Investment Budget Constraints: The total investment in each year cannot exceed the available budget for investments. For each year ($t = 1, 2, 3$):

$$I_t \leq Budget_t$$

Non-Negativity Constraints:

$X_t, Y_t, I_t \geq 0$ for all $t = 1, 2, 3$.

Objective Function:

$$Maximize\ Z = Profit_{1X_1} + Profit_{2X_2} + Profit_{3X_3} + Profit_{1Y_1} + Profit_{2Y_2} + Profit_{3Y_3}$$

Where $Profit_{1X_1}$, $Profit_{2X_2}$, and $Profit_{3X_3}$ are the profit margins for product *A* in years 1, 2, and 3, respectively, and $Profit_{1Y_1}$, $Profit_{2Y_2}$, and $Profit_{3Y_3}$ are the profit margins for product *B* in years 1, 2, and 3, respectively.

✍ *By formulating this problem as a Multi-Period LP, the manufacturing company can make informed decisions on how much to produce and invest in each year, considering the changing market demands, production capacities, and investment budgets over time. The optimization process will result in a production and investment plan that maximizes the company's profits throughout the three-year planning horizon.*

Software tools used for Linear Programming (LP): One example of a software tool used for Linear Programming (LP) in economics is IBM ILOG CPLEX Optimization Studio. CPLEX is a widely used optimization software suite that provides powerful tools for solving large-scale LP and Mixed-Integer Linear Programming (MILP) problems in various economic applications.

Features of IBM ILOG CPLEX Optimization Studio:

- Linear and Nonlinear Optimization: CPLEX can handle both linear and nonlinear optimization problems, allowing economists to model a wide range of economic scenarios.
- Integer and Mixed-Integer Programming: It supports solving Integer Linear Programming (ILP) and Mixed-Integer Linear Programming (MILP) problems, making it suitable for discrete decision-making in economic contexts.
- Convex Optimization: CPLEX efficiently handles convex optimization problems, which are common in economic models due to their convexity properties.
- Multi-Objective Optimization: CPLEX supports multi-objective optimization, enabling economists to explore trade-offs between conflicting objectives and find Pareto-optimal solutions.
- Network Optimization: It includes specialized algorithms for solving network optimization problems, such as transportation and flow optimization, which are essential in economic applications.
- Sensitivity Analysis: CPLEX provides sensitivity analysis tools that allow users to examine how changes in problem parameters affect the optimal solution.
- Stochastic Optimization: It supports modeling and solving stochastic optimization problems, which are essential in addressing uncertainty in economic decision-making.
- High Performance: CPLEX is designed for high-performance computing, capable of handling large-scale optimization problems with thousands of variables and constraints.
- Integration with Programming Languages: CPLEX can be integrated into various programming languages like C++, Java, Python, and MATLAB, allowing for seamless integration with existing economic modeling and simulation frameworks.

Example 3.6.c: A company wants to optimize its production and distribution process to minimize costs. They need to decide the optimal production levels at different plants and the distribution plan to various warehouses while considering demand and transportation costs.

By formulating the problem as a linear programming model and using IBM ILOG CPLEX Optimization Studio, the company can find the optimal production and distribution plan that minimizes overall costs while satisfying production capacity and demand constraints. CPLEX efficiently solves the large-scale optimization problem, providing the company with a cost-effective and efficient production and distribution strategy.

✍ *In this example, CPLEX demonstrates its capability to handle complex economic optimization problems, making it a valuable tool for decision-makers in various economic sectors.*

△ *Overall, the study of LP Extensions and Variations, along with the use of sophisticated software tools, empowers economists and decision-makers to tackle intricate and dynamic economic problems effectively and make informed decisions in various sectors.*

Questions you should be able to answer after studying the topic of ”LP Extensions and Variations. Software Tools for LP”

What are some common extensions of linear programming (LP) in economics, and how are they applied to real-world economic problems?

How does Integer Linear Programming (ILP) differ from standard LP, and what are its applications in economics?

What is Mixed-Integer Linear Programming (MILP), and how can it be used to model complex economic optimization problems?

How does Nonlinear Programming (NLP) differ from linear programming, and what are some examples of its applications in economics?

What is the significance of Quadratic Programming (QP) in economic modeling, and where is it commonly used?

How can Convex Optimization techniques be applied to economic decision-making and resource allocation?

What are some applications of Multi-Objective Linear Programming (MOLP) in economics, and how can it help in decision-making under multiple conflicting objectives?

How does Stochastic Linear Programming address uncertainty in economic models, and when is it appropriate to use in real-world scenarios?

What is Robust Optimization, and how can it handle uncertainties and variations in economic systems?

How does Sensitivity Analysis help in understanding the impact of changes in input parameters on the optimal solution in economic models?

What is Duality Theory in linear programming, and what insights does it provide in the context of economic optimization problems?

How do Decomposition Techniques and Lagrangian Relaxation help in solving large-scale economic optimization problems more efficiently?

What are some metaheuristic algorithms (e.g., Genetic Algorithms, Simulated Annealing) used in economic optimization, and what are their advantages and limitations?

How does the Branch and Bound algorithm help in solving Integer Linear Programming (ILP) problems optimally?

What are Interior-Point Methods, and how do they contribute to efficiently solving convex optimization problems in economics?

How do Cutting Plane Algorithms enhance the efficiency of solving LP problems with a large number of constraints?

What are the applications of network optimization techniques in various economic sectors, such as transportation and communication networks?

How can LP extensions and software tools be used in optimizing supply chains, production planning, and resource allocation in economic systems?

What role does Linear Programming play in portfolio optimization and revenue management in finance and business?

What are some commonly used LP software tools and optimization libraries (e.g., CPLEX, Gurobi) in economics, and how can they be integrated into decision support systems?

EXERCISES

3.6.1 Fill in the blanks with the appropriate words to complete each sentence related to "LP Extensions and Variations" and "Software Tools for LP" in the context of economic mathematics. These exercises will reinforce your understanding of LP concepts and their applications in economics:

a) LP Extensions and Variations Consider the following LP problem:

Maximize: $5x + 3y$. Subject to: $2x + y \leq 10,\ x + 3y \leq 15,\ x, y \geq 0$

a.1) This is an example of a _____ Linear Programming problem, where the objective is to maximize the profit (or value) of a particular quantity. a.*2)* The constraints $2x + y \leq 10$ and $x + 3y \leq 15$ represent _____ constraints, which restrict the availability of resources for the

production of goods. *a.3)* The LP problem contains _____ variables, representing the quantities of two goods that need to be determined to optimize the objective function. *a.4)* The LP problem can be solved using various LP techniques, such as the Simplex Method or the _____ Method.

b) Software Tools for LP In modern economic mathematics, various software tools are available to solve LP problems efficiently. Some popular LP software tools include Excel Solver, GNU Linear Programming Kit (GLPK), and LINGO.

b.1) Excel Solver is a widely used spreadsheet-based tool that can be used to find optimal solutions for LP problems. It is part of _____, which is commonly used for data analysis and optimization. *b.2)* GNU Linear Programming Kit (GLPK) is an open-source software package that provides _____ tools for solving large-scale linear programming problems. *b.3)* LINGO is a software tool specifically designed for optimization problems, including Linear Programming. It provides an easy-to-use interface and allows users to model complex _____ programming problems effectively.

c) LP Extensions LP can be extended to address more complex real-world scenarios. Some common LP extensions include Integer Linear Programming (ILP), Mixed-Integer Linear Programming (MILP), and _____ Linear Programming (QP).

c.1) Integer Linear Programming (ILP) allows the decision variables to take only _____ values, making it suitable for problems that require integer solutions. *c.2)* Mixed-Integer Linear Programming (MILP) is a combination of LP and ILP, where some variables are restricted to integer values while others can take _____ values. *c.3)* Quadratic Linear Programming (QP) extends LP to problems with a quadratic _____ function and linear constraints, commonly found in optimization problems with squared variables.

d) LP in Economic Applications Linear Programming finds extensive applications in economics, such as production planning, resource allocation, portfolio optimization, and transportation. LP helps optimize decision-making processes, improve efficiency, and maximize _____ in various economic activities.

3.6.2 For each exercise, fill in the blanks with the appropriate words to complete the sentences or the missing information in the given problem. These exercises cover LP extensions, software tools, applications, and sensitivity analysis to further enhance your understanding of linear programming in economic mathematics:

a) LP Extensions and Variations: *a.1)* In Integer Linear Programming (ILP), the decision variables are required to take on _____ values. *a.2)* In Binary Integer Linear Programming (BILP), the decision variables are restricted to be either _____ or _____. *a.3)* In Mixed-Integer Linear Programming (MILP), some decision variables are _____ and others are _____. *a.4)* Quadratic Programming (QP) involves an objective function that is _____ and constraints that are _____. *a.4)* In Dynamic Programming, the problem is divided into _____ stages, and decisions are made sequentially.

b) LP Software Tools: *b.1)* _____ is a popular open-source LP solver that can be used to solve large-scale linear programming problems. *b.2)* _____ is a powerful commercial LP solver known for its efficiency and robustness. *b.3)* _____ is a software tool for modeling, solving, and analyzing optimization problems, including linear programming. *b.4)* _____ is a Python

library that provides various optimization tools, including linear and mixed-integer programming. e) _____ is an optimization modeling language that allows users to define and solve complex LP problems.

c) Transportation Problem with Multiple Constraints: Consider the following transportation problem with multiple constraints:

	D1	D2	D3	Supply
S1	5	2	6	100
S2	4	5	8	150
S3	3	6	4	200
Demand 1	120	100	180	
Demand 2	80	120	100	

c.1) Formulate the transportation problem mathematically with the objective of minimizing transportation costs. *c.2)* Identify the decision variables, objective function, and constraints.

d) LP Sensitivity Analysis: In LP, sensitivity analysis is used to assess the impact of changes in the problem's parameters on the optimal solution.

d.1) The shadow price of a constraint represents the change in the optimal objective value per unit increase in the constraint's _____. *d.2)* The reduced cost of a decision variable in a maximization problem represents the amount by which the objective function coefficient of that variable can be _____ before a different variable enters the solution.

e) LP Applications in Finance: Linear programming is widely used in finance for portfolio optimization, asset allocation, and risk management.

e.1) In portfolio optimization, LP can help investors find the _____ allocation of assets to maximize returns for a given level of risk. *e.2)* LP is utilized in _____ management to allocate resources efficiently and optimize spending decisions. *e.3)* LP can be used to construct an _____ frontier that shows the optimal trade-off between risk and return for a portfolio.

3.6.3 These exercises integrate the concepts of planes in three-dimensional space with economic problems, such as linear programming, investment portfolio optimization, and resource utilization. Show all the steps involved in solving the exercises to demonstrate your understanding of how to apply planes in economic mathematics.

a) Plane Orientation and Resource Allocation A company has three resources: labor (L), raw materials (R), and capital (C). The company has the following resource constraints: $3L + 2R - C \leq 10$, $L + R + 2C \leq 15$: *a.1)* Write the two resource constraints as planes in three-dimensional space. *a.2)* Determine the feasible region for resource allocation. *a.3)* If the company wants to maximize its production output (P) subject to the resource constraints, formulate the objective function for profit maximization. *a.4)* Find the optimal resource allocation (L, R, C) to maximize the production output *P*.

b) Orientation of a Plane and Profit Maximization A manufacturing company produces two products, A and B. The production of A requires 2 units of labor and 3 units of raw materials, while the production of B requires 4 units of labor and 1 unit of raw materials. The company has 100 units

of labor and 60 units of raw materials available: *b.1)* Formulate the constraint equations for labor and raw materials as planes in three-dimensional space. *b.2)* Determine the region in the 3D space that represents feasible production levels of products A and B. *b.3)* If the profit per unit of A is $8 and for B is $6, find the objective function for profit maximization. *b.4)* Determine the optimal production levels of products A and B to maximize the profit, considering the feasible region

c) Linear Programming and Plane Orientation A company produces two products, X and Y. The profit per unit of product X is $10, and the profit per unit of product Y is $8. The company has two production departments, A and B. Department A can produce 4 units of X and 3 units of Y per hour, while Department B can produce 2 units of X and 5 units of Y per hour. The company has a total of 40 hours available in Department A and 30 hours available in Department B: *c.1)* Write the constraints for production levels of products X and Y as planes in three-dimensional space. *c.2)* Determine the feasible region for production levels of products X and Y. *c.3)* Formulate the objective function for profit maximization. *c.4)* Find the optimal production levels of products X and Y to maximize the profit, considering the feasible region.

d) Plane Intersection and Investment Portfolio An investor allocates their money into three investment options: stocks (S), bonds (B), and real estate (R). The expected annual return for stocks is 12%, for bonds is 8%, and for real estate is 10%. The investor wants to allocate their money in such a way that the expected return is at least 9% per year. Additionally, the investor wants to invest at least 30% of their money in stocks and at most 40% of their money in real estate: *d.1)* Write the constraints for the investment allocation as planes in three-dimensional space. *d.2)* Determine the feasible region for investment allocation. *d.3)* Formulate the objective function to maximize the expected return on investment. *d.4)* Find the optimal allocation of the investor's money into stocks, bonds, and real estate to maximize the expected return, considering the feasible region.

e) Plane Orientation and Resource Utilization A factory produces two products, P1 and P2, using three resources: labor (L), materials (M), and machines (Mach). The resource usage per unit of each product is as follows: Product P1: 2L + 3M + 1Mach = 10 Product P2: 4L + 2M + 3Mach = 15: *e.1)* The factory has a total of 100 units of labor, 60 units of materials, and 80 units of machine time available: Write the resource constraints as planes in three-dimensional space. *e.2)* Determine the feasible region for production of products P1 and P2. *e.3)* Formulate the objective function to maximize the total production output (P1 + P2). *e.4)* Find the optimal production levels of products P1 and P2 to maximize the total production output, considering the feasible region.

4 FUNCTIONS OF SEVERAL VARIABLES

CONTENT UNITS: In economics, the study of "Function of Several Variables" involves various content units that are relevant to economic analysis and decision-making. The key content units in this context include:

Multivariable Functions in Economics: Definition and representation of multivariable functions in economic models, Economic interpretation of independent and dependent variables, Multivariable functions in utility functions, production functions, and cost functions.

Partial Derivatives in Economic Analysis: Calculating partial derivatives to measure marginal changes in economic variables, Applications in elasticity analysis and marginal cost estimation, Interpretation of partial derivatives in economic contexts.

Optimization Problems in Economics: Maxima and minima of multivariable functions in economic optimization, Unconstrained and constrained optimization in economic decision-making, Applications in profit maximization, utility maximization, and resource allocation.

Economic Applications of Gradient and Directional Derivatives: Using gradients to identify the direction of steepest ascent and descent in economic models, Directional derivatives in sensitivity analysis and marginal utility estimation.

Economic Applications of Total Differential and Tangent Planes: Linear approximations and their relevance in economic forecasting, Tangent planes in marginal analysis and optimization of economic functions.

Chain Rule and Implicit Function Theorem in Economics: Chain rule for economic functions with multiple variables, Implicit function theorem and its applications in economic equilibrium analysis.

Economic Applications of Double and Triple Integrals: Double integrals in computing economic measures like consumer surplus and producer surplus, Triple integrals in economic applications such as volume calculation and production planning.

Optimization with Constraints in Economics: Lagrange multipliers in constrained optimization problems in economics, Applications in utility maximization subject to budget constraints and production optimization with resource limitations.

Applications in Economics and Econometrics: Economic modeling using multivariable functions and optimization techniques, Econometric estimation and analysis of economic models with multiple variables.

Vector Calculus in Economics: Gradient, divergence, and curl in the analysis of economic vector fields, Applications in studying economic flows, resource allocation, and economic dynamics.

Real-World Economic Case Studies: Applying multivariable functions and optimization methods to real-world economic problems, Case studies in areas such as market equilibrium, cost minimization, and optimal resource allocation.

✍ *These content units provide economists with the necessary tools to analyze complex economic relationships involving multiple variables and optimize economic outcomes efficiently. The study of functions of several variables is fundamental to understanding and addressing various economic challenges and opportunities in a rigorous and systematic manner.*

KEYWORDS: Multivariable Functions, Partial Derivatives, Gradient, Directional Derivatives, Total Differential, Tangent Planes, Chain Rule, Extrema, Local Extrema, Global Extrema, Critical Points, Second Derivative Test, Lagrange Multipliers, Line Integrals, Green's Theorem, Surface Integrals, Divergence Theorem, Gradient, Divergence, Constrained Optimization, Implicit Function Theorem, Taylor Series, Optimization in Economics, Utility Maximization, Cost Function Optimization, Real-World Applications in Economics.

✍*These keywords represent the core concepts and applications of functions of several variables in economics, covering optimization, derivatives, integrals, and their real-world implications in economic decision-making, modeling, and analysis.*

LEARNING OBJECTIVES:

Understanding Multivariable Functions: Students should be able to comprehend the concept of functions with multiple variables, including their domains, ranges, and graphical representation.

Analyzing Partial Derivatives: Students should be capable of computing partial derivatives of multivariable functions and interpreting their meanings in an economic context.

Exploring Optimization: Learners should be able to identify local and global extrema of multivariable functions, as well as apply the second derivative test for critical points.

Applying Chain Rule and Gradient: Students should be able to utilize the chain rule for multivariable functions and calculate gradients to analyze directional derivatives and optimize economic models.

Understanding Total Differential: Learners should grasp the concept of total differential and its significance in economic applications.

Applying Line Integrals and Green's Theorem: Students should be able to apply line integrals and Green's theorem in economic contexts, such as calculating work and flux.

Applying Vector Calculus: Students should be capable of using vector calculus concepts, including gradient, divergence, and curl, in economic models and real-world scenarios.

Exploring Optimization in Economics: Learners should analyze and solve optimization problems in economics using functions of several variables to maximize utility, profits, or production.

Analyzing Real-World Applications: Learners should be able to recognize and analyze real-world economic problems that involve functions of several variables and apply appropriate mathematical techniques for optimization.

Using Software Tools: Students should be proficient in using mathematical software tools (e.g., MATLAB, Python) to solve and analyze economic problems related to functions of several variables.

Demonstrating Critical Thinking: Learners should demonstrate critical thinking skills by connecting mathematical concepts to economic theories and applying them to solve complex real-world economic problems.

✍ *By achieving these learning objectives, students will gain a strong foundation in the study of functions of several variables and their application in various economic contexts, enabling them to analyze and optimize economic models effectively.*

SOME BASIC CONCEPTS THAT CAN BE CONNECTED TO ECONOMIC MATHEMATICS:

Utility Functions: In economics, utility functions are used to represent consumer preferences for different goods and services. These functions depend on several variables (goods or commodities), and their optimization is crucial in determining the consumer's optimal consumption bundle.

Production Functions: Economic production processes often involve multiple inputs (e.g., labor, capital, raw materials), and production functions describe the relationship between inputs and output. Analyzing production functions involves understanding partial derivatives and optimizing output with respect to different input combinations.

Cost Functions: Cost functions play a vital role in economics, as they represent the relationship between inputs and production costs. Finding the minimum cost for a given level of output involves optimization techniques in multivariable functions.

Profit Maximization: In microeconomics, firms aim to maximize their profits by optimizing production levels and cost structures. This involves working with multivariable functions and identifying critical points to determine the optimal production quantity.

Input Allocation: Allocating scarce resources efficiently is crucial in economic decision-making. Multivariable functions help determine the best allocation of resources to maximize output or utility, subject to constraints.

Elasticity: Elasticity measures the responsiveness of one variable to changes in another. In economic models involving multiple variables, elasticity calculations are essential for analyzing market behavior, pricing strategies, and consumer behavior.

Consumer Surplus and Producer Surplus: These economic concepts rely on integrating multivariable functions to find the area under a demand or supply curve, which represents the welfare gained by consumers or producers.

Portfolio Optimization: In finance, optimizing investment portfolios involves working with multivariable functions to maximize returns while managing risk and considering correlations between assets.

Multi-Objective Optimization: Economic decision-making often involves balancing multiple objectives, such as maximizing profits while minimizing environmental impact. Multi-objective optimization techniques are used to find Pareto-optimal solutions.

Input-Output Models: In macroeconomics, input-output models describe the relationships between different sectors of an economy. These models use multivariable functions to analyze the interdependencies between sectors and the effects of shocks or changes.

✍ *Overall, "Function of Several Variables" provides a robust mathematical foundation for economic analysis, allowing economists to model, optimize, and make informed decisions in complex real-world scenarios. Understanding these fundamentals is essential for addressing various economic challenges and exploring opportunities for economic growth and efficiency.*

4.0 INITIATION IN FUNCTION OF SEVERAL VARIABLES

Initiation in "Function of Several Variables" in Economic Mathematics involves understanding the fundamentals of multivariable functions and their applications in economic analysis. Here's a brief overview of the key concepts to get started:

Multivariable Functions: In Economic Mathematics, a multivariable function is a mathematical function that depends on two or more variables. For example, in microeconomics, utility

functions, production functions, and cost functions are essential multivariable functions used to model consumer behavior and firm production decisions.

Example 4.0.a Let's consider a simple example of a *production function* for a firm that produces two goods, *X* and *Y*. The production function can be represented as:

$$Q = f(L, K)$$

where: *Q* is the quantity of output (goods *X* and *Y*) produced, *L* is the quantity of labor input used, *K* is the quantity of capital input used, and $f(.)$ is the production function that relates the inputs (*L* and *K*) to the output *(Q)*.

✍ *The production function tells us how much output (goods X and Y) a firm can produce based on the amounts of labor and capital it employs. It is assumed that the production function is continuous and differentiable, allowing for the application of calculus techniques to analyze its properties.*

Using multivariable calculus, economists can compute partial derivatives of the production function with respect to labor (L) and capital (K) to determine how changes in each input affect output. For example, the partial derivative ∂Q/∂L represents the marginal product of labor, which indicates how much additional output is produced when one more unit of labor is employed, holding capital constant.

✍ *Analyzing the production function allows economists to make informed decisions about resource allocation, determining the optimal combination of labor and capital to achieve maximum output or profit. It is a fundamental concept in microeconomic theory and forms the basis for understanding firm behavior and production choices in various economic settings.*

Partial Derivatives: Partial derivatives represent the rate of change of a multivariable function concerning one variable while keeping others constant. In economic applications, partial derivatives are used to analyze how changes in variables impact economic outcomes.

Example 3.0.b One example of using partial derivatives in Economic Mathematics is in the analysis of production functions. Production functions represent the relationship between inputs (such as labor and capital) and output (goods or services) in the production process of a firm.

Suppose we have a Cobb-Douglas production function given by:

$$Q = A * L^{\alpha} * K^{\beta}$$

where: Q = Output (quantity of goods produced), L = Labor input, K = Capital input, A = Total factor productivity (a constant), α and β = Elasticities of output with respect to labor and capital, respectively.

In this production function, the exponents α and β determine the responsiveness of output to changes in labor and capital inputs. To understand how output changes concerning each input, we can use partial derivatives.

Partial Derivative with respect to Labor (L): $\partial Q/\partial L = \alpha * A * L^{\alpha-1} * K^{\beta}$

The partial derivative with respect to labor (∂Q/∂L) represents the rate of change of output concerning labor, while holding capital constant. It shows how much additional output (Q) is produced for a one-unit increase in labor (L).

Partial Derivative with respect to Capital (K): $\partial Q/\partial K = \beta * A * L^{\alpha} * K^{\beta-1}$

The partial derivative with respect to capital ($\partial Q/\partial K$) represents the rate of change of output concerning capital, while holding labor constant. It shows how much additional output (Q) is produced for a one-unit increase in capital (K).

Interpreting the Partial Derivatives:

- If $\alpha > 1$, the production function exhibits increasing returns to scale concerning labor, indicating that a proportionate increase in labor will lead to a more than proportionate increase in output.
- If $\alpha = 1$, the production function exhibits constant returns to scale concerning labor, indicating that a proportionate increase in labor will lead to a proportionate increase in output.
- If $\alpha < 1$, the production function exhibits decreasing returns to scale concerning labor, indicating that a proportionate increase in labor will lead to a less than proportionate increase in output.

✍ *The same interpretation applies to the partial derivative with respect to capital (β) in relation to returns to scale concerning capital.*

Understanding these partial derivatives is crucial for firms to make production decisions, optimize resource allocation, and maximize output efficiently, considering the responsiveness of output to changes in labor and capital inputs.

Optimization: In Economic Mathematics, optimization involves finding the optimal values of variables to maximize or minimize an objective function, subject to constraints. This is crucial in economic decision-making, such as profit maximization for firms or utility maximization for consumers.

Example 4.0.c: One example of optimization in Economic Mathematics is profit maximization for a firm. Let's consider a simplified scenario: *Suppose a manufacturing company produces two products, Product A and Product B. The profit per unit of Product A is $10, and the profit per unit of Product B is $15. The company has limited resources, such as raw materials and labor hours, available for production. The production process requires 2 units of raw materials and 3 labor hours for each unit of Product A, and 4 units of raw materials and 2 labor hours for each unit of Product B. The company has a total of 50 units of raw materials and 30 labor hours available.*

The goal of the firm is to determine the number of units of each product to produce to maximize their total profit while staying within the resource constraints.

Mathematically, let:

$x = the\ number\ of\ units\ of\ Product\ A\ to\ produce$

$y = the\ number\ of\ units\ of\ Product\ B\ to\ produce$

The objective function to maximize is the total profit (Z) given by: $Z = 10x + 15y$

Subject to the following constraints:

$2x + 4y \leq 50$ (resource constraint for raw materials)

$3x + 2y \leq 30$ (resource constraint for labor hours)

$x \geq 0, y \geq 0$ (non-negativity constraints)

✍ *The firm can use the concept of optimization to find the values of x and y that maximize the total profit (Z) while satisfying the resource constraints. This involves solving the linear programming problem using techniques like the Simplex Method or optimization software tools.*

The optimal solution will provide the values of x and y that maximize the total profit, indicating the number of units of each product the company should produce to achieve the highest profitability while efficiently using its limited resources.

Gradient and Directional Derivatives: The gradient of a multivariable function represents the vector of its partial derivatives. It points in the direction of the steepest increase of the function. Directional derivatives provide information about the rate of change in a specific direction, which is useful in economic modeling.

Example 3.0.d: Let's consider an example in Economic Mathematics where the concept of gradient and directional derivatives is applied: Profit Maximization for a Firm.

Suppose a firm produces two products, *A* and *B*. The total profit function (in thousands of dollars) for this firm is given by:

$$P(x, y) = 3x + 2y - 2x^2 - y^2$$

where: x = quantity of product A produced (in units), y = quantity of product B produced (in units).

The firm wants to determine the production levels of products A and B that will maximize their total profit, subject to the following constraints:

Production Capacity Constraint: The total production of products A and B combined cannot exceed 100 units: $x + y \leq 100$.

Non-negativity Constraint: The production quantities must be non-negative: $x \geq 0$, $y \geq 0$.

To find the maximum profit and the optimal production levels, we need to optimize the profit function $P(x, y)$ subject to the given constraints.

Step 1: Calculating the Gradient: The gradient of the profit function P(x, y) is a vector that consists of its partial derivatives with respect to x and y:

$$\underbrace{\boldsymbol{\nabla P}(x, y)}_{gradient\ of\ P(x,y)} = (\partial P/\partial x, \partial P/\partial y) = (3 - 4x, 2 - 2y)$$

Step 2: Finding the Critical Points: Critical points are the points where the gradient is zero (∇P(x, y) = 0). Setting each component of the gradient to zero and solving for x and y:

$3 - 4x = 0 \Rightarrow x = \frac{3}{4}$, $2 - 2y = 0 \Rightarrow y = 1$. The critical point is $(\frac{3}{4}, 1)$.

Step 3: Evaluating the Hessian: The Hessian matrix is a square matrix of second-order partial derivatives. It is used to determine whether the critical point is a maximum, minimum, or saddle point. For the profit function, the Hessian matrix is:

$$H(x, y) = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}$$

Step 4: Determining the Nature of the Critical Point: Since the Hessian is diagonal and both eigenvalues are negative (i.e., -4 and -2), the critical point (3/4, 1) is a local maximum.

Step 5: Considering Constraints: Now, we need to check if the critical point satisfies the constraints. For x = 3/4 and y = 1, the production capacity constraint is satisfied ($3/4 + 1 = 7/4 \leq 100$) and the non-negativity constraint is also satisfied ($3/4 \geq 0$, $1 \geq 0$).

Step 6: Interpreting Results: The critical point (3/4, 1) represents the optimal production levels for products A and B that will maximize the firm's total profit, subject to the given constraints. The firm should produce 3/4 units of product A and 1 unit of product B to achieve the maximum profit of 9/4 thousand dollars.

✍ *This example illustrates how the concepts of gradient and directional derivatives are used in Economic Mathematics to optimize economic outcomes, such as profit maximization for a firm under certain production constraints.*

Total Differential and Tangent Planes: Total differential allows us to approximate changes in a multivariable function. Tangent planes provide linear approximations of a function at a given point, which is valuable in economic analysis.

Example 3.0.e: One example of using total differentials and tangent planes in Economic Mathematics is in cost analysis and production optimization.

Consider a manufacturing firm that produces two goods, *X* and *Y*, using two inputs, labor *(L)* and capital *(K)*. The cost function $C(L, K)$ represents the total cost of production, where *L* and *K* are the quantities of labor and capital employed, respectively.

Total Differential: The total differential of the cost function C(L, K) can be represented as follows: $dC = \partial C/\partial L * dL + \partial C/\partial K * dK$

where *dC* is the total differential of the cost function, *∂C/∂L* represents the partial derivative of *C* with respect to labor *(L)*, *∂C/∂K* represents the partial derivative of *C* with respect to capital *(K)*, and *dL* and *dK* represent small changes in the quantities of labor and capital, respectively.

The total differential provides an approximation of how the total cost changes when there are small changes in the quantities of labor and capital.

Tangent Planes: Suppose the firm wants to optimize its production to minimize total costs, subject to certain production constraints. At the optimal production level *(L*, K*)*, the firm seeks to find the tangent plane to the cost surface. The equation of the tangent plane to the cost function C(L, K) at the point (L^*, K^*) is given by:

Tangent plane equation: $z - C(L^*, K^*) = \partial C/\partial L * (L - L^*) + \partial C/\partial K * (K - K^*)$

where *z* represents the elevation of the tangent plane above the cost surface.

The tangent plane represents a linear approximation of the cost function near the optimal production point (L^*, K^*). This allows the firm to estimate the change in cost for small deviations from the optimal production level and helps make cost-effective decisions.

✍ *By analyzing the total differential and tangent plane, the firm can optimize production levels of goods X and Y while minimizing costs and ensuring that production remains within the feasible region defined by input constraints (e.g., available labor and capital resources).*

Overall, the use of total differentials and tangent planes in Economic Mathematics provides valuable insights into cost analysis and production optimization, helping firms make informed decisions to achieve efficiency and profitability.

Applications in Economics: The concepts of "Function of Several Variables" find numerous applications in economics. For instance, analyzing consumer preferences, production decisions, cost structures, utility maximization, and firm behavior all require working with multivariable functions.

Example 3.0.f: One example of the application of "Function of Several Variables" in Economic Mathematics is the utility maximization problem for a consumer. Let's consider a consumer who wants to allocate their income between two goods, *X* and *Y*, to maximize their utility (satisfaction)

subject to their budget constraint. The consumer's utility function, $U(X,Y)$, represents their preferences for consuming different quantities of goods *X* and *Y*.

The budget constraint is given by the total amount of income (I) the consumer has to spend on goods *X* and *Y*, and the prices of goods $X(P_X)$ and $Y(P_Y)$. The budget constraint equation is: $P_X * X + P_Y * Y = I$.

The consumer's goal is to choose the quantities of goods *X* and *Y* (X^* and Y^*) that maximize their utility while satisfying the budget constraint.

Mathematically, this problem can be formulated as follows:

$Maximize\ U(X,Y)$

Subject to: $P_X\ *\ X + P_Y * Y = I$

To solve this problem, the consumer needs to find the critical points of the utility function (partial derivatives equal to zero) and check for the feasibility of the solutions with the budget constraint. The critical points represent potential optimal consumption bundles that maximize the consumer's utility.

✍ *By using the concepts of "Function of Several Variables," the consumer can determine the optimal combination of goods X and Y that maximizes their utility within the budget constraint. This type of analysis is fundamental in microeconomics to understand consumer behavior and make predictions about market demand for different goods.*

Moreover, this optimization problem can be extended to consider multiple goods and additional constraints, such as income changes or other preferences, allowing economists to analyze more complex consumer choices and market dynamics.

Multivariable Integration: Integrating multivariable functions is used to calculate areas, volumes, and welfare measures in economic analysis.

Example 3.0.g: One example of multivariable integration in Economic Mathematics is calculating the total welfare or consumer surplus in a market given a demand function. Suppose we have a market with a demand function for a commodity represented as:

$$Qd\ =\ 100\ -\ 2P\ +\ 3I$$

where *Qd* is the quantity demanded, *P* is the price of the commodity, and *I* is the income of consumers.

Additionally, let's assume the market supply function is given by:

$$Qs\ =\ 20\ +\ 4P$$

To find the equilibrium price and quantity in the market, we set the demand equal to the supply:

$$100\ -\ 2P\ +\ 3I\ =\ 20\ +\ 4P$$

Now, we solve for the equilibrium price (P) in terms of income (I):

$$6P\ =\ 80\ -\ 3I$$

$$P\ =\ (80\ -\ 3I)\ /\ 6$$

Once we have the equilibrium price, we can calculate the equilibrium quantity (*Q*) by substituting the price back into either the demand or supply function:

$$Q\ =\ 100\ -\ 2P\ +\ 3I$$

Now, to find the total welfare or consumer surplus (*CS*) in the market, we need to calculate the area under the demand curve up to the equilibrium quantity. This involves integrating the demand function over the range of quantity from 0 to the equilibrium quantity (Q):

$$CS = \int_0^Q (100 - 2P + 3I)\, dQ$$

To perform this multivariable integration, we use the equilibrium price equation ($P = (80 - 3I) / 6$) to express the price (P) in terms of quantity (Q) and income (I). Substituting this back into the consumer surplus equation:

$$CS = \int_0^Q (100 - 2((80 - 3I) / 6) + 3I)\, dQ$$

After performing the integration, we obtain the total welfare or consumer surplus in the market.

✍ *This example illustrates how multivariable integration is used to calculate welfare measures in economic analysis. In real-world scenarios, similar integration techniques are applied to calculate the areas under demand or supply curves to quantify consumer surplus or producer surplus, respectively. These welfare measures provide valuable insights into the economic efficiency and distribution of benefits in a market.*

Elasticity: Elasticity measures the responsiveness of one variable to changes in another. In Economic Mathematics, elasticity calculations play a vital role in demand and supply analysis, pricing decisions, and market behavior.

Example 3.0.h: Elasticity is a crucial concept in Economic Mathematics that measures the responsiveness of one economic variable to changes in another variable. Let's consider an example of price elasticity of demand: Price Elasticity of Demand. Suppose a company manufactures and sells smartphones in the market. The company wants to determine the impact of a price change on the quantity of smartphones sold. The price elasticity of demand measures how sensitive the quantity demanded is to changes in the price of the smartphones.

Let: P be the price of smartphones, Q be the quantity of smartphones sold.

The price elasticity of demand (PED) is defined as:

$$PED = (\%\ Change\ in\ Quantity\ Demanded)/(\%\ Change\ in\ Price)$$

Mathematically, the formula for price elasticity of demand is:

$$PED = (\Delta Q/Q)/(\Delta P/P)$$

where ΔQ is the change in quantity demanded, ΔP is the change in price, Q is the initial quantity demanded, and P is the initial price.

Now, suppose the company conducted a study and found that when the price of smartphones was $500, the quantity demanded was 10,000 units. After reducing the price to $450, they observed an increase in quantity demanded to 12,000 units.

Using the formula, we can calculate the price elasticity of demand as follows:

Initial quantity demanded, Q = 10,000 units Change in quantity demanded,

$$\Delta Q = 12{,}000 - 10{,}000 = 2{,}000\ units$$

Initial price, P = $500 Change in price, $\Delta P = \$450 - \$500 = -\$50$

$PED = (\Delta Q\ /\ Q)/(\Delta P/P) PED = (2{,}000/10{,}000)/(-50/500) PED = (0.2)/(-0.1) PED = -2$

Interpretation: The price elasticity of demand in this example is -2. This negative value indicates that the quantity demanded is highly responsive to changes in price. Specifically, a 1%

decrease in price results in a 2% increase in the quantity demanded. The negative sign indicates that demand is elastic, meaning consumers are highly sensitive to price changes, and a decrease in price leads to a proportionally larger increase in quantity demanded.

✍ *Understanding price elasticity of demand is vital for businesses to make informed pricing decisions, forecast changes in revenue, and respond to market conditions effectively*.

Multi-Objective Optimization: Economic decisions often involve multiple conflicting objectives. Multi-objective optimization techniques help identify trade-offs and Pareto-optimal solutions.

Example 3.0.i: One example of Multi-Objective Optimization in Economic Mathematics is the portfolio optimization problem for an investor. *Suppose an investor wants to allocate their funds among different assets, such as stocks, bonds, and real estate, to achieve multiple objectives simultaneously. The investor's objectives may include maximizing the expected return, minimizing the risk (measured by variance or standard deviation), and satisfying certain constraints, such as budget constraints or target allocations for each asset*.

The Multi-Objective Optimization problem can be formulated as follows:

Objective 1: Maximize the expected return of the portfolio: $Maximize: Return = w_1 * Return(stock) + w_2 * Return(bond) + w_3 * Return(real\ estate)$

Objective 2: Minimize the risk (variance) of the portfolio: $Minimize: Risk = w_1^2 * Variance(stock) + w_2^2 * Variance(bond) + w_3^2 * Variance(real\ estate) + 2 * w_1 * w_2 * Covariance(stock, bond) + 2 * w_1 * w_3 * Covariance(stock, real\ estate) + 2 * w_2 * w_3 * Covariance(bond, real\ estate)$

Subject to the following constraints:

$w_1 + w_2 + w_3 = 1$ (Total allocation constraint)

$w_1, w_2, w_3 \geq 0$ (Non-negativity constraints)

Other specific constraints, such as budget constraints or target allocations for each asset.

In this example, the investor wants to find the best allocation of weights (w_1, w_2, w_3) to different assets to achieve the trade-off between maximizing the expected return and minimizing the risk. This is a classic multi-objective optimization problem, as there is a trade-off between the two objectives, and there is no single solution that is optimal for both objectives simultaneously.

To solve this problem, specialized multi-objective optimization techniques, such as Pareto-based approaches or evolutionary algorithms, can be employed to identify the set of Pareto-optimal solutions. These solutions represent the best trade-offs between the conflicting objectives and help the investor make an informed decision based on their risk tolerance and return expectations.

✍ *The Multi-Objective Optimization approach in portfolio optimization allows the investor to consider multiple objectives simultaneously and gain insights into the risk-return trade-offs to create a well-balanced and efficient investment portfolio.*

Input-Output Models: In macroeconomics, input-output models involve multivariable functions to represent the relationships between different sectors of an economy.

Example 3.0.j: An example of an Input-Output Model in Economic Mathematics is the Leontief Input-Output Model. This model was developed by Wassily Leontief and is widely used to analyze the interdependencies between different sectors of an economy.

In the Leontief Input-Output Model, the economy is divided into various sectors, such as agriculture, manufacturing, services, etc. The model considers how each sector uses inputs from other sectors and how it produces outputs that are used as inputs by other sectors. This creates a system of linear equations that represent the relationships between the sectors.

Let's consider a simplified example with three sectors: agriculture (A), manufacturing (M), and services (S). The input-output coefficients represent the amount of input required from each sector to produce one unit of output in another sector.

The input-output coefficients are represented in a matrix called the "input-output matrix." For example, consider the following input-output matrix:

A	M	S	
0.2	0.3	0.1	Agriculture (A)
0.1	0.4	0.2	Manufacturing (M)
0.3	0.1	0.5	Services (S)

In this matrix, the (i, j)-th entry represents the proportion of output from sector i that is used as an input by sector j.

Using this input-output matrix, we can analyze the relationships between the sectors. For example:

To produce one unit of output in agriculture, 0.2 units of agriculture, 0.3 units of manufacturing, and 0.1 units of services are required.

To produce one unit of output in manufacturing, 0.1 units of agriculture, 0.4 units of manufacturing, and 0.2 units of services are required.

To produce one unit of output in services, 0.3 units of agriculture, 0.1 units of manufacturing, and 0.5 units of services are required.

✍ *The Leontief Input-Output Model is used to study various economic scenarios, such as the effects of changes in final demand (e.g., changes in consumer spending or exports) on the output levels of each sector, the calculation of multiplier effects, and the impact of changes in production technology. Input-Output Models are particularly valuable in macroeconomic analysis and policy-making, as they provide insights into the interconnections between different sectors and the economy's overall performance.*

△ *Initiating into "Function of Several Variables" in Economic Mathematics requires a solid foundation in calculus and linear algebra. Understanding these concepts will enable economists to model and analyze complex economic scenarios, make informed decisions, and optimize economic outcomes effectively.*

4.1 THE LIMIT AND CONTINUITY OF FUNCTIONS IN MULTIVARIABLE CALCULUS

Keywords: *Multivariable Functions, Function Domain, Function Codomain, Input Variables, Output Variable, Euclidean Space, Limit of a Multivariable Function, Limit Point, Approach, Existence of Limit, Path Dependence, Continuity of Multivariable Functions, Point Continuity, Domain Continuity, Discontinuity, Removable Discontinuity, Jump Discontinuity, Infinite Discontinuity, Multivariable Continuity Criteria, Continuous Multivariable Function, Discontinuous Multivariable Function, Open Set, Closed Set, Boundary Point, Neighborhood of a Point, Partial Continuity.*

The definition of a multivariable function: The definition of a multivariable function is a mathematical function that takes multiple input variables and maps them to an output variable. In the context of economic mathematics, multivariable functions are used to model relationships between multiple economic variables, allowing economists to analyze and understand complex economic systems. Mathematically, a multivariable function can be represented as:

$$f: R^n \to R$$

where R^n represents the *n-dimensional* Euclidean space, and R is the set of real numbers, which is the codomain of the function.

Example 4.1.a: One classic example of a multivariable function used in economic mathematics is the *Cobb-Douglas production function*. This function is commonly used to model the relationship between multiple inputs (usually labor and capital) and the output of a production process. It is widely employed in microeconomics and macroeconomics for analyzing production and economic growth. The Cobb-Douglas production function can be represented as follows:

$$f: R^2 \to R, \qquad Q = f(L, K) = A * L^{\alpha} * K^{\beta}$$

where: Q is the output (quantity produced), L is the output (quantity produced), K represents the quantity of capital input, A is the total factor productivity (a constant representing technological progress and efficiency), α is the output elasticity of labor (a parameter measuring the responsiveness of output to changes in labor input), β is the output elasticity of capital (a parameter measuring the responsiveness of output to changes in capital input), respectively.

◘ *In this production function, the output Q is a function of both labor L and capital K, with the parameters A, α, and β determining the relationship between the inputs and output.*

Economists use the Cobb-Douglas production function to analyze various aspects of production, such as the optimal combination of labor and capital inputs to maximize output, the impact of changes in technology or factor prices, and the long-term growth of economies.

✍*The function's multivariable nature allows economists to study the joint effects of labor and capital on production and make informed decisions in the context of resource allocation and economic policy.*

The domain and codomain of a function: The domain and codomain are both important concepts in the study of functions, particularly in economic mathematics. Let's define each term and provide an example in economic mathematics to illustrate the difference:

***Domain of a Function*:** The domain of a function is the set of all possible input values for which the function is defined. In other words, it is the set of independent variables for which the function can be evaluated and produce meaningful output. Not all values can be input into a function, as some values might lead to undefined or nonsensical results.

***Codomain of a Function*:** The codomain of a function is the set of all possible output values that the function can take. It represents the set of dependent variable values that the function can produce based on its input. The codomain is a superset of the actual range, which is the set of values that the function actually outputs for the given inputs.

Example 4.1.b: Let's consider a production function in economic mathematics. A common production function used in economics is the Cobb-Douglas production function, which relates the inputs of capital (K) and labor (L) to the output of goods or services (Q). The Cobb-Douglas production function can be represented as follows:

$$f: R^2 \to R, \qquad Q = f(L, K) = A * L^{\alpha} * K^{\beta}$$

where, Q represents the quantity of output (goods or services) produced, *K* represents the amount of capital used in production, *L* represents the amount of labor used in production, *A* is a positive constant representing total factor productivity, α and β are positive constants representing the output elasticities of capital and labor, respectively.

Domain: In the context of the Cobb-Douglas production function, the domain consists of all valid combinations of capital (*K*) and labor (*L*) values for which the function can be evaluated. It means that both *K* and *L* must be non-negative values since negative values of capital or labor do not make sense in this production setting.

Codomain: The codomain of the Cobb-Douglas production function is the set of all possible output values (*Q*) that can be produced based on the given inputs of capital (*K*) and labor (*L*). The codomain includes all non-negative real numbers because output quantities should be positive or zero (the absence of output).

◘ *It's important to note that the actual range of the production function may be limited by market demand, resource constraints, or other economic factors. While the codomain covers all non-negative real numbers, the actual range will depend on the specific values of A, α, β, K, and L used in real-world scenarios.*

The input variables and an output variable*:* In the context of multivariable functions in economic mathematics, input variables are the independent variables, also known as the "inputs" or "factors," that influence the value of the dependent variable, which is the output. The multivariable function takes multiple input variables and maps them to a single output variable. The relationship between the inputs and the output is defined by the function.

Example 4.1.c: Let's consider a production function, which is a fundamental concept in economics used to describe the relationship between inputs (factors of production) and the output (quantity of goods or services produced). Suppose we have a simplified Cobb-Douglas production function:

$$f: R^2 \to R, \qquad Q = f(L, K) = A * L^{\alpha} * K^{\beta}$$

where: *Q* represents the quantity of output produced, *A* is a constant representing the total factor productivity (the overall efficiency of production, including technology and management), *L* denotes

the quantity of labor input, K denotes the quantity of capital input, α and β are the output elasticities of labor and capital, respectively.

They represent the responsiveness of output to changes in labor and capital inputs.

◘ *In this production function, L and K are the input variables because they can be controlled and adjusted by the firm. The firm can choose how much labor (L) and capital (K) to use in the production process.*

On the other hand, Q is the output variable because it depends on the values of L, K, A, α, and β. The output variable Q represents the total quantity of goods or services produced based on the chosen levels of labor and capital inputs.

✍ *Economists use such multivariable production functions to analyze the optimal combination of inputs (labor and capital) that maximize output or profit while considering the constraints and costs involved. By varying the input variables (labor and capital) in the production function, economists can determine the impact on the level of output, providing valuable insights for decision-making and resource allocation in firms and industries.*

The limit of a multivariable function*:* The limit of a multivariable function is defined similarly to the limit of a single-variable function but in higher dimensions. It describes the behavior of the function as the input variables approach a particular point in the domain of the function. The limit is said to exist if the function values converge to a specific value as the input variables get arbitrarily close to the given point, regardless of the path or direction of approach.

Mathematically, the limit of a multivariable function $f(x, y)$ as (x, y) approaches the point (a, b) is denoted by:

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

Example 4.1.d: Now, let's look at an example of how limits are used in economic mathematics: Production Function. In economics, a production function represents the relationship between inputs (usually labor and capital) and the output of a production process. Let's consider a Cobb-Douglas production function, a commonly used production function in economic modeling:

$$f: R^2 \to R, \qquad Q = f(L, K) = A * L^{\alpha} * K^{\beta}$$

where: Q is the output quantity, L is the quantity of labor input, K is the quantity of capital input, A is a positive constant representing the total factor productivity, α and β are positive constants representing the output elasticity with respect to labor and capital, respectively.

Now, let's say we want to find the limit of this production function as both labor (L) and capital (K) approach zero. This situation might be relevant in an economic context when a company is considering the output level when resources are scarce or nearly depleted.

$$\lim_{(L,K)\to(0,0)} Q(L, K) = \lim_{(L,K)\to(0,0)} \left(A * L^{\alpha} * K^{\beta}\right)$$

To evaluate this limit, we need to consider various paths of approach, such as along the axes (i.e., $L = 0$ or $K = 0$), or along some specific curves. Depending on the values of α and β, the limit might exist or not.

If, for instance, $\alpha > 0$ and $\beta > 0$, and both L and K approach zero, the limit will be 0 because the output will tend to zero as resources become negligible. On the other hand, if $\alpha = 0$ and $\beta = 0$,

the limit might yield a non-zero value, indicating that the production function could produce some output even with one of the inputs approaching zero.

◘ *In this example, the limit of the production function as the inputs approach zero provides insights into the behavior of the production process under extreme resource constraints and can be useful in making decisions regarding resource allocation and production strategies.*

Existence of the limit for a multivariable function: For a limit to exist for a multivariable function, the function's output must approach a particular value as the input variables get arbitrarily close to a specific point in the domain. In other words, as the input variables approach the specified point, the function's values should approach a unique finite value or infinity. If this condition holds true, we say that the limit of the multivariable function exists at that point.

In mathematical notation, the limit of a multivariable function $f(x, y)$ as (x, y) approaches (a, b) is denoted as:

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

where (a, b) is the specific point to which the input variables (x, y) are approaching.

Example 4.1.e: Let's consider a simple economic example using a multivariable function to represent the demand for a product in a market. The demand *D* for a product is often influenced by both its own price *P* and the income of consumers *Y*. We can represent this relationship using a multivariable function:

$$f: R^2 \to R, \qquad D = f\ (P, Y) = \frac{a}{P} + bY$$

where: *a* and *b* are constants representing the sensitivity of demand to changes in price and income, respectively, *P* is the price of the product, *Y* is the income of consumers.

Now, let's consider the limit of this demand function as the price of the product *P* approaches zero. In economic terms, this means we are examining the behavior of demand as the product's price approaches "free".

$$\lim_{P\to 0} D(P, Y)$$

As *P* approaches zero, the first term $\frac{a}{P}$ in the demand function becomes increasingly large (approaching infinity), while the second term bY remains constant. Hence, the overall demand *D* also approaches infinity.

◘ *In this example, we have shown that as the price of the product gets closer and closer to zero, the demand for the product becomes unbounded and tends towards infinity. This indicates a highly elastic demand for the product, where consumers are willing to buy more of the product as the price decreases significantly.*

The limit of a multivariable function at a specific point: To determine the limit of a multivariable function at a specific point, you need to analyze the behavior of the function as it approaches that point. In other words, you evaluate the function as the input variables get arbitrarily close to the given point. If the function approaches a unique value regardless of the direction from which it approaches the point, then the limit exists.

Example 4.1.f: Let's consider an economic example to illustrate this concept. Suppose we have a production function that represents the output (Q) of a company based on two input factors, labor (L) and capital (K):

$$f: R^2 \to R, \qquad Q = f(L,K) = 5L^2 + 3K^2 - 2LK$$

Now, let's find the limit of this production function as (L,K) approaches the point $(2,3)$.

To find the limit, we calculate the following:

$$\lim_{(L,K)\to(2,3)} Q = \lim_{(L,K)\to(2,3)} (5L^2 + 3K^2 - 2LK)$$

To determine if the limit exists, we need to check if the function approaches a unique value as (L, K) approaches (2, 3) from any direction.

One way to do this is to evaluate the function along different paths approaching the point $(2,3)$ and see if they all lead to the same value. Let's consider two paths: the path along the line *L=2* and the path along the line *K=3*.

Path 1: *L=2*

Substitute *L=2* into the production function:

$Q(2,K) = 5 * 2^2 + 3K^2 - 2 * 2 * K = 20 + 3K^2 - 4K.$

Path 2: *K=3*

Substitute *K=3* into the production function:

$Q(L,3) = 5 * L^2 + 3 * 3^2 - 2 * L * 3 = 5 * L^2 + 27 - 6L.$

Now, we evaluate each path as we approach the point (2, 3):

Approaching along the line L=2:

$$\lim_{(L,K)\to(2,3)} Q = \lim_{K\to 3} (20 + 3K^2 - 4K) = 20 + 3 * 3^2 - 4 * 3 = 35$$

*Approaching along the line K=3***:**

$$\lim_{(L,K)\to(2,3)} Q = \lim_{L\to 2} (5 * L^2 + 27 - 6L) = 5 * 2^2 + 27 - 6 * 2 = 35$$

◘ *Since the function approaches by equal values along different paths, the limit of the production function exist and as (L, K) approaches (2, 3) it equal to 35.*

✍ *This example demonstrates how to find the limit of a multivariable function in economic mathematics and highlights the importance of checking the behavior along different paths to determine the existence of the limit.*

The continuity for multivariable function:

Definition: A multivariable function $f(x,y)$ is said to be continuous at a point (a,b) in its domain if three conditions are met:

Existence of the Function: The function $f(x,y)$ is defined at (a,b). In other words, the point (a,b) belongs to the domain of the function.

Existence of the Limit: The limit of the function as (x,y) approaches (a,b) exists. Mathematically, this means:

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is finite.

Limit Equals Function Value: The limit value is equal to the value of the function at (a,b). In mathematical terms:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

◘ *In summary, a multivariable function is continuous at a point* (a, b) *if the function exists at that point, the limit of the function exists and is finite as we approach* (a, b)*, and the limit value is equal to the value of the function at* (a, b).

Definition*:* If the function satisfies these conditions for every point in its domain, then it is continuous throughout its domain, and it exhibits smooth and well-behaved behavior.

✍ *Continuity is an essential property in multivariable calculus, allowing for more straightforward analysis and predictions in various mathematical and real-world applications, including economic modeling, optimization problems, and physical sciences.*

The discontinuity for multivariable function: In economic mathematics, discontinuities in functions can arise due to various factors, and they have specific implications for economic models. Here is a summary of the types of discontinuities commonly encountered in economic mathematics:

Removable Discontinuity**:** A removable discontinuity occurs when there is a hole in the graph of the function at a specific point, but the limit of the function exists at that point. Economically, this could represent situations where the function has a different behavior for a specific range of inputs but can be made continuous by redefining the function's value at that point. In a demand function, the quantity demanded may have a different equation for a specific price range, but it can be made continuous by adjusting the function's value at the boundary price.

Example 4.1.g: Consider a production function that represents the relationship between the quantity of output (Q) and the quantity of capital (K) and labor (L) used in a manufacturing process:

$$Q = f(K, L) = \frac{K^2 L}{K^2 + L^2}$$

At the point $(K, L) = (0, 0)$, the production function is not defined since division by zero is not allowed. However, we can redefine the function at this point by setting $f(0,0) = 0$. Now, the function becomes continuous at $(0, 0)$, and the discontinuity is removed.

Jump Discontinuity: A jump discontinuity occurs when the function has different finite values from the left and right sides of a particular point. Economically, this could represent scenarios where there is a sudden change in the output or behavior of a variable at a specific point. In a revenue function, the revenue generated by a company may have a different value for a particular quantity sold, resulting in a jump in revenue at that quantity.

Example 4.1.h: Consider a demand function for a product:

$$P = f(Q) = \begin{cases} 10, & if\ Q < 100 \\ 15, & if\ Q \geq 100 \end{cases}$$

In this case, when the quantity demanded (Q) crosses 100 units, there is a jump in the price (P) from \$10 to \$15. The function is discontinuous at Q = 100 because the left-hand limit (as Q approaches 100 from the left) is \$10, while the right-hand limit (as Q approaches 100 from the right) is \$15.

Infinite Discontinuity**:** An infinite discontinuity happens when the limit of the function approaches either positive or negative infinity at a specific point. Economically, this could represent situations where the function becomes unbounded at a particular input value.

Example 4.1.i: Consider a profit function for a company:

$$Profit = f(Q) = \frac{1000}{Q - 50}$$

At $Q = 50$, the profit function is not defined since it results in division by zero. As Q approaches 50 from either side, the profit approaches positive or negative infinity, depending on the direction of the approach. This situation represents an infinite discontinuity at $Q = 50$.

***Asymptotic Discontinuity*:** An asymptotic discontinuity occurs when the function approaches a vertical asymptote, and the limit of the function either approaches infinity or negative infinity. Economically, this could represent situations where certain inputs result in unattainable or impractical outcomes.

Example 4.1.j: An example of a multivariable function with an asymptotic discontinuity can be found in the context of an investment model, specifically for a capital investment project. Consider a profit function (P) that represents the annual profit (in thousands of dollars) generated by an investment project with the initial capital investment (C, in thousands of dollars) and the project duration (T, in years). The profit function can be represented as follows:

$$P = f(C, T) = \frac{100C}{T}$$

In this profit function, the capital investment (C) is in the numerator, while the project duration (T) is in the denominator. This means that as the project duration (T) approaches zero (i.e., the project is extremely short-term), the profit (P) becomes unbounded and tends to positive infinity.

Asymptotic Discontinuity Explanation**:** When the project duration (T) approaches zero, the profit function will have an asymptote. An asymptote is a line that the graph of the function approaches but never crosses. In this case, as the duration (T) gets very close to zero, the profit (P) increases without bound.

Interpretation in Economic Mathematics**:** This investment model illustrates an asymptotic discontinuity in the profit function when the project duration becomes extremely small. It suggests that, in theory, if a capital investment project were to last for an infinitesimally short time, the profit generated would tend towards infinity. However, in practical terms, the duration of an investment project cannot be exactly zero, and there are likely to be other constraints and factors that limit the profitability of very short-term projects.

◘ *Understanding asymptotic discontinuities in investment models is essential for analyzing the behavior of profit functions and gaining insights into the optimal duration and scale of capital investments in various economic contexts. It helps economists and businesses make informed decisions about investment strategies, taking into account the trade-offs between risk, return, and time horizon.*

✍ *In economic mathematics, identifying and understanding these types of discontinuities are essential for interpreting economic models accurately. They can provide insights into critical points, market behaviors, and potential problems in economic decision-making processes. Proper handling of discontinuities is crucial for building robust and reliable economic models.*

△ *Overall, a solid grasp of the "Definition, Limit, and Continuity of Multivariable Functions" is essential for economists and researchers to develop, analyze, and interpret complex economic*

models and make informed decisions based on mathematical principles. The application of these concepts allows for a deeper understanding of economic behavior, providing valuable insights into various economic phenomena.

Questions you should be able to answer

How can multivariable functions be used to model economic relationships?

What are some examples of multivariable functions used in economic analysis?

How are multivariable functions applied in production functions to determine output levels?

How do multivariable functions help in analyzing utility functions and consumer preferences?

What role do multivariable functions play in cost functions and profit maximization for firms?

How are multivariable functions utilized in analyzing market equilibrium and supply-demand interactions?

How do economists use the concept of limits to analyze the behavior of economic variables as they approach certain values?

Can you explain the continuity of economic functions and its significance in economic models?

In what ways can multivariable functions be employed to solve economic optimization and constrained optimization problems?

Can you provide examples of real-world economic situations where the study of multivariable functions is crucial for decision-making?

How do economists use multivariable functions to analyze the impact of changes in multiple variables on economic outcomes?

EXERCISES

4.1.1 In each exercise, choose the appropriate term to fill in the blank based on the context of "*The Limit and Continuity of Functions in Multivariable Calculus*":

a) The limit of a multivariable function $f(x, y)$ as (x, y) approaches a point (a, b) is denoted as _____.

b) If the limit of a function $f(x, y)$ as (x, y) approaches a point (a, b) exists and is equal to a constant L, then we say that $f(x, y)$ is _____ at (a, b).

c) A function $f(x, y)$ is continuous at a point (a, b) if three conditions are met: $f(a, b)$ is _____, the limit of $f(x, y)$ as (x, y) approaches (a, b) exists, and the limit is equal to $f(a, b)$.

d) To evaluate the limit of a multivariable function as (x, y) approaches a point (a, b), we can use _____ along different paths to check if the limit exists and is unique.

e) If a multivariable function $f(x, y)$ is continuous at every point in its domain, we call it a _____ function.

f) At a point where a function is not continuous, one or more of the three conditions for continuity ($f(a, b)$ exists, the limit as (x, y) approaches (a, b) exists, and the limit is equal to $f(a, b)$) may be _____.

g) The concept of a _____ point is crucial in understanding limits and continuity of functions in multivariable calculus.

h) The limit of a function as (x, y) approaches a point (a, b) can be different along different _____ to that point.

i) A multivariable function $f(x, y)$ is said to have a _____ at a point (a, b) if it has a limit at that point but is not continuous there.

j) The squeeze theorem can be applied to evaluate the limit of a multivariable function when it is _____ between two other functions with known limits.

4.1.2 In each exercise, choose the appropriate term to fill in the blank based on the context of "*The Limit and Continuity of Functions in Multivariable Calculus*", related to economic mathematics:

a) The concept of limit is essential in multivariable calculus and economic mathematics. For a function of two variables, $f(x, y)$, we say that the _____ of $f(x, y)$ as (x, y) approaches (a, b) exists if the value of $f(x, y)$ gets arbitrarily close to a unique number L as (x, y) gets arbitrarily close to (a, b).

b) The limit of a function at a point (a, b) can be expressed mathematically as follows: $\lim_{(x,y)\to(a,b)} f(x, y) = L$. In economic applications, this means that as the variables *x* and *y* approach certain values (a, b), the function $f(x, y)$ _____ to the value L.

c) A function $f(x, y)$ is said to be continuous at a point (a, b) if three conditions are satisfied: $f(a, b)$ is defined, the limit of $f(x, y)$ as (x, y) approaches (a, b) exists, and the limit is equal to $f(a, b)$. In the context of economics, continuity is crucial because it ensures that small changes in the variables *x* and y result in small changes in the _____ of the function.

d) A function $f(x, y)$ may not be continuous at all points within its domain. Discontinuities can occur due to jump discontinuities, removable discontinuities, or essential discontinuities. In economic models, discontinuities can have significant implications, such as abrupt changes in _____ and unexpected outcomes.

e) In economic optimization problems, we often deal with functions of several variables. The concept of partial derivatives is essential in such cases. A partial derivative represents the rate of change of a function with respect to one variable while holding other variables _____. This concept allows economists to analyze how changes in one variable affect the overall behavior of the function.

f) When dealing with economic functions, it is important to examine their continuity and differentiability. A function that is continuous and differentiable throughout its domain is considered to be _____. In economic applications, differentiability ensures that marginal changes in variables can be accurately analyzed.

g) Economists often use the concept of directional derivatives to study the sensitivity of economic functions in specific directions. The directional derivative of a function $f(x, y)$ at a point (a, b) in the direction of a unit vector $u = [u1, u2]$ is denoted by $D_u f(a, b)$ and measures the rate of change of f in the direction of u. In economic decision-making, directional derivatives help analyze how functions respond to _____ changes in variables.

h) A function $f(x,y)$ is said to have a local minimum at a point (a,b) if there exists an open neighborhood around (a,b) where $f(a,b)$ is the smallest value. Similarly, a function has a local maximum at a point (a,b) if there exists an open neighborhood around (a,b) where $f(a,b)$ is the largest value. In economic models, local extrema are crucial as they indicate potential _____ points or optimal solutions.

4.1.3 For each exercise, try to evaluate the limit by approaching the indicated point along different paths (along the coordinate axes, along straight lines, etc.) to check for the existence of the limit. If the limit exists, provide the value of the limit. If the limit does not exist, explain why:

a) Evaluate the limit of the function $f(x,y) = \frac{2x^2 - y^2}{x - y}$ as (x,y) approaches $(2,2)$.

b) Evaluate the limit of the function $g(x,y) = \frac{xy}{x^2 + y^2}$ as *(x, y)* approaches *(0, 0)*.

c) Evaluate the limit of the function $h(x,y) = \frac{x^2 - y^2}{x + y}$ as *(x, y)* approaches *(1, -1)*.

d) Evaluate the limit of the function $k(x,y) = \frac{x^3 - y^3}{x^2 - y^2}$ as *(x, y)* approaches *(1, 1)*.

e) Evaluate the limit of the function $m(x,y) = \frac{e^{xy} - 1}{x^2 + y^2}$ as *(x, y)* approaches *(0, 0)*.

f) Evaluate the limit of the function $n(x,y) = \frac{sin(x) + cos(y)}{x - y}$ as *(x, y)* approaches $(\frac{\pi}{4}, \frac{\pi}{4})$.

g) Evaluate the limit of the function $p(x,y) = \frac{ln(x) + ln(y)}{x - y}$ as *(x, y)* approaches *(1, 1)*.

h) Evaluate the limit of the function $q(x,y) = \frac{x^2 - 4y^2}{x - 2y}$ as *(x, y)* approaches *(2, 1)*.

i) Evaluate the limit of the function $r(x,y) = \frac{x^2 - y^2}{xy - y^2}$ as *(x, y)* approaches *(1, 1)*.

j) Evaluate the limit of the function $s(x,y) = \frac{2x^3 + 3y^3}{x^2 + y^2}$ as *(x, y)* approaches *(1, -1)*.

4.1.4 For each exercise, check the continuity of the multivariable function at the indicated points. If the function is continuous at the given points, explain why it is continuous. If it is not continuous, state the reason for the discontinuity. Make sure to consider different paths and the behavior of the function around the given points.

a) Determine the points (if any) where the following function is continuous: $f(x,y) = \frac{3x^2 - 2y}{x - y}$. Hint: Check the continuity along different paths and at isolated points.

b) Investigate the continuity of the function $f(x,y) = sin(x + y)$ at the origin *(0, 0)*.

c) Consider the function $f(x,y) = (x^2 + y^2)/(x + y)$ for $(x,y) \neq (0,0)$ and $f(x,y) = 0$ at *(0, 0)*. Determine if the function is continuous at the origin.

d) Examine the continuity of the function $f(x,y) = ln(xy)$ at the point *(1, 2)*.

e) For the function $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$, determine whether it is continuous at the point *(0, 0)*.

f) Study the continuity of the function $f(x,y) = e^{x-y} + x^2y^2$ at the origin *(0, 0)*.

g) Investigate the continuity of the function $f(x,y) = \frac{x^2 - y^2}{x - y}$ at $(x,y) \neq (0,0)$ and $f(x,y) = 1$ at *(0, 0)*.

h) For the function $f(x,y) = \frac{x^3 - y^3}{x - y}$, determine if it is continuous at *(0, 0)*.

i) Consider the function $f(x,y) = sqrt(|xy|)$ and investigate its continuity at the origin *(0, 0)*.

j) Study the continuity of the function $f(x,y) = (2x^2y)/(x^4 + y^2)$ at the origin *(0, 0)*.

4.1.5 For each exercise, first analyze the behavior of the given multivariable function at the indicated point. Determine if the function is discontinuous at that point and classify the type of discontinuity, if applicable (removable, jump, essential). Provide a clear explanation of your findings for each case.

a) Consider the function $f(x,y) = \frac{x^2 - y^2}{x - y}$. Determine if the function is discontinuous at the point *(1, 1)*. If it is discontinuous, classify the type of discontinuity (removable, jump, essential).

b) Given the function $g(x,y) = \frac{sin(x+y)}{x+y}$, examine the continuity at the origin *(0, 0)*. Identify the type of discontinuity, if any, that occurs at this point.

c) The function $h(x,y) = \frac{xy}{x^2 + y^2}$ is defined for all points except the origin *(0, 0)*. Analyze the behavior of the function as *(x, y)* approaches the origin. Identify the type of discontinuity that occurs at *(0, 0)*.

d) Consider the function $f(x,y) = \frac{x^2 + y^2 - 1}{x^2 + y^2}$. Investigate the continuity at the point *(0, 0)* and classify the type of discontinuity, if any.

e) The function $g(x,y) = \frac{x^3 - y^3}{x - y}$ is defined for all points except *(1, 1)*. Examine the continuity of the function at *(1, 1)* and classify the type of discontinuity that occurs at this point.

f) Given the function $h(x,y) = \frac{xy}{x - y}$, analyze the continuity at the point *(2, 2)*. Determine the type of discontinuity, if any, at this point.

g) Consider the function $f(x,y) = \frac{x^2 - y^2}{x + y}$. Investigate the continuity at the point *(2, -2)* and classify the type of discontinuity, if present.

h) The function $g(x,y) = \frac{x^3 - y^3}{x + y}$ is defined for all points except the origin *(0, 0)*. Examine the behavior of the function as *(x, y)* approaches the origin and identify the type of discontinuity that occurs at *(0, 0)*.

4.2 PARTIAL DERIVATIVES AND DIFFERENTIALS FOR MULTIVARIABLE FUNCTIONS

***Keywords**: Multivariable Functions, Partial Derivatives, Differentials, Marginal Analysis, Total Derivative, Production Functions, Consumption Functions, Profit Maximization, Cost Minimization, Elasticity, Optimization, Utility Functions, Indifference Curves, Budget Constraints, Lagrangian Multipliers.*

The partial derivatives in multivariable functions: In multivariable calculus, a partial derivative is a way of measuring how a multivariable function changes with respect to one of its input variables while keeping all other variables constant. It calculates the rate of change of the function concerning a specific input variable, similar to how ordinary derivatives measure the rate of change of single-variable functions.

Mathematically, if we have a multivariable function $f(x, y)$ with two input variables x and y, then the partial derivative of *f* with respect to *x* is denoted by $\frac{\partial f}{\partial x}$, and the partial derivative of *f* with respect to *y* is denoted by $\frac{\partial f}{\partial y}$.

Example 4.2.a: Consider a utility function that models an individual's utility (satisfaction) from consuming two goods, *x* and *y*. Let the utility function be given by:

$$U: R^2 \to R, \qquad U\ (x, y) = x^{\alpha} * y^{\beta}$$

where α and β are positive constants representing the individual's preferences for each good.

Now, let's find the partial derivatives of *U* with respect to *x* and *y* separately.

Partial derivative with respect to *x*:

To find $\frac{\partial U}{\partial x}$, we treat *y* as a constant and differentiate *U* with respect to *x*:

$$\frac{\partial U}{\partial x} = \left(x^{\alpha} * y^{\beta}\right)'_x = \alpha * x^{\alpha-1} * y^{\beta}$$

The result $\frac{\partial U}{\partial x}$ represents the marginal utility of good *x*. It tells us how much additional utility an individual gains when they consume one more unit of good *x* while holding the consumption of good *y* constant.

Partial derivative with respect to *y*:

To find $\frac{\partial U}{\partial y}$, we treat *y* as a constant and differentiate *U* with respect to *x*:

$$\frac{\partial U}{\partial y} = \left(x^{\alpha} * y^{\beta}\right)'_y = \beta * x^{\alpha} * y^{\beta-1}$$

The result $\frac{\partial U}{\partial y}$ represents the marginal utility of good *y*. It tells us how much additional utility an individual gains when they consume one more unit of good *y* while holding the consumption of good *x* constant.

✍ *In economic analysis, partial derivatives play a vital role in understanding consumer preferences, producer behavior, and optimizing economic decisions. They provide information about the sensitivity of economic variables to changes in specific factors and are used to derive important concepts like marginal utility, marginal cost, elasticity, and production functions, among others.*

Differentiability of Multivariable Functions: In economic mathematics, the concept of differentiability of multivariable functions plays a crucial role in analyzing and optimizing various economic relationships and decision-making processes. Differentiability allows economists to study how small changes in multiple input variables of a function affect the output, and it helps in identifying optimal solutions for economic problems.

Definition of differentability: A multivariable function $f(x_1, x_2, \dots, x_n)$ is considered differentiable at a point $(a_1, a_2, \dots, a_n)$ if all its partial derivatives exist and are continuous at that point. In other words, the function must have well-defined rates of change with respect to each input variable, and these rates of change should vary smoothly in the neighborhood of the point.

Example 4.2.b: Let's consider an example of a profit function for a firm that involves multiple input variables in economic mathematics. Suppose a firm produces two goods, *X* and *Y*, using two input factors: labor (*L*) and capital (*K*). The firm's profit function, $P(L, K)$, represents the total profit generated by producing and selling *X* and *Y*, given the amounts of labor and capital used.

The profit function is defined as:

$$P: R^2 \to R, \qquad P(L, K) = 10L^2 + 15K^2 - 2LK$$

where: *L* represents the amount of labor used, *K* represents the amount of capital used, The profit function $P(L, K)$ is measured in monetary units (e.g., dollars).

Differentiability**:** To check the differentiability of the profit function $P(L, K)$, we need to find its partial derivatives with respect to both input variables, *L* and *K*:

Partial derivative with respect to *L*: $\frac{\partial P}{\partial L} = 20L - 2K$

Partial derivative with respect to *K*: $\frac{\partial P}{\partial K} = 30K - 2K$

Both partial derivatives exist and are continuous for all values of *L* and *K*. Hence, the profit function $P(L, K)$ is differentiable at all points in its domain.

Economic Interpretation**:** The differentiability of the profit function has important economic implications. The partial derivatives represent the marginal rates of change of profit concerning changes in labor (*L*) and capital (*K*) inputs, respectively.

$\frac{\partial P}{\partial L}$ represents the rate of change of profit concerning labor input. It indicates how much additional profit the firm earns for each additional unit of labor used, while holding the capital constant.

$\frac{\partial P}{\partial K}$ represents the rate of change of profit concerning capital input. It indicates how much additional profit the firm earns for each additional unit of capital used, while holding the labor constant.

By analyzing these partial derivatives, the firm can determine the optimal combination of labor and capital inputs that maximizes profit. The firm would aim to set the values of *L* and *K* in such a way that both $\frac{\partial P}{\partial L}$ and $\frac{\partial P}{\partial K}$ are equal to zero or satisfy any relevant optimization constraints.

✍ *In summary, the differentiability of the profit function is crucial in determining the optimal input levels for a firm to maximize its profit and make efficient production decisions in a competitive economic environment.*

***Economic Interpretation*:** In economic mathematics, differentiability is especially relevant in cases where multiple input variables influence an economic outcome (e.g., production output, cost, utility, profit). Here are some examples of how differentiability is applied in economic analysis:

Production Functions**:** Differentiability of production functions allows economists to examine how small changes in input factors (e.g., labor and capital) impact the production output. This information is valuable for determining the optimal combination of inputs to maximize output or minimize costs.

Cost Functions: Cost functions relate the cost of production to the quantities of inputs used. Differentiability of cost functions helps economists analyze the sensitivity of costs to changes in input prices or production levels. This analysis aids firms in optimizing their production processes and minimizing costs.

Utility Functions: Utility functions represent consumer preferences and satisfaction levels for various combinations of goods and services. Differentiability of utility functions allows economists to analyze how changes in consumption impact utility and, consequently, consumer behavior and demand patterns.

Profit Functions: Profit functions in microeconomics show the relationship between output levels and profits. By examining the differentiability of profit functions, economists can determine the production level that maximizes profit for a firm.

General Equilibrium Models: In complex economic models that involve multiple interrelated markets and variables, differentiability is essential for understanding the stability and dynamics of general equilibrium.

Optimization Problems: Differentiability is fundamental for solving optimization problems in economics. For example, economists often encounter constrained optimization problems when resources or budgets are limited. By differentiating the objective function and the constraints, economists can identify optimal solutions.

✍ *In economic mathematics, differentiability of multivariable functions provides valuable insights into how small changes in input variables affect economic outcomes. It is a fundamental concept used in various economic models, optimization problems, and decision-making processes to find optimal solutions, analyze consumer behavior, and understand the behavior of economic systems.*

The connection between partial derivatives with continuity and differentiability: Partial derivatives play a significant role in both continuity and differentiability of multivariable functions in economic mathematics. Let's explore this relationship with an example related to a production function.

Example 4.2.c: In economics, the Cobb-Douglas production function is commonly used to represent the relationship between the inputs (capital, K, and labor, L) and the output (goods or services, Y) in a production process. The general form of the Cobb-Douglas production function is given as:

$$f: R^2 \to R, \qquad Y = f(L, K) = A * L^{\alpha} * K^{\beta}$$

where: *Y* represents the output (goods or services) produced, *K* is the amount of capital used in the production process, *L* is the amount of labor employed in the production process, *A* is a positive

constant representing the total factor productivity, α and β are positive constants representing the output elasticities of capital and labor, respectively.

Continuity: To determine the continuity of the Cobb-Douglas production function, we need to check if it is continuous at all points in its domain. A function is continuous if it has no breaks or jumps and can be drawn without lifting the pen. The Cobb-Douglas production function is a polynomial function with exponents α and β. Since polynomials are continuous everywhere, the Cobb-Douglas production function is continuous for all values of K and L in its domain.

Differentiability: Differentiability refers to the ability of a function to have well-defined partial derivatives at each point in its domain. For the Cobb-Douglas production function, we need to find the partial derivatives with respect to K and L:

Partial derivative with respect to K: $\frac{\partial Y}{\partial K} = \beta * L^{\alpha} * K^{\beta-1}$

Partial derivative with respect to L: $\frac{\partial Y}{\partial L} = \alpha * L^{\alpha-1} * K^{\beta}$

◘ *Both partial derivatives exist and are well-defined for all values of K and L in the domain of the Cobb-Douglas production function. Thus, the function is differentiable at all points in its domain.*

✍ *In this example of the Cobb-Douglas production function, we have seen how partial derivatives relate to both continuity and differentiability. The function is continuous because it is a polynomial function. It is differentiable because the partial derivatives with respect to K and L are well-defined and exist for all values in its domain. These properties make the Cobb-Douglas production function a useful model in economic mathematics to analyze the relationship between inputs and output in the production process and to optimize production decisions.*

The total derivative of a multivariable function: To find the total derivative of a multivariable function, you need to consider how the function changes with respect to all its variables. The total derivative accounts for both the direct changes in the function due to the changes in each variable and the indirect changes resulting from the interdependence of the variables.

Mathematical Formulation:

Let $f(x_1, x_2, \dots, x_n)$ be a multivariable function with n variables $x_1, x_2, \dots, x_n$. The total derivative df of f with respect to its variables $x_1, x_2, \dots, x_n$ is given by:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

where: df is the total derivative of f, $\frac{\partial f}{\partial x_i}$ represents the partial derivative of f with respect to variable x_i, dx_i represents the small change in variable x_i.

The formula for the total derivative can be expressed more compactly using vector notation. If $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is the vector of variables, and $\boldsymbol{\nabla f}$ is the gradient vector of f, then the total derivative df can be written as:

$$df = \boldsymbol{\nabla f} \cdot dx$$

where: df is the total derivative of f, $\boldsymbol{\nabla f}$ is the gradient vector of f, containing all the partial derivatives of f with respect to its variables, $\boldsymbol{dx} = (dx_1, dx_2, \dots, dx_n)$ is the vector of small changes in the variables $x_1, x_2, \dots, x_n$.

The vector notation simplifies the expression and is particularly useful when dealing with functions involving many variables.

Example 4.2.d: Cobb-Douglas Production Functn:

$$f: R^2 \to R, \qquad Y = f(L, K) = A * L^{\alpha} * K^{\beta}$$

Total Derivative: To find the total derivative of the production function, we need to consider how the total output Y changes with respect to both inputs L and K. The total derivative dY is given by:

$$dY = \frac{\partial Y}{\partial L} dL + \frac{\partial Y}{\partial K} dK$$

where: dL and dK are the small changes in the labor and capital inputs, respectively.

Partial Derivatives: Now, let's find the partial derivatives of the production function with respect to L and K:

$$\frac{\partial Y}{\partial L} = \alpha * L^{\alpha - 1} * K^{\beta}$$

$$\frac{\partial Y}{\partial K} = \beta * L^{\alpha} * K^{\beta - 1}$$

Interpretation: The total derivative dY represents the change in total output Y resulting from small changes in both labor dL and capital dK. The partial derivatives $\frac{\partial Y}{\partial L}$ and $\frac{\partial Y}{\partial K}$ give the direct changes in output with respect to changes in labor and capital, respectively. The total derivative also takes into account how these two inputs interact, as captured by the product of the partial derivatives and the changes dL and dK.

✍ *In economics, understanding the total derivative is crucial for analyzing production behavior, input substitution, and the impact of changes in inputs on the overall output level. It helps economists make informed decisions about resource allocation and production efficiency in various industries.*

The higher-order partial derivatives: Higher-order partial derivatives refer to taking multiple derivatives of a multivariable function with respect to the same or different variables. For example, the second partial derivative is the derivative of the first partial derivative, and the third partial derivative is the derivative of the second partial derivative, and so on.

In microeconomics, the Cobb-Douglas utility function is commonly used to model consumer preferences. It represents the relationship between the consumption of two goods, x and y, and the total utility U obtained from consuming those goods:

$$f: R^2 \to R, \;\; U = f(x, y) = x^{\alpha} y^{\beta}$$

where: *x* and *y* are the quantities of goods consumed, α and β are positive constants representing the marginal utilities of goods *x* and *y* respectively. Let's consider this economic example to illustrate higher-order partial derivatives.

Example 4.2.a: Cobb-Douglas Utility Function

First Partial Derivatives: To find the first partial derivatives of the utility function with respect to x and y, we differentiate the function with respect to each variable separately, treating the other variable as a constant:

$$\frac{\partial U}{\partial x} = \alpha * x^{\alpha-1} y^{\beta}$$
$$\frac{\partial U}{\partial y} = \beta * x^{\alpha} y^{\beta-1}$$

Second Partial Derivatives:

Now, let's find the second partial derivatives, which are derivatives of the first partial derivatives:

$$\frac{\partial^2 U}{\partial x^2} = \alpha * (\alpha - 1) * x^{\alpha-2} y^{\beta}$$
$$\frac{\partial^2 U}{\partial y^2} = \beta * (\beta - 1) * x^{\alpha} y^{\beta-2}$$

Mixed Partial Derivatives:

The mixed partial derivatives are derivatives taken with respect to one variable first and then with respect to the other. For the Cobb-Douglas utility function, they are equal due to the symmetry of the function:

$$\frac{\partial^2 U}{\partial x \partial y} = \alpha * \beta * x^{\alpha-1} y^{\beta-1}$$
$$\frac{\partial^2 U}{\partial y \partial x} = \beta * \alpha * x^{\alpha-1} y^{\beta-1}$$

◘ *In this example, the second partial derivatives give us information about the rate of change of the marginal utilities of goods x and y. The mixed partial derivatives indicate how the marginal utility of one good changes with respect to changes in the quantity of the other good.*

Higher-order partial derivatives can be used in various economic applications, such as analyzing consumer behavior, production functions, and cost functions to understand how variables interact and impact each other.

Optimization problems*: Partial derivatives* and *gradients* are essential tools in optimization problems for finding critical points and determining maximum and minimum values of multivariable functions. In economic mathematics, these concepts are often used to analyze production functions, cost functions, utility functions, and other economic models.

The critical points: The critical points of a multivariable function are the points in the domain where all the first-order partial derivatives are equal to zero or do not exist. In other words, critical points are the locations where the rate of change of the function with respect to each input variable is zero or undefined, indicating potential candidates for local extrema (maximum or minimum) or saddle points. To find the critical points of a multivariable function, follow these steps:

- Find the first-order partial derivatives of the function with respect to each variable.

- Set all the partial derivatives equal to zero and solve the resulting system of equations to find the values of the variables at the critical points.

Let the function of n-variables be given: $f(x_1, x_{2,} \dots, x_n)$. The system for searching for critical points is:

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} = 0 \\ \cdots \\ \dfrac{\partial f}{\partial x_n} = 0 \end{cases}$$

and as a special case, the case of the function of two variables:

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} = 0 \end{cases}$$

◘ *If a partial derivative is not defined at certain points, investigate those points separately as they might also be critical points*.

◘ *Once the critical points are identified, further analysis, such as using the Hessian matrix and eigenvalues, can be performed to determine the nature of each critical point.*

◘ *In some cases, it's possible to have critical points that are not local extrema or saddle points, but rather points where the function has horizontal or vertical tangents, or where the function has a cusp or a point of inflection*.

✍ *To summarize, critical points are important locations to investigate when optimizing multivariable functions, and they play a significant role in identifying local extrema or saddle points in mathematical and economic applications*.

The Hessian Matrix and it's eigenvalues: The Hessian matrix is a square matrix containing the second-order partial derivatives of a multivariable function. It plays a crucial role in optimization problems, helping to determine critical points and their nature, such as local maxima, local minima, or saddle points.

<u>The form of the Hessian matrix is as follows:</u>

$$H_{f(x_1,x_2,\dots,x_n)} = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2 \partial x_2} & & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

and as a special case, the case of the function of two variables:

$$H_{f(x_1,x_2)} = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix}$$

The eigenvalues of the Hessian matrix: To find the eigenvalues of the Hessian matrix, one needs to compute the characteristic equation, which involves subtracting a scalar variable from the diagonal elements of the Hessian matrix and taking the determinant. The eigenvalues are the solutions to this characteristic equation.

To calculate the eigenvalues for the Hessian matrix, you need to solve the characteristic equation associated with the matrix. For a given Hessian matrix H of size $n \times n$, the characteristic equation is given by:

$$det(H - \lambda I) = 0$$

where I is the identity matrix of size $n \times n$, and λ is the eigenvalue you want to find.

Steps to Calculate Eigenvalues:

Step 1: Form the Characteristic Equation: Subtract λ times the identity matrix I from the Hessian matrix H to get $H - \lambda I$. Then, calculate the determinant of this resulting matrix.

Step 2: Solve the Characteristic Equation: Set the determinant of $H - \lambda I$ equal to zero and solve for λ. This will give you the eigenvalues of the Hessian matrix.

Step 3: Repeat for Each Eigenvalue: For a $n \times n$ Hessian matrix, you will typically have n eigenvalues. Solve the characteristic equation for each value of λ to find all the eigenvalues.

Step 4: Calculate Eigenvalues: The solutions obtained for λ from the characteristic equation are the eigenvalues of the Hessian matrix.

The eigenvalues provide essential information about the local behavior of the function at the critical points. For example, if all eigenvalues are positive, the critical point is a local minimum. If all eigenvalues are negative, it is a local maximum. If the eigenvalues have both positive and negative values, the critical point is a saddle point.

In summary, the Hessian matrix and its eigenvalues are valuable tools in multivariable optimization, assisting in classifying critical points and determining the nature of extrema or saddle points in various mathematical and economic applications.

Second Derivative Test: For a function $f(x_1, x_2, \dots, x_n)$ with critical point $(x_1^*, x_2^*, \dots, x_n^n)$, the Hessian matrix evaluated at this critical point, denoted as H^*, is a symmetric matrix. The eigenvalues of H^* are real numbers, and they provide important information about the curvature of the function at that critical point.

The *Second Derivative Test* is a method used to determine the nature of critical points based on the eigenvalues of the Hessian matrix.

- If the eigenvalues of H^* are all positive, the function has a local minimum at the critical point.
- If the eigenvalues are all negative, the function has a local maximum.
- If the eigenvalues have both positive and negative values, the function has a saddle point.
- Indeterminate Cases: If any of the eigenvalues are zero, the Second Derivative Test is inconclusive, and additional analysis is required to determine the nature of the critical point.

Example 4.2.e: Profit Maximization

Consider a firm with a production function that represents the total ptofit obtained from selling x unit of good A and y units of good y. The profit function is given by function:relationship between the quantities of labor (x) and capital (y) used in the production process and the resulting output (P):

$$f: R^2 \to R, \qquad P = f(x,y) = x^3 - x + y^3 - y$$

The profit function depends on both inputs, x and y, the goal of the firm is to find the combination of x and y that maximizes profit.

Step 1: Finding Critical Points: To find the critical points, we first need to take the partial derivatives of the profit function with respect to both *L* and *K* and set them equal to zero:

$$\begin{cases} \frac{\partial P}{\partial x} = 0 \\ \frac{\partial P}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 1 = 0 \\ 3y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \mp\frac{1}{\sqrt{3}} \\ y = \mp\frac{1}{\sqrt{3}} \end{cases}$$

Solving these equations, we get the four critical pointss. So, the critical points are $(x^{*1}, y^{*1}) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $(x^{*2}, y^{*2}) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $(x^{*3}, y^{*3}) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$,. $(x^{*4}, y^{*4}) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

These equations represent the conditions for the firm to maximize profit by choosing the right combination of x and y inputs.

Step 2: Using the Gradient: The gradient of the profit function, denoted as ∇P, is a vector that points in the direction of the steepest increase of the profit function at any given point. In other words, it points towards the direction of maximum increase in profit:

$$\nabla P = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}\right) = \begin{pmatrix} 3x^2 - 1 \\ 3y^2 - 1 \end{pmatrix}$$

$$\nabla P(x^*, y^*) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

O *At the critical points, the gradient ∇P will be the zero vector, indicating that there is no direction of maximum increase or decrease in profit.*

Step 3: Second Partial Derivatives and Hessian Matrix**:** To determine whether the critical points correspond to a maximum or minimum profit, we need to use the second partial derivatives and the Hessian matrix.

Calculate the second partial derivatives:

$\frac{\partial^2 U}{\partial x^2} = 6x$

$\frac{\partial^2 U}{\partial y^2}$ =6y

$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} = 0$

Form the Hessian matrix:

$$H_{f(x,y)} = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

Step 4: Nature of Critical Points: Now, evaluate the Hessian matrix at the critical points, $H(x^*, y^*)$.

To calculate the eigenvalues for the Hessian matrix, we need to solve the characteristic equation associated with the matrix:

$$det(H - \lambda I) = 0 \quad \Rightarrow \quad det\left(\begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \quad det\begin{pmatrix} 6x - \lambda & 0 \\ 0 & 6y - \lambda \end{pmatrix} = 0$$

$$(6x - \lambda) * (6x - \lambda) = 0.$$

Calculating the eigenvalues for each critical point, we obtain the next values:

$$(x^{*1}, y^{*1}) = \left(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right): \ \left(6 * \left(-\tfrac{1}{\sqrt{3}}\right) - \lambda\right) * \left(6 * \left(-\tfrac{1}{\sqrt{3}}\right) - \lambda\right) = 0 \Rightarrow \begin{bmatrix} 6 * \left(-\frac{1}{\sqrt{3}}\right) - \lambda = 0 \\ 6 * \left(-\frac{1}{\sqrt{3}}\right) - \lambda = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 = -2\sqrt{3} \\ \lambda_2 = -2\sqrt{3} \end{bmatrix}$$

$$(x^{*2}, y^{*2}) = \left(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right): \ \left(6 * \left(-\tfrac{1}{\sqrt{3}}\right) - \lambda\right) * \left(6 * \left(\tfrac{1}{\sqrt{3}}\right) - \lambda\right) = 0 \Rightarrow \begin{bmatrix} 6 * \left(-\frac{1}{\sqrt{3}}\right) - \lambda = 0 \\ 6 * \left(\frac{1}{\sqrt{3}}\right) - \lambda = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 = -2\sqrt{3} \\ \lambda_2 = 2\sqrt{3} \end{bmatrix}$$

$$(x^{*3}, y^{*3}) = \left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right): \ \left(6 * \left(\tfrac{1}{\sqrt{3}}\right) - \lambda\right) * \left(6 * \left(-\tfrac{1}{\sqrt{3}}\right) - \lambda\right) = 0 \Rightarrow \begin{bmatrix} 6 * \left(\frac{1}{\sqrt{3}}\right) - \lambda = 0 \\ 6 * \left(-\frac{1}{\sqrt{3}}\right) - \lambda = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 = 2\sqrt{3} \\ \lambda_2 = -2\sqrt{3} \end{bmatrix}$$

$$(x^{*4}, y^{*4}) = \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right): \quad \left(6 * \left(\tfrac{1}{\sqrt{3}}\right) - \lambda\right) * \left(6 * \left(\tfrac{1}{\sqrt{3}}\right) - \lambda\right) = 0 \Rightarrow \begin{bmatrix} 6 * \left(\frac{1}{\sqrt{3}}\right) - \lambda = 0 \\ 6 * \left(\frac{1}{\sqrt{3}}\right) - \lambda = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 = 2\sqrt{3} \\ \lambda_2 = 2\sqrt{3} \end{bmatrix}$$

Use the Second Derivative Test to determine the nature of critical points based on the eigenvalues of the Hessian matrix:

- In the critical point $(x^{*1}, y^{*1}) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, the function has a local maximum, because the eigenvalues are all negative,
- In the critical point $(x^{*2}, y^{*2}) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, the function has a saddle point, because the eigenvalues have both positive and negative values,
- In the critical point $(x^{*3}, y^{*3}) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, the function has a saddle point, because the eigenvalues have both positive and negative values,
- In the critical point $(x^{*4}, y^{*4}) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, the function has a local minimum, because the eigenvalues are all positive.

✍ *In more complex cases with larger Hessian matrices, finding the eigenvalues may require numerical methods or specialized software tools, but the underlying principle remains the same—solving the characteristic equation to determine the eigenvalues.*

△ *Conclusion: By using partial derivatives, gradients, and the Hessian matrix, firms can analyze their profit functions and determine the optimal combination of labor and capital inputs that lead to maximum or minimum profits. This optimization process plays a crucial role in economic decision-making and resource allocation.*

Questions you should be able to answer

What is a multivariable function, and how does it differ from a single-variable function in the context of economic mathematics?

How do you calculate partial derivatives for a multivariable function with respect to individual input variables? How are these derivatives interpreted in an economic context?

What are higher-order partial derivatives, and how can they be useful in understanding economic relationships and behaviors?

How do you find the gradient of a multivariable function, and what does it represent in the context of economics?

What are differentials, and how are they used to approximate changes in economic variables?

How can you apply partial derivatives and differentials to analyze production functions in economics? What do the output elasticities represent in such functions?

What is the concept of the Marginal Product of Labor (MPL) and Marginal Product of Capital (MPK) in relation to production functions? How are these concepts relevant to economic decision-making?

How do you use partial derivatives to determine critical points and optimize multivariable functions in economic contexts?

How can economic applications benefit from the study of partial derivatives and differentials, and what are some specific examples of their use in economic models and analysis?

Can you provide real-world examples of economic scenarios where the concepts of partial derivatives and differentials are employed to understand economic behaviors and make informed decisions?

EXERCISES

4.2.1 In each exercise, fill in the missing words to complete the sentences related to partial derivatives and differentials for multivariable functions:

a) The _____ of a multivariable function $f(x, y)$ with respect to x is denoted by $\frac{\partial f}{\partial x}$ and measures the rate of change of *f* with respect to changes in *x*, while holding y _____.

b) For a function $f(x, y, z)$, the _____ of *f* with respect to y is denoted by $\frac{\partial f}{\partial y}$ and represents the rate of change of *f* with respect to changes in *y*, keeping *x* and *z* _____.

c) To calculate the _____ of a multivariable function $g(x, y, z)$ with respect to *z*, we use the notation $\frac{\partial g}{\partial z}$, and it shows how *g* changes as *z* changes while x and *y* _____.

d) The _____ of a multivariable function $h(x, y, z)$ with respect to *x* is denoted by *dh* and is given by $dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}\,dz$. It represents the _____ in h resulting from changes in *x*, *y*, and *z*.

e) For a function $u(x, y, z)$, the _____ differential *du* is given by $du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz$. It quantifies the approximate change in *u* resulting from small changes in *x*, *y*, and *z*, known as _____.

f) The _____ of a multivariable function $w(x, y)$ with respect to *x* and *y* is denoted by $\frac{\partial^2 w}{\partial x \partial y}$. It measures how the _____ derivative of *w* with respect to *y* changes as *x* changes.

g) The mixed partial derivatives, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$, are equal if the function $f(x, y)$ is _____ differentiable, which means the order of taking the partial derivatives doesn't _____ the result.

h) The _____ derivative $\frac{\partial^3 g}{\partial x^2 \partial y}$ of a function $g(x, y)$ represents the rate of change of the _____ derivative $\frac{\partial^2 g}{\partial x^2}$ with respect to *y*.

4.2.2 Fill in the missing words in each exercise to complete the sentences related to partial derivatives and differentials for multivariable functions in the context of economic mathematics:

a) The _____ of a multivariable function $f(x, y)$ with respect to x is denoted by $\frac{\partial f}{\partial x}$ and represents the _____ of the function with respect to changes in the variable *x* while keeping *y* constant.

b) The _____ of a multivariable function $g(x, y, z)$ with respect to z is denoted by $\frac{\partial g}{\partial z}$ and represents the _____ of the function with respect to changes in the variable *z* while keeping *x* and *y* constant.

c) The _____ of a multivariable function $h(x, y, z)$ with respect to y is denoted by $\frac{\partial h}{\partial y}$ and represents the _____ of the function with respect to changes in the variable *y* while keeping *x* and *z* constant.

d) The total _____ of a multivariable function $F(x, y, z)$ with respect to x is denoted by dF and represents the combined effect of changes in all variables *(x, y, and z)* on the function.

e) The total _____ of a multivariable function $G(x, y, z)$ with respect to *z* is denoted by *dG* and represents the combined effect of changes in all variables *(x, y, and z)* on the function.

f) The _____ rule allows us to find the partial derivative of a multivariable function by taking the derivative with respect to one variable while treating the other variables as _____.

g) In economic applications, partial derivatives are used to analyze the _____ of a function on multiple variables, such as the impact of changes in price and quantity on _____.

h) In optimization problems, partial derivatives are used to find the _____ of a multivariable function subject to certain constraints, such as maximizing profit while considering _____ limitations.

i) The concept of partial derivatives is essential in understanding the _____ of economic models that involve multiple variables, such as production functions, utility functions, and _____ functions.

j) Partial derivatives are also used in economic mathematics to calculate _____, such as marginal revenue, marginal cost, and _____ of substitution, which provide valuable insights for decision-making and analysis.

4.2.3 In each exercise, calculate the partial derivatives as indicated and use the given multivariable functions. If required, also calculate second partial derivatives ($\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}$) for the functions with three variables.:

a) Given the function $f(x, y) = 2x^2 + 3xy - y^2$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

b) Consider the function $g(x, y) = sin(x) + cos(y)$. Calculate $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

c) For the function $h(x, y) = e^{xy}$, find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

d) Suppose the function $u(x, y) = x^3 + y^3 - 3xy$. Determine $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

e) Given the function $f(x, y, z) = 2x^2 + 3xy - y^2 + 4z$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

f) Consider the function $g(x, y, z) = x^2 + y^2 + z^2$. Calculate $\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$, and $\frac{\partial g}{\partial z}$.

g) For the function $h(x, y, z) = e^{xyz}$, find $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$, and $\frac{\partial h}{\partial z}$.

h) Suppose the function $u(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. Determine $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$, and $\frac{\partial^2 u}{\partial z^2}$.

4.2.4 In each exercise, apply the concept of partial derivatives to find the given derivatives with respect to the specified variables and interpret their meanings in the context of the economic functions:

a) Consider a production function for a firm: $Q = 1000L^{0.5}K^{0.5}$, where *Q* is the quantity of output, *L* is labor input, and *K* is capital input. Find the partial derivative $\frac{\partial Q}{\partial L}$ and interpret its meaning in the context of the firm's production.

b) Suppose a utility function for a consumer is given by $U(x, y) = 2x + 3y$, where *x* is the quantity of good *X* consumed, and *y* is the quantity of good *Y* consumed. Find the partial derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ and interpret their meanings in the context of the consumer's utility.

c) Consider a Cobb-Douglas production function for a firm: $Q = 200L^{0.3}K^{0.6}M^{0.1}$, where Q is the quantity of output, *L* is labor input, *K* is capital input, and *M* is material input. Find the partial derivative $\frac{\partial Q}{\partial L}$ and interpret its meaning in the context of the firm's production.

d) Suppose a utility function for a consumer is given by $U(x, y, z) = 3x + 2y + 4z$, where *x* is the quantity of good *X* consumed, *y* is the quantity of good *Y* consumed, and *z* is the quantity of good *Z* consumed. Find the partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$, and $\frac{\partial U}{\partial z}$ and interpret their meanings in the context of the consumer's utility.

e) Consider a production function for a firm in a competitive market: $Q = 500L^{0.4}K^{0.3}H^{0.2}$, where *Q* is the quantity of output, *L* is labor input, *K* is capital input, and *H* is the quantity of high-skilled labor input. Find the partial derivatives $\frac{\partial Q}{\partial L}, \frac{\partial Q}{\partial K}$, and $\frac{\partial Q}{\partial H}$ and interpret their meanings in the context of the firm's production.

f) Suppose a utility function for a consumer is given by $U(x, y, z) = x^{0.5} + y^{0.3} + 2z^{0.2}$, where *x* is the quantity of good *X* consumed, *y* is the quantity of good *Y* consumed, and *z* is the quantity of good *Z* consumed. Find the partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$, and $\frac{\partial U}{\partial z}$ and interpret their meanings in the context of the consumer's utility.

4.2.5 In each exercise, use the definition of differentiability for multivariable functions to verify if the function is differentiable at the given point. If the function is differentiable, find the linear approximation (tangent plane) to the function at that point:

a) Consider the function $f(x, y) = 3x^2 + 2y^2$. Verify whether the function is differentiable at the point *(1, 2)*.

b) Given the function $g(x, y) = xy + 2x - 3y$, determine if the function is differentiable at the point *(2, -1)*.

c) For the function $h(x, y) = e^{xy}$, verify if the function is differentiable at the origin *(0, 0)*.

d) Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Determine whether the function is differentiable at the point $(1, -2, 3)$.

e) Given the function $g(x, y, z) = xyz + x^2 - z^2$, verify if the function is differentiable at the point *(1, 1, -1)*.

f) For the function $h(x, y, z) = ln(x^2 + y^2 + z^2)$, check whether the function is differentiable at the point *(-1, 2, -3)*.

4.2.6 In each exercise, calculate the total derivative of the given multivariable function with respect to the specified variables, and use it to estimate the change in the function for given increments in the variables at a specific point:

a) For the function $f(x, y) = 3x^2y - 2xy^2$, calculate the total derivative *df*, given that $dx = 2$ and $dy = -3$. Evaluate *df* at the point $(x, y) = (1, 2)$.

b) Given the function $g(x, y) = e^x + ln(y)$, find the total derivative *dg*. Use the total derivative to approximate the change in *g* when *x* increases by *0.1* and *y* decreases by *0.2* at the point $(x, y) = (0, 1)$.

c) For the function $h(x, y, z) = x^2 + y^2 + z^2 - 3xy$, calculate the total derivative *dh*, where $dx = 0.1, dy = -0.2$, and $dz = 0.3$. Find the total change in *h* at the point $(x, y, z) = (1, 2, -1)$.

d) Consider the function $u(x, y, z) = xyz + e^{x+y} - z^2$. Determine the total derivative *du*. Use the total derivative to estimate the change in *u* when *x* increases by *0.2*, *y* increases by *0.3*, and *z* decreases by *0.1* at the point $(x, y, z) = (0, 1, 2)$.

e) Given the function $w(x, y, z) = x^3 + 2y^2z + 3xz^2$, find the total derivative *dw*. Evaluate the total derivative at the point $(x, y, z) = (2, -1, 3)$ when $dx = 0.5, dy = -0.2$, and $dz = 0.1$.

f) For the function $f(x, y, z) = 4xy + z^3 - e^{xyz}$, compute the total derivative *df*. Use the total derivative to approximate the change in *f* when *x* increases by *0.1*, *y* decreases by *0.2*, and *z* increases by *0.3* at the point $(x, y, z) = (1, 2, 0)$.

4.2.7 In each exercise, first find the critical points of the given multivariable function by finding the partial derivatives and solving for when they are equal to zero. Then, determine the nature of each critical point by using the second partial derivative test or other appropriate methods:

a) Consider the function $f(x,y) = x^2 + 2y^2 - 4x - 6y + 8$: *a.1)* Find the critical points of the function $f(x,y)$; *a.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

b) For the function $g(x,y) = xy + 2x^2 - 3y^2$, follow these steps: *b.1)* Find the critical points of the function $g(x,y)$; *b.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

c) Consider the function $h(x,y,z) = x^3 + y^3 + z^3 - 6xyz$: *c.1)* Find the critical points of the function $h(x,y,z)$; *c.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

d) For the function $w(x,y,z) = x^2 + 2y^2 + 3z^2 - 2xy - 4yz - 6xz$, follow these steps: d.*1)* Find the critical points of the function $w(x,y,z)$; *w.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

e) Consider the function $u(x,y,z) = x^2 - y^2 + 2z^2 + 4x - 2y - 6z + 1$: *e.1)* Find the critical points of the function $u(x,y,z)$; *e.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

f) For the function $f(x,y) = x^3 - 3xy + 4y^3$, follow these steps: *f.1)* Find the critical points of the function $f(x,y)$; *f.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

g) Consider the function $g(x,y,z) = x^2 + y^2 + z^2 + 2xyz$: *g.1)* Find the critical points of the function $g(x,y,z)$; *g.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

h) For the function $h(x,y) = xy^2 - 2x^2y + x^3$, follow these steps: *h.1)* Find the critical points of the function $h(x,y)$; *h.2)* Determine the nature (maximum, minimum, or saddle point) of each critical point.

4.3 DERIVATIVES OF COMPOSITE AND IMPLICIT FUNCTIONS IN MULTIVARIABLE CALCULUS

Keywords: *Derivative, Composite Function, Chain Rule, Implicit Function, Multivariable Calculus, Partial Derivative, Total Derivative, Differentiability, Tangent Plane, Implicit Differentiation, Multivariate Chain Rule, Jacobian Matrix, Directional Derivative, Gradient, Critical Points, Extrema, Lagrange Multipliers, Implicit Function Theorem, Hessian Matrix.*

In economic mathematics, understanding derivatives of composite and implicit functions in multivariable calculus is essential for analyzing economic models and making informed decisions.

Here are some key concepts related to this topic:

Derivatives of composite functions in multivariable calculus: In economic mathematics, composite functions often arise when multiple economic variables are interrelated. If you have

functions like $f(x)$ and $g(x)$, then the composite function $(g \circ f)(x) = g\big(f(x)\big)$ represents the combination of these functions, where the output of $f(x)$ serves as the input for $g(x)$. Understanding how to find the derivative of composite functions is crucial for optimizing economic models that involve multiple variables.

In economics, composite functions are frequently used to model interrelated economic variables or processes. Let's take an example of a composite function in economics related to consumption and income.

The concept of Composition of Functions: Let $f: R^3 \to R$ be a function of the form $w = f(x, y, z)$. That is, w is the only dependent variable (output), and x, y, z are the independent variables (inputs). Next, let $g: R^2 \to R^3$ be a function of the form $g(s,t) = (x(s,t), y(s,t), z(s,t))$. In this case, x, y, z are the dependent variables, and s, t are the independent variables. The components of $g(s,t)$, $x(s,t), y(s,t)$ and $z(s,t)$, are called the component functions of the vector-valued function g.

It follows that the composition of f with g is a new function h, defined by:

$$w = h(s,t) = (f \circ g)(s,t) = f\big(g(s,t)\big) = f(x(s,t), y(s,t), z(s,t))$$

Please observe that the function h is identical to the outer function f, with its independent variables corresponding to those of the inner function g.

Example 4.3.a: Let $f(x,y,z) = x^2 + y^2 + z^2$ and let
$g(s,t) = (x(s,t), y(s,t), z(s,t)) = (e^8 cost, e^8 sint, t)$.

Then

$h(s,t) = f\big(g(s,t)\big) = f(x(s,t), y(s,t), z(s,t)) = \big(x(s,t)\big)^2 + \big(y(s,t)\big)^2 + \big(z(s,t)\big)^2 =$
$(e^8 cost)^2 + (e^8 sint)^2 + t^2 = e^{16} + t^2$.

Example 4.3.b: Consumption Function: In economics, the consumption function represents the relationship between disposable income (Y) and the amount of consumption (C) in an economy. It is often assumed to be a composite function of the form $C(Y) = f(g(Y))$, where $g(Y)$ represents the disposable income, and $f(g(Y))$ represents the corresponding consumption level.

Let's assume the following consumption function:

$$C(Y) = 0.8 * Y$$

In this example, the consumption function is linear, where consumption (C) is equal to 80% (0.8) of disposable income (Y). The composite function $C(Y)$ combines the disposable income (Y) as an input to the function $g(Y) = 0.8 * Y$, and the output of this function is used as an input to the function $f(x) = x$.

Now, let's see how this consumption function can be used in an economic context:

Suppose we have an economy where the disposable income of households (Y) is *$10,000*. We can use the consumption function to determine the corresponding level of consumption (C):

$$C(Y) = 0.8 * \$10{,}000 \Rightarrow C(Y) = \$8{,}000$$

According to the consumption function, when disposable income is $10,000, households will consume $8,000. This implies that 80% of the disposable income is spent on consumption.

O *The consumption function is essential in economics because it helps to understand how changes in income influence consumption behavior. By using the concept of composite functions,*

economists can model and analyze complex relationships between various economic variables and study their impact on the overall economic system.

***Derivatives of composition of functions in multivariable calculus*:** In multivariable calculus, derivatives of composite functions involve finding the rate of change of a function that depends on multiple variables, where each variable can, in turn, depend on other variables. The Chain Rule is a fundamental concept used to compute these derivatives.

***Derivatives of compound functions using the Gradient*:** Let $f: R^3 \to R$ be a function of the form $w = f(x, y, z)$. That is, w is the only dependent variable (output), and x, y, z are the independent variables (inputs). Next, let $g: R^2 \to R^3$ be a function of the form $(x(s,t), y(s,t), z(s,t)) = g(s,t)$. In this case, x, y, z are the dependent variables, and s, t are the independent variables. The components of $g(s,t)$, witc are $x(s,t), y(s,t)$ and $z(s,t)$, are called the component functions of the vector-valued function g.

It follows that the composition of f with g is a new function h, defined by:

$$w = h(s,t) = (f \circ g)(s,t) = f\big(g(s,t)\big) = f(x(s,t), y(s,t), z(s,t))$$

Because f has one output (dependent variable) and three inputs (independent variables), its derivative, or Jacobian Matrix, has one row and three columns:

$$J_f(x,y,z) = \nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

In general, when a function f has only one output, the derivative of f is known as the gradient of f, wich is denoted by ∇f.

Because g has three outputs and two inputs, its derivative has three rows and two columns:

$$J_g(s,t) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial s} & \frac{\partial g}{\partial t} \end{pmatrix}$$

Note that each row is a gradient of one of the component functions of g, and each column is a derivative of g with respect to one of the independent variables.

Example 4.3.c: Let $f(x,y,z) = x^2 + y^2 + z^2$ and let $g(s,t) = (x(s,t), y(s,t), z(s,t)) = (e^8 cost, e^8 sint, t)$.

The derivatives of f and g are:

$$J_f(x,y,z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} = (2x \;\; 2y \;\; 2z)$$

$$J_f(x,y,z) = \begin{pmatrix} \nabla x(s,t) \\ \nabla y(s,t) \\ \nabla z(s,t) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} e^8 cost & -e^8 sint \\ e^8 sint & e^8 cost \\ 0 & 1 \end{pmatrix}$$

Derivatives of compound functions using the Chain Rule: Let $f: R^3 \to R$ be a function of the form $w = f(x,y,z)$. That is, w is the only dependent variable (output), and x, y, z are the independent variables (inputs). Next, let $g: R^2 \to R^3$ be a function of the form $g(s,t). = (x(s,t), y(s,t), z(s,t))$. In this case, x, y, z are the dependent variables, and s, t are the independent variables. The components of $g(s,t)$, witch are $x(s,t), y(s,t)$ and $z(s,t)$, are called the component functions of the vector-valued function g.

Using the multivariable Chain Rule, $J_{f^\circ \boldsymbol{g}} = \big(J_f{}^\circ g\big) * J_{\boldsymbol{g}}$, the derivative of $h(s,t) = f\big(g(s,t)\big)$ is

$$J_h(s,t) = \nabla f(x(s,t), y(s,t), z(s,t)) * J_{\boldsymbol{g}}(s,t) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} * \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix}$$

Because the outer function f has one output and the inner function g has two inputs, the derivarive of $h = f^\circ g$, which has one output and two inputs, has one row and two columns.

Each entry of this product can be expressed as a dot product. Recall that the dot product of two vectors $\boldsymbol{u} = (u_1 \ u_2 \ \dots \ u_n)$ and $\boldsymbol{v} = (v_1 \ v_2 \ \dots \ v_n)$ is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

We then have: $J_h(s,t) = \Big(\nabla f \cdot \frac{\partial \boldsymbol{g}}{\partial s}, \quad \nabla f \cdot \frac{\partial \boldsymbol{g}}{\partial t}\Big)$.

In general, to obtain the entry in row i, column j of the derivative of h, we take the dot product of row i of the derivative of the outer function f with column j of the derivative of the inner function g.

In this scenario, calculating these dot products results in

$$J_h(s,t) = \begin{pmatrix} \frac{\partial h}{\partial s} & \frac{\partial h}{\partial t} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{pmatrix} & \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} * \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} * \frac{\partial z}{\partial s} & \frac{\partial f}{\partial x} * \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} * \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} * \frac{\partial z}{\partial t} \end{pmatrix}$$

Example 4.3.d: Using the functions from the preceding examples, we obtain

$$\frac{\partial h}{\partial s} = \nabla f \cdot \frac{\partial \boldsymbol{g}}{\partial s} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{pmatrix} = \begin{pmatrix} 2x(s,t) & 2y(s,t) & 2z(s,t) \end{pmatrix} \cdot \begin{pmatrix} e^8 cost \\ e^8 sint \\ 0 \end{pmatrix} = 2 * x(s,t) *$$

$$e^8 cost + 2 * y(s,t) * e^8 sint + 2 * z(s,t) * 0 = 2 * e^{16} cos^2 t + 2 * e^{16} sin^2 t = 2 * e^{16}.$$

Similarly

$$\frac{\partial h}{\partial t} = \nabla f \cdot \frac{\partial \boldsymbol{g}}{\partial t} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} 2x(s,t) & 2y(s,t) & 2z(s,t) \end{pmatrix} \cdot \begin{pmatrix} -e^8 sint \\ e^8 cost \\ 1 \end{pmatrix} = -2 * x(s,t) *$$

$$e^8 sint + 2 * y(s,t) * e^8 cost + 2 * z(s,t) * 1 = -2 * e^{16} * cost * sint + 2 * e^{16} * sint * cost + 2t = 2t.$$

So, the derivatives of compound functions using the Chain Rule is

$$J_h(s,t) = (2e^{16} \quad 2t)$$

✪ *The Chain Rule allows economists to understand how changes in one variable (e.g., capital) affect another variable (e.g., output) in complex economic relationships, making it a powerful tool for analyzing economic models and decision-making processes.*

Derivatives of implicit functions in multivariable calculus: In economic mathematics, implicit functions are commonly encountered when there is no explicit expression for one variable in terms of another. Instead, there is an implicit relationship between the variables. For example, the demand and supply functions in an economic market are often represented implicitly. Finding the derivatives of implicit functions is crucial for understanding how changes in one variable affect another.

The concept of Implicit Functions: In multivariable calculus, an implicit function is a function defined implicitly by an equation that relates multiple variables without explicitly expressing one variable in terms of the others. In other words, the dependent and independent variables are not explicitly isolated on opposite sides of the equation, making it difficult to solve explicitly for one variable in terms of the others.

The general form of an implicit equation involving $n - variables$, $x_1, x_2, \dots, x_n$, is often written as:

$$F(x_1, x_2, \dots, x_n) = 0$$

Here, F is a function of $n - variables$, and the equation represents the implicit relationship between $x_1, x_2, \dots, x_n$.

Example 4.3.e: Examples of implicit functions in economic mathematics:

- Isoquants: In production theory, isoquants represent combinations of inputs (such as capital and labor) that yield the same level of output. An isoquant is described by an implicit function, where the production function is set equal to a constant level of output.

Consider a Cobb-Douglas production function $Q = A * K^{\alpha} * L^{\beta}$, where Q represents output, K is capital, L is labor, and A, α, and β are parameters. The isoquant equation for a constant output level Q_0 would be: $A * K^{\alpha} * L^{\beta} = Q_0$. This equation implicitly defines the combinations of capital and labor that produce the same level of output Q_0.

- Indifference Curves: In microeconomics, indifference curves represent different combinations of goods that provide the same level of utility or satisfaction to a consumer. The indifference curve equation is an implicit function relating the quantities of two goods.

Let's assume a consumer has a utility function $U(X,Y) = X^{\alpha} * Y^{\beta}$, where X and Y are the quantities of two goods, and α and β are parameters. An indifference curve equation representing a constant level of utility U_0 would be: $X^{\alpha} * Y^{\beta} = U_0$. This equation implicitly defines the combinations of goods that yield the same level of utility U_0.

- Production Possibility Frontier (PPF): The PPF represents the maximum possible output that an economy can produce given its resources and technology. The PPF is described by an implicit function involving the quantities of different goods produced.

Consider an economy that can produce two goods, X and Y. The PPF equation would be $X + Y = C$, where C represents the total amount of resources available in the economy. This equation implicitly defines the combinations of goods X and Y that exhaust all available resources.

- Phillips Curve: The Phillips Curve shows the inverse relationship between inflation and unemployment in an economy. It is described by an implicit equation involving the inflation rate and the unemployment rate.

A common Phillips Curve equation is $\pi = \pi_0 - \alpha * u$, where π represents the inflation rate, u is the unemployment rate, π_0 is the expected inflation rate, and α is a constant parameter. This equation implicitly shows the trade-off between inflation and unemployment levels.

✍ *These examples illustrate how implicit functions are used in economic mathematics to model complex relationships and analyze various economic phenomena. By leveraging implicit differentiation, economists can derive important insights and make informed decisions in economic analysis and policy-making.*

Derivatives of Implicit Functions (method 1): In such situations, we may wish to know how to compute the partial derivatives of one of the variables with respect to the other variables. To do so, we have to do something quite subtle. On one hand, we want to treat the variables as independent in order to find the partial derivatives of the function F. On the other hand, we want to take into account the dependence of the variables on one another, via the equation $F(x,y,z) = 0$.

To find derivatives of implicit functions, we use implicit differentiation. The process involves differentiating both sides of the equation with respect to the independent variable (usually denoted by t or some other letter). This technique allows us to find the rate of change of one variable concerning another variable without explicitly solving for one variable in terms of the others.

The steps for implicit differentiation are as follows:

Step 1: Differentiate both sides of the equation with respect to the independent variable (e.g., t).

Step 2: When differentiating terms that involve the dependent variable (e.g., y), use the chain rule, which treats y as a function of x and applies the derivative dy/dx.

Example 4.3.f: Let's consider an example of an implicit function involving three variables in economic mathematics, Cobb-Douglas Production Function. The Cobb-Douglas production function is a widely used model in economics to represent the relationship between inputs and outputs in production. It is an example of an implicit function involving three variables: capital (K), labor (L), and output (Q). The Cobb-Douglas production function can be represented as follows:

$$Q = A * K^{\alpha} * K^{\beta}$$

where: Q is the quantity of output produced, K is the amount of capital used in production, L is the amount of labor employed, A is a positive constant representing total factor productivity, α and β are constants representing the output elasticities of capital and labor, respectively.

The Cobb-Douglas production function implicitly relates the three variables (Q, K, and L) through the power exponents α and β.

Now, suppose we want to find the implicit partial derivatives of the Cobb-Douglas production function with respect to K and L. These partial derivatives measure the rate of change of output (Q) concerning changes in capital (K) and labor (L) while keeping the other variable constant.

To find the implicit partial derivatives, we differentiate the production function with respect to K and L, treating Q as a function of K and L. The partial derivatives are represented as $\frac{\partial Q}{\partial K}$ and $\frac{\partial Q}{\partial L}$.

Step 1: Take the partial derivative with respect to K:

$$\frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}(A * K^{\alpha} * L^{\beta})$$

Therefore,

$$\frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}(A * \alpha * K^{\alpha-1} * L^{\beta})$$

Step 2: Take the partial derivative with respect to L:

$$\frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}(A * \beta * K^{\alpha} * L^{\beta-1})$$

The implicit partial derivatives $\frac{\partial Q}{\partial K}$ and $\frac{\partial Q}{\partial L}$ give us valuable information about how output changes concerning changes in capital and labor, respectively, in the Cobb-Douglas production function. These derivatives are essential for economic analysis, such as understanding factor productivity, input substitution, and optimizing production processes.

Derivatives of Implicit Functions (method 2): The chain rule says that if F is a function of 'old' variables x, y, z, each of which is a function of 'new' variables s, t, then

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s} = 0$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t} = 0$$

The mind-warping element of implicit differentiation is that we're going to take s and t to be two of the three of x, y, z. (Doing so is not just a cosmetic difference, it affects the differentiation!) Indeed, the 'old' variables will be x, y, z treated as if they're independent, from the perspective of someone looking at the function without the information that $F(x, y, z) = 0$. The 'new' variables

will be whatever's left after we choose to consider one as a function of the other two, in such a way that the equation $F(x,y,z) = 0$ is satisfied. Hence the $\frac{\partial F}{\partial s}$ above will always be zero, because after we take the dependence $F(x,y,z)$ into account, we're left with a constantly zero function!

How it's done. Suppose x, y, z are related by $F(x,y,z) = 0$ and you want to compute $\frac{\partial z}{\partial x}$. To do this, we treat z as if it were a function of x and y. Thus, in the language of the chain rule: The 'old' variables are x, y, z, treated as being completely independent; The 'new' variables are just x and y, because we want to treat z as a function of x and y. In the chain rule, when we differentiate with respect to the 'old' variables we do so as if they had no dependence on the 'new' variables. To distinguish between 'old' vs 'new', write $s = x$ and $t = y$ when they're considered as new variables. Thus

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s} = 0$$

Now, $x = s$, so $\frac{\partial x}{\partial s} = 1$. And $y = t$, so $\frac{\partial y}{\partial s} = 0$. So, really we have $\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}$.

We can then write s as x once again to obtain

$$\frac{\partial z}{\partial s} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Simillary

$$\frac{\partial z}{\partial t} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

This result is known as the implicit function theorem.

Example 4.3.g: Suppose x, y, z are variables related by the equation $x^4 + y^4 + z^4 + x^2y^2z^2 = 0$, and that we want to finde $\frac{\partial y}{\partial z}$.

We thus treat y as a function of x and z. So, the „old" variables are x, y, z and the „new" variables are x, z with $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 1$.

Write $F(x,y,z) = x^4 + y^4 + z^4 + x^2y^2z^2$ for the function where x, y, z are comsidered unrelated. Then

$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial z} = 0 \Rightarrow \frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} * 0 + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} * 1 = 0 \Rightarrow \frac{\partial F}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} \Rightarrow \frac{\partial y}{\partial z} = -\frac{4z^3 + 2x^2y^2z}{4y^3 + 2x^2yz^2} = -\frac{z(2z^2 + x^2y^2)}{y(2y^2 + x^2z^2)}$$

✍ *Implicit differentiation is a powerful tool in multivariable calculus, particularly when dealing with complex relationships where it may be challenging to solve explicitly for one variable in terms of the others. It is commonly used in economics, physics, engineering, and other fields where implicit relationships between variables are prevalent.*

Total Derivatives: Total derivatives take into account changes in all variables of an economic model simultaneously. They provide a complete picture of how the entire system responds to changes in the variables, considering both direct and indirect effects.

The concept of total derivatives: In multivariable calculus, the concept of total derivatives is crucial when dealing with functions of multiple variables. A total derivative quantifies the variation of a function with respect to all of its variables, considering both direct and indirect influences. It represents the combined effect of small changes in all variables on the function's output.

The total derivative of a multivariable function $f(x_1, x_2, \ldots, x_n)$ with respect to an independent variable t (often representing time) is denoted by $\frac{df}{dt}$. It is also sometimes referred to as the total differential.

The total derivative can be calculated using partial derivatives. For a function of *n-variables*, the total derivative $\frac{df}{dt}$ is given by the sum of the partial derivatives of f with respect to each variable multiplied by the corresponding rate of change of that variable with respect to t.

The total derivative can be written as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} * \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} * \frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n} * \frac{dx_n}{dt}$$

where $\frac{df}{dt}$ is the total derivative of f with respect to t, $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ are the partial derivatives of f with respect to each variable $x_1, x_2, \ldots, x_n$, respectively, $\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}$ are the rates of change of each variable $x_1, x_2, \ldots, x_n$ with respect to t, respectively.

The total derivative provides valuable information about how the function responds to changes in all of its variables simultaneously. It is particularly useful when dealing with complex functions involving multiple input factors, such as economic models.

Total derivatives in multivariable calculus refer to finding the total rate of change of a function concerning all its variables. It takes into account both the direct effects of each variable's change and the indirect effects due to the variables' interdependencies. Total derivatives are essential in economic mathematics for understanding how changes in multiple economic variables simultaneously impact the overall economic system.

Example 4.3.h: Cost Function in Economics: Let's consider a cost function $C(q, w, r)$ that represents the total cost (C) of producing a certain quantity of output (q) given the prices of labor (w) and capital (r). The cost function may be defined as:

$$C(q, w, r) = w * L(q) + r * K(q)$$

where: C is the total cost of production, q is the quantity of output produced, w is the price of labor per unit, r is the price of capital per unit, L(q) is the amount of labor used in producing q units of output, K(q) is the amount of capital used in producing q units of output.

Now, we want to find the total derivative $\frac{\partial C}{\partial q}$, which represents the change in total cost concerning changes in the quantity of output q while accounting for the indirect effects of changes in labor and capital.

Total Derivative: The total derivative of $C(q, w, r)$ with respect to q is given by:

$$\frac{dC}{dq} = \frac{\partial C}{\partial q} + \frac{\partial C}{\partial w} * \frac{dw}{dq} + \frac{\partial C}{\partial r} * \frac{dr}{dq}$$

Step 1: Find the partial derivatives:

$\frac{\partial C}{\partial q}$ represents the direct effect of changes in q on the total cost. It is the partial derivative of C with respect to q.

$\frac{\partial C}{\partial w}$ represents the direct effect of changes in the price of labor on the total cost. It is the partial derivative of C with respect to w.

$\frac{\partial C}{\partial r}$ represents the direct effect of changes in the price of capital on the total cost. It is the partial derivative of C with respect to r.

Step 2: Find the derivatives of labor and capital with respect to q:

$\frac{dw}{dq}$ represents the indirect effect of changes in q on the price of labor. It is the derivative of w with respect to q.

$\frac{dr}{dq}$ represents the indirect effect of changes in q on the price of capital. It is the derivative of r with respect to q.

Step 3: Substitute the values and evaluate: Once you have the partial derivatives and the derivatives of labor and capital with respect to q, you can substitute the values into the total derivative formula and evaluate it for specific values of q, w, and r to find the total change in the total cost concerning changes in the quantity of output q.

⭘ *Understanding total derivatives is crucial for analyzing economic functions that involve multiple interrelated variables. In this cost function example, the total derivative helps economists understand how changes in output quantity affect the overall production cost, considering the prices of labor and capital. This information is valuable for cost optimization, profit maximization, and decision-making in economic production processes.*

✍ *In economic mathematics, the concept of total derivatives is essential for understanding how economic variables interact and influence one another. For example, in macroeconomic models, the total derivative of the GDP function with respect to time can help economists understand the overall growth or contraction of the economy due to changes in various factors like consumption, investment, government spending, and net exports.*

Economic Applications: Understanding derivatives of composite and implicit functions is essential for various economic applications, such as cost and production functions, utility maximization, profit maximization, and optimization problems in general.

Sensitivity Analysis: Derivatives play a crucial role in sensitivity analysis in economic models. Sensitivity analysis helps economists understand how changes in input variables impact the model's output and aids in making informed policy decisions.

△ *Overall, mastering the derivatives of composite and implicit functions in multivariable calculus is essential for economists to model complex economic systems accurately and analyze their behavior under different conditions.*

Questions you should be able to answer:

What are composite functions in multivariable calculus, and how are they represented mathematically?

What is the Chain Rule, and how is it used to find derivatives of composite functions?

How do you find the derivative of a composite function with respect to one of its variables?

Provide an example of an economic application where the Chain Rule is used to find the derivative of a composite function.

What are implicit functions in multivariable calculus, and how are they different from explicit functions?

What is implicit differentiation, and how is it used to find derivatives of implicit functions?

How do you calculate partial derivatives, and what do they represent in multivariable calculus?

Provide an example of an economic application involving a function with implicit differentiation.

How do you find the total derivative of a multivariable function, and what does it represent in the context of economic mathematics?

Explain the importance of understanding derivatives of composite and implicit functions in analyzing economic models and making informed decisions in economics.

How do you interpret the results of derivatives of composite and implicit functions in real-world economic scenarios?

In what situations would you choose to use the Chain Rule versus implicit differentiation to find derivatives in economic mathematics?

EXERCISES

4.3.1 In the context of "Derivatives of Composite and Implicit Functions in Multivariable Calculus.", fill in the missing words in each sentence:

a) The derivative of a composite function is computed using the _____ rule, which states that the derivative of the outer function multiplied by the derivative of the inner function.

b) In the chain rule, if $y = f(g(x))$, then $\frac{dy}{dx} =$ _____.

c) For implicit functions, to find $\frac{dy}{dx}$, we use the _____ with respect to *x*.

d) To apply implicit differentiation to a given equation, differentiate both sides of the equation with respect to _____.

e) In multivariable calculus, the partial derivative of a function of two or more variables is taken with respect to _____ variable, keeping all other variables constant.

f) The partial derivative of a function $f(x, y)$ with respect to *x* is denoted by _____.

g) When taking the derivative of an implicit function, if *y* is a function of *x*, then $\frac{\partial y}{\partial x}$ is denoted as _____.

h) The chain rule can be extended to functions of more than two variables using the _____.

i) In multivariable calculus, the total derivative of a function $f(x, y, z)$ is given by _____.

j) When differentiating a function of two variables, such as $z = f(x, y)$, the notation $\frac{\partial z}{\partial x}$ represents the _____ partial derivative.

4.3.2 Fill in the missing words in each sentence:

a) The production function for a firm is given by $Q = 5K^2L + 3K$, where *Q* represents the quantity of output, *K* is capital, and *L* is labor. The _____ of *Q* with respect to *K* is $\frac{dQ}{dK} =$ _____.

b) In an economic model, the demand function for a product is represented as $Q = 2000 - 10P$, where *Q* is the quantity demanded, and *P* is the price. The _____ of *Q* with respect to *P* is $\frac{dQ}{dP} =$ _____.

c) Consider a utility function $U = 2x^2 + 3y^3$, where *U* is the utility of a consumer from consuming goods *x* and *y*. The _____ of *U* with respect to *x* is $\frac{dU}{dx} =$ _____.

d) In a cost function analysis, the cost of production *C* is given as $C = 1000 + 5Q^2$, where *Q* represents the quantity produced. The _____ of *C* with respect to *Q* is $\frac{dC}{dQ} =$ _____.

e) The profit function for a company is given by $\pi = TR - TC$, where π represents profit, *TR* is total revenue, and *TC* is total cost. The _____ of π with respect to *Q* (quantity produced) is $\frac{d\pi}{dQ} =$ _____.

f) An economist is analyzing a Cobb-Douglas production function, given by $Q = 10L^{0.5}K^{0.5}$, where *Q* is the output, L is labor, and *K* is capital. The _____ of *Q* with respect to *L* is $\frac{dQ}{dL} =$ _____.

g) In an economic model, the supply function for a product is represented as $Q = 500 + 8P$, where *Q* is the quantity supplied, and *P* is the price. The _____ of *Q* with respect to *P* is $\frac{dQ}{dP} =$ _____.

h) Consider a utility function $U = x^2 + 2xy + y^2$, where *U* is the utility of a consumer from consuming goods *x* and y. The _____ of *U* with respect to *y* is $\frac{dU}{dy} =$ _____.

4.3.3 Finding the derivatives of composite functions by applying the chain rule in Multivariable Calculus. Perform the calculations step-by-step, and simplify the results where necessary.

a) Given the functions $f(x) = x^2$ and $g(x) = e^x$, find the derivative of the composite function $(g \circ f)(x)$ and evaluate it at $x = 1$.

b) Consider the functions $f(u,v) = u^3 + v^2$ and $g(x) = sin(x)$. Find the derivative of the composite function $(f \circ g)(x)$ and simplify the result.

c) Let $f(x,y) = x^2 + y^3$ and $g(t) = t^2 - 1$. Find the partial derivatives $\frac{\partial(f \circ g)}{\partial t}$ and $\frac{\partial(f \circ g)}{\partial x}$.

d) Given the functions $f(u) = u^2 + 1$ and $g(x,y) = 2xy$, find the derivative of the composite function $(f \circ g)(x,y)$ with respect to *x*.

e) Consider the functions $f(x) = ln(x)$ and $g(u,v) = u^2 - v^2$. Find the partial derivative $\frac{\partial(g \circ f)}{\partial x}$.

f) Given the functions $f(x,y) = x^2 + y^2$ and $g(t) = e^t$, find the derivative of the composite function $(f \circ g)(t)$ and evaluate it at $t = 0$.

g) Let $f(u,v) = u^3 + v^3$ and $g(x) = 3x$. Find the derivative of the composite function $(f \circ g)(x)$ and simplify the result.

h) Consider the functions $f(x) = sin(x)$ and $g(u,v) = u + v$. Find the derivative of the composite function $(g \circ f)(x)$ and evaluate it at $x = \frac{\pi}{2}$.

i) Given the functions $f(x) = e^x$ and $g(u,v) = u^2 + v^2$, find the derivative of the composite function $(g \circ f)(x)$.

j) Let $f(x,y) = x^2 + y^3$ and $g(t) = 2t + 1$. Find the partial derivatives $\frac{\partial(f \circ g)}{\partial t}$ and $\frac{\partial(f \circ g)}{\partial x}$.

4.3.4 Finding the derivatives of implicit functions in multivariable calculus. Perform the calculations step-by-step and simplify the results where necessary:

a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if the equation of the implicit function is $x^2 + y^2 + z^2 = 25$.

b) For the implicit function $x^2 - xy + y^2 + z^2 = 9$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

c) Consider the equation $x^2 + y^3 - 2xy + z = 7$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

d) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit function $e^z = xy + x^2$.

e) For the implicit function $x^3 - 3x^2y + y^3 + z^2 = 10$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

f) Consider the equation $x^2 + y^2 + z^3 - 2xyz = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

g) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit function $ln(z) = x^2 + y^2$.

h) For the implicit function $x^3 + y^3 + z^3 = 6xyz$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

i): Consider the equation $e^x + e^y - z = x^2 + y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

j) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit function $sin(z) = xy + x^2 - y^2$.

Note: *In each exercise, apply the chain rule step-by-step to find the partial derivatives* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *of the given implicit function. Simplify the results as necessary. The chain rule involves*

differentiating each term with respect to x and y, and you may need to use implicit differentiation techniques.

4.4 THE DIRECTIONAL DERIVATIVE AND GRADIENT OF MULTIVARIABLE FUNCTIONS

Keywords*: Multivariable Functions, Partial Derivatives, Chain Rule, Directional Derivative, Gradient, Directional Vector, Unit Vector, Tangent Plane, Rate of Change, Directional Gradient, Directional Derivative Formula, Differentiability, Vector Calculus, Level Curves, Level Surfaces, Jacobian Matrix, Directional Derivative in the Direction of Maximum Increase, Directional Derivative in the Direction of Maximum Decrease, Hessian Matrix, Second Partial Derivatives.*

The directional derivative of a multivariable function measures the rate of change of the function concerning a specific direction or vector in the input space. It tells us how the function changes as we move along a given direction from a particular point. The directional derivative is a powerful tool in multivariable calculus and has applications in various fields, including economics.

Mathematical concept of the directional derivative: Let's consider a multivariable function $f(x, y, z)$ that depends on three variables *x*, *y*, and *z*. The directional derivative of f at a point (x_0, y_0, z_0) in the direction of a unit vector $\boldsymbol{v} = (a, b, c)$ is denoted by $D_v f(x_0, y_0, z_0)$, and it is defined as:

$$D_v f(x_0, y_0, z_0) = \boldsymbol{\nabla f}(x_0, y_0, z_0) \cdot v$$

where: $\boldsymbol{\nabla f}(x_0, y_0, z_0)$ is the gradient vector of f at the point (x_0, y_0, z_0), given by $\boldsymbol{\nabla f} = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, $\cdot$ denotes the dot product between the gradient vector and the unit direction vector **v**.

Definition: The directional derivative represents the slope of the tangent line to the graph of the function f in the direction of $\boldsymbol{v}$ at the point (x_0, y_0, z_0).

Key points to note about directional derivatives*:*

- The directional derivative exists only if the partial derivatives of f exist at the point (x_0, y_0, z_0).
- The direction vector $\boldsymbol{v}$ must be a unit vector (i.e., its magnitude must be 1) to obtain the correct directional derivative.
- The directional derivative is maximum when the direction vector $\boldsymbol{v}$ is aligned with the gradient vector $\boldsymbol{\nabla f}$, and it is minimum when $\boldsymbol{v}$ is opposite to $\boldsymbol{\nabla f}$.
- The directional derivative can be used to find the maximum rate of change of the function at a given point and direction.

Applications in Economics: In economics, directional derivatives are used in various applications, such as production optimization, cost minimization, and utility maximization. For instance:

- In production optimization, directional derivatives can help determine the most efficient direction to increase output given the constraints of inputs.
- In cost minimization, directional derivatives can be used to find the direction in which a firm should adjust its input usage to minimize production costs.

➢ In utility maximization, directional derivatives can help determine the direction of change in consumption to maximize utility, subject to budget constraints.

✍ *By calculating and understanding directional derivatives, economists can gain valuable insights into how economic functions change concerning specific directions and make informed decisions for optimization and resource allocation in various economic scenarios.*

Explanation of the significance of the gradient vector with regard to direction of change along a surface: The gradient vector is a crucial concept in multivariable calculus, especially concerning the direction of change along a surface. The gradient vector points in the direction of the steepest increase of a function at a specific point. It provides important information about how the function changes concerning each input variable and in which direction the function increases most rapidly.

In the context of a surface represented by a function $f(x, y)$, the gradient vector $\nabla f(x, y)$ at a point (x, y) provides the direction of the steepest increase of the function *f*. Its magnitude represents the rate of change or slope of the function in that direction.

Key Significance of the Gradient Vector:

➢ Direction of Steepest Ascent: The gradient vector points in the direction of the steepest ascent or increase of the function. If you move along this direction, the function will increase most rapidly at that point. Conversely, the negative gradient points in the direction of the steepest descent, where the function decreases most rapidly.

➢ Orthogonality to Level Curves: The gradient vector is orthogonal (perpendicular) to the level curves of the function. Level curves are curves along which the function takes constant values. This means that at any point on a level curve, the gradient vector points directly outward, perpendicular to the curve.

➢ Magnitude and Rate of Change: The magnitude of the gradient vector $\|\boldsymbol{\nabla f}(x, y)\|$ represents the maximum rate of change of the function f at the point (x, y). The larger the magnitude, the steeper the slope and the faster the function is changing at that point.

➢ Directional Derivative: The gradient vector is directly related to the directional derivative of the function. The directional derivative in the direction of a unit vector $\boldsymbol{v}$ is given by the dot product of the gradient vector and the unit direction vector: $D_{\boldsymbol{v}} f(x, y) = \boldsymbol{\nabla f}(x, y) \cdot \boldsymbol{v}$. This tells us the rate of change of the function along the specified direction.

Example 4.4.a: Let's consider an economic example involving a utility function and the gradient vector to understand the direction of steepest increase in utility, Cobb-Douglas Utility Function. The Cobb-Douglas utility function is used to model the relationship between the consumption of two goods $(x \text{ and } y)$ and the utility (U) derived from that consumption. It is represented as follows:

$$U(x, y) = x^{\alpha} * y^{\beta}$$

where: *U* is the utility obtained from consuming goods *x* and *y*, *x* and *y* are the quantities of the two goods consumed, α and β are positive constants representing the utility elasticities of *x* and *y*, respectively.

Let's find the gradient vector of the Cobb-Douglas utility function $U(x, y)$ at a specific point (x_0, y_0) and understand the direction of steepest increase in utility at that point.

Step 1: Compute the Gradient Vector ***(∇U)***: The gradient vector of the utility function $U(x, y)$ is given by:

$$\boldsymbol{\nabla U} = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)$$

where $\frac{\partial U}{\partial x}$ is the partial derivative of *U* with respect to *x*, and $\frac{\partial U}{\partial y}$ is the partial derivative of *U* with respect to *y*.

$\frac{\partial U}{\partial x} = \alpha * x^{\alpha-1} * y^{\beta}$

$\frac{\partial U}{\partial y} = \beta * x^{\alpha} * y^{\beta-1}$.

Step 2: Evaluate the Gradient Vector at a Specific Point: Let's assume we want to find the gradient vector at the point (x_0, y_0). Substitute x_0 and y_0 into the partial derivatives to obtain the components of the gradient vector at that point.

$$\boldsymbol{\nabla U}(x_0, y_0) = \left(\alpha * x_0^{\alpha-1} * y_0^{\beta}, \beta * x_0^{\alpha} * y_0^{\beta-1}\right)$$

Step 3: Interpretation of the Gradient Vector: The gradient vector $\boldsymbol{\nabla U}(x_0, y_0)$ provides valuable insights into the direction of steepest increase in utility at the point (x_0, y_0). The direction of $\boldsymbol{\nabla U}(x_0, y_0)$ gives the direction in the (x, y) plane that will result in the fastest increase in utility at that specific consumption point.

For example, if the components of the gradient vector are both positive, then an increase in both *x* and *y* will result in a steeper increase in utility. Similarly, if the components are both negative, a decrease in both *x* and *y* will lead to a steeper increase in utility. If one component is positive and the other negative, then an increase in one good and a decrease in the other will result in a steeper increase in utility.

⭘ *In this way, the gradient vector helps economists understand the optimal direction to change consumption of goods to maximize utility subject to budget constraints or other constraints.*

When dealing with utility maximization problems, it is essential to consider the budget constraint and other relevant constraints to determine the exact combination of goods that will lead to the highest utility.

✍ *In economic applications, the gradient vector is valuable for understanding the sensitivity of economic models to changes in input variables. For example, in production functions or utility functions, the gradient vector can provide insights into the optimal direction for maximizing output or utility, considering the constraints of input resources or budgets.*

Determining the directional derivative in a given direction for a function of two variables: To determine the directional derivative of a function of two variables in a given direction, follow these steps: Let's consider a function $f(x, y)$ that depends on two variables, *x* and *y*.

Step 1: Identify the Direction Vector: First, identify the direction in which you want to find the directional derivative. The direction is represented by a unit vector, $\boldsymbol{v} = (a, b)$, where 'a' and 'b' are the components of the direction vector.

O *For the direction vector to be a unit vector, its magnitude must be 1. So, make sure to normalize the direction vector if needed.*

Step 2: Compute the Gradient Vector: Next, compute the gradient vector of the function $f(x, y)$. The gradient vector $\nabla f(x, y)$ is a vector that points in the direction of the steepest increase of the function at each point (x, y) and is given by:

$$\boldsymbol{\nabla f}(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial derivatives of f with respect to x and y, respectively.

Step 3: Calculate the Dot Product: Once you have the direction vector $\boldsymbol{v} = (a, b)$ and the gradient vector $\boldsymbol{\nabla f}(x, y)$, calculate the dot product between them. The directional derivative of f in the direction of $\boldsymbol{v}$ at the point (x, y) is given by:

$$D_v f(x, y) = \boldsymbol{\nabla f}(x, y) \cdot \boldsymbol{v}$$

where · denotes the dot product.

Step 4: Evaluate the Directional Derivative: Substitute the components of the direction vector and the partial derivatives of f into the dot product expression to calculate the directional derivative $D_v f(x, y)$.

The resulting directional derivative will give you the rate of change of the function $f(x, y)$ in the specified direction at the point (x, y).

Example 4.4.b: Let's consider an economic example involving a production function and calculating the directional derivative in a specific direction, Cobb-Douglas Production Function. The Cobb-Douglas production function is commonly used in economics to model the relationship between inputs (capital, *K*, and labor, *L*) and output *(Q)* in a production process. It is represented as follows:

$$Q(K, L) = A * K^{\alpha} * L^{\beta}$$

where: *Q* is the quantity of output produced, *K* is the amount of capital used in production, *L* is the amount of labor employed, *A* is a positive constant representing total factor productivity, α and β are constants representing the output elasticities of capital and labor, respectively.

Let's find the directional derivative of the Cobb-Douglas production function at a specific point (K_0, L_0) in the direction of a unit vector $\boldsymbol{v} = (a, b)$.

Step 1: Compute the Gradient Vector (∇Q): The gradient vector of the production function $Q(K, L)$ is given by:

$$\boldsymbol{\nabla Q} = \left(\frac{\partial Q}{\partial K}, \frac{\partial Q}{\partial L}\right)$$

where $\frac{\partial Q}{\partial K}$ is the partial derivative of *Q* with respect to *K*, and $\frac{\partial Q}{\partial L}$ is the partial derivative of *Q* with respect to *L*:

$$\frac{\partial Q}{\partial K} = A * \alpha * A * K^{\alpha-1} * L^{\beta} \text{ and } \frac{\partial Q}{\partial L} = A * \beta * A * K^{\alpha} * L^{\beta-1}$$

Step 2: Normalize the Direction Vector: The given direction vector is $\boldsymbol{v} = (a, b)$. To ensure it is a unit vector (magnitude = 1), divide each component by its magnitude:

$$||v|| = \sqrt{a^2 + b^2}$$

Normalized direction vector:

$$\hat{\boldsymbol{v}} = \left(\frac{a}{\|\boldsymbol{v}\|}, \frac{b}{\|\boldsymbol{v}\|}\right)$$

Step 3: Calculate the Dot Product ($\boldsymbol{\nabla Q} \cdot \hat{\boldsymbol{v}}$): Now, calculate the dot product between the gradient vector and the normalized direction vector:

$$\boldsymbol{\nabla Q} \cdot \hat{\boldsymbol{v}} = \frac{\partial Q}{\partial K} * \frac{a}{\|\boldsymbol{v}\|} + \frac{\partial Q}{\partial L} * \frac{b}{\|\boldsymbol{v}\|}$$

Step 4: Evaluate the Directional Derivative: Finally, substitute the partial derivatives of *Q* and the normalized components of $\boldsymbol{v}$ into the dot product expression, Directional Derivative $D_v\boldsymbol{Q}$ at (K_0, L_0) in the direction of $\boldsymbol{v} = (a, b)$:

$$D_v\boldsymbol{Q}(K_0, L_0) = \frac{\partial Q}{\partial K}(K_0, L_0) * \frac{a}{\|\boldsymbol{v}\|} + \frac{\partial Q}{\partial L}(K_0, L_0) * \frac{b}{\|\boldsymbol{v}\|}$$

O *This result gives you the rate of change of the Cobb-Douglas production function at the point (K_0, L_0) in the direction of the unit vector $\boldsymbol{v} = (a, b)$. It represents how the output changes concerning changes in capital and labor input in the specified direction.*

Determining the directional derivative in a given direction for a function of three variables: Let's calculate directional derivatives and gradients in three dimensions. In three-dimensional space, functions typically depend on three variables, represented as $f(x, y, z)$. We will calculate the directional derivative and gradient in a specific direction represented by a unit vector $\boldsymbol{v} = (a, b, c)$.

Directional Derivative in Three Dimensions: The directional derivative $D_v f(x, y, z)$ of a function $f(x, y, z)$ at a point (x_0, y_0, z_0) in the direction of the unit vector $\boldsymbol{v} = (a, b, c)$ is given by the dot product of the gradient vector $\nabla f(x_0, y_0, z_0)$ and the unit vector $\boldsymbol{v}$:

$$D_v f(x_0, y_0, z_0) = \boldsymbol{\nabla f}(x_0, y_0, z_0) \cdot \boldsymbol{v}$$

where $\boldsymbol{\nabla f}(x_0, y_0, z_0)$ is the gradient vector of f at the point (x_0, y_0, z_0).

Gradient in Three Dimensions: The gradient vector ∇f(x, y, z) of a function f(x, y, z) is a vector that points in the direction of the steepest increase of the function at each point (x, y, z). It is given by:

$$\boldsymbol{\nabla f}(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

where ∂f/∂x, ∂f/∂y, and ∂f/∂z are the partial derivatives of f with respect to x, y, and z, respectively.

Example 4.3.c: Let's go through an example to calculate the directional derivative and gradient of a function in three dimensions. Consider the function $f(x, y, z) = 2x^2 + 3y^2 + z^2$.

Step 1: Calculate the Gradient Vector $\boldsymbol{\nabla f}(x, y, z)$:

$$\begin{cases} \partial f/\partial x = 4x \\ \partial f/\partial y = 6y \\ \partial f/\partial z = 2z \end{cases} \Rightarrow \boldsymbol{\nabla f}(x, y, z) = (4x, 6y, 2z)$$

Step 2: Calculate the Directional Derivative $D_v f(x_0, y_0, z_0)$: Let's assume we want to calculate the directional derivative of f at the point $(1, 2, 3)$ in the direction of the unit vector $\boldsymbol{v} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

$$D_v f(1,2,3) = \boldsymbol{\nabla f}(1,2,3) \cdot \boldsymbol{v}$$

Substitute the components of $\boldsymbol{\nabla f}(1,2,3)$ and $\boldsymbol{v}$ into the dot product:

$$D_v f(1,2,3) = \underbrace{(4*1\,, \quad 6*2, \quad 2*3)}_{\boldsymbol{\nabla f}(1,2,3)} * \underbrace{(\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}})}_{\boldsymbol{v}}$$

$$D_v f(1,2,3) = 4 * \frac{1}{\sqrt{3}} + 12 * \frac{1}{\sqrt{3}} + 6 * \frac{1}{\sqrt{3}} = \frac{22}{\sqrt{3}} \approx 12{,}702$$

So, the directional derivative of f at the point $(1,2,3)$ in the direction of the unit vector $\boldsymbol{v}$ is approximately *12.702*.

✍ *The gradient vector* ($\boldsymbol{\nabla f}$) *represents the direction of steepest increase of the function at any given point, while the directional derivative gives the rate of change of the function concerning a specific direction at that point. These concepts are crucial for understanding the behavior of functions in three-dimensional space and have various applications in optimization and analyzing complex surfaces.*

In conclusion, the directional derivative of multivariable functions is a valuable concept in economic mathematics, enabling economists to analyze how a function changes concerning a specific direction or vector in its domain. Understanding directional derivatives is crucial for studying economic models, optimizing economic systems, and making informed decisions in various economic contexts.

Key points about *directional derivatives* in economic mathematics:

Definition: The directional derivative of a multivariable function measures the rate of change of the function concerning a specific direction or vector in its domain. It quantifies how the function changes as one moves along a particular direction.

Notation: The directional derivative of a function $f(x,y,z)$ concerning a direction specified by the unit vector $\boldsymbol{v} = (a,b,c)$ at a point (x,y,z) is denoted by $D_v f(x,y,z)$.

Formula: The formula for the directional derivative in three dimensions is given by:

$$D_v f(x,y,z) = \boldsymbol{\nabla f}(x,y,z) \cdot \boldsymbol{v}$$

where $\boldsymbol{\nabla f}(x,y,z)$ represents the gradient vector of the function $f(x,y,z)$ at the point (x, y, z), and "·" denotes the dot product.

Interpretation: The directional derivative provides information about the slope or rate of change of the function f along the specified direction. If the directional derivative is positive, the function increases in that direction, while a negative directional derivative indicates a decrease. A directional derivative of zero means the function is constant along that direction.

Economic Applications: In economic mathematics, directional derivatives are used to analyze various economic models and scenarios. For example, economists may study the sensitivity of a utility function to changes in the quantities of goods consumed or the effect of input changes on the production function.

Optimization: Directional derivatives play a significant role in optimization problems in economics. Economists can use them to find the steepest ascent or descent directions, which are essential for maximizing or minimizing economic objectives.

Decision-Making: Understanding directional derivatives helps economists make informed decisions by analyzing how small changes in input variables impact the output or utility in a particular direction. This information is crucial for formulating effective economic policies.

✍ *In economic mathematics, directional derivatives are used to analyze the sensitivity of economic models to changes in specific directions. For example, in a production function, directional derivatives can help determine the optimal direction for increasing output while holding input constraints constant. In utility maximization problems, directional derivatives can be used to find the direction of change in consumption to maximize utility subject to budget constraints. By calculating directional derivatives, economists can gain valuable insights into the behavior of economic models and make informed decisions in various economic scenarios.*

Questions you should be able to answer:

What is the directional derivative of a multivariable function?

How is the directional derivative computed with respect to a specific direction?

What is the significance of the directional derivative in multivariable calculus and applications?

How is the directional derivative related to the gradient of a multivariable function?

What is the definition of the gradient of a multivariable function?

How do you compute the gradient of a function with respect to multiple variables?

What is the geometric interpretation of the gradient?

In what direction does the gradient of a function point?

How can the gradient be used to find the direction of steepest increase of a function at a specific point?

What is the relationship between the gradient and level curves (contour lines) of a function?

How can the gradient be used to find critical points and optimize multivariable functions?

What are the applications of directional derivatives and gradients in real-world problems, especially in fields like economics, engineering, and physics?

EXERCISES

4.4.1 Fill in the missing words to complete the sentences related to the theme "The Directional Derivative and Gradient of Multivariable Functions":

a) The _____ derivative of a multivariable function measures the rate of change of the function in the direction of a given vector.

b) For a function $f(x, y, z)$, the _____ at a point P is a vector that points in the direction of the steepest increase of the function at that point.

c) The _____ of a function $f(x, y)$ at a point (a, b) is the directional derivative of *f* at that point in the direction of vector $v = < c, d >$.

d) The gradient vector points in the direction of the _____ increase of the function at a given point.

e) For a function $f(x, y)$ in two variables, the gradient is given by $\nabla f = <$_____, _____$>$.

f) The gradient vector at a point (x, y, z) in three-dimensional space is given by ∇f = <_____, _____, _____>.

g) The _____ is a vector that points in the direction of the maximum increase of a function at a given point.

h) If the gradient vector is zero at a point, it means the function has a _____ at that point.

i) The directional derivative of a function $f(x, y)$ at a point P in the direction of a unit vector $u = < a, b >$ is given by the dot product of the gradient of f at P and vector u, that is, $D_u f(P)$ = _____.

j) The directional derivative of a function $f(x, y, z)$ at a point P in the direction of a vector $v = < a, b, c >$ is given by the dot product of the gradient of f at P and the unit vector in the direction of v, that is, $D_v f(P)$ = _____.

4.4.2 Fill in the missing words to complete the sentences related to the theme "The Directional Derivative and Gradient of Multivariable Functions" in the context of Economic Mathematics:

a) The directional derivative of a multivariable function $f(x, y)$ at a point P in the direction of a unit vector u is given by the dot product of the _____ and the gradient of f at point P.

b) The gradient of a multivariable function $f(x, y, z)$ represents the _____ of steepest increase of the function at a given point.

c) To find the directional derivative of a function $f(x, y)$ at a point P in the direction of a vector u, we need to take the dot product of the _____ of u and the gradient of f at point P.

d) The directional derivative is the rate of change of a multivariable function in a specific _____ at a given point.

e) If the gradient of a function $f(x, y)$ is $(3, 4)$ at a point P, then the function increases _____ at that point in the direction of the gradient vector.

f) The directional derivative of a multivariable function $f(x, y, z)$ in the direction of a unit vector $\boldsymbol{u}$ is defined as the dot product of the gradient of f and the _____ of $\boldsymbol{u}$.

g) The gradient of a function $f(x, y, z)$ can be thought of as a _____ that points in the direction of the steepest increase of the function at each point.

h) If the gradient of a function $f(x, y)$ is $(2, -1)$ at a point P, then the function decreases _____ at that point in the direction opposite to the gradient vector.

i) The directional derivative is used to find the rate of change of a multivariable function along a specific _____.

j) The gradient vector of a function $f(x, y, z)$ can be represented as $\boldsymbol{\nabla f} = (f_x, f_y, f_z)$, where $f_x, f_y, and\ f_z$ are the _____ derivatives of f with respect to *x, y, and z,* respectively.

4.4.3 For each exercise, after computing the gradient vector at the given point, provide an interpretation of the gradient vector. Explain what the components of the gradient vector represent in terms of the rate of change and the direction of steepest increase of the function at that point.

a) Compute the gradient vector for the function $f(x, y) = 3x^2 + 2y$ at the point (2, 4).

b) Compute the gradient vector for the function $g(x, y) = e^x + ln(y)$ at the point (1, 2).

c) Compute the gradient vector for the function $h(x, y, z) = x^2 + y^2 + z^2$ at the point (1, 2, 3).

d) Compute the gradient vector for the function $f(x, y) = xy + \frac{1}{x}$ at the point (3, 2).

e) Compute the gradient vector for the function $g(x, y) = sin(x) + cos(y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{6})$.

f) Compute the gradient vector for the function $h(x, y, z) = xyz$ at the point (2, 3, 4).

g) Compute the gradient vector for the function $f(x, y) = x^3 + y^3$ at the point (1, 1).

h) Compute the gradient vector for the function $g(x, y) = x^2y^3$ at the point (2, 3).

i) Compute the gradient vector for the function $h(x, y, z) = e^{xyz}$ at the point (0, 1, 2).

j) Compute the gradient vector for the function $f(x, y) = x^2 + 2xy + y^2$ at the point (1, -1).

4.4.4 In each exercise, determine the directional derivative of the given function of two varables, in the specified direction and provide an interpretation of the result in the context of the problem:

a) Consider the function $f(x, y) = 3x^2 + 2y$. Calculate the directional derivative of *f* at the point *(1, 2)* in the direction of the vector $\boldsymbol{u} = (1, 1)$.

b) A manufacturing company has a production function given by $f(x, y) = 5x + 2y^2$, where *x* represents labor input and *y* represents capital input. Determine the rate of change of production at the point *(2, 3)* in the direction of the vector $\boldsymbol{u} = (3, 4)$.

c) A temperature distribution in a region is given by $T(x, y) = 50x - 20y + 30$, where *T* represents temperature in degrees Celsius and *x, y* represent spatial coordinates. Find the directional derivative of temperature at the point *(2, 1)* in the direction of the vector $\boldsymbol{u} = (1, -2)$.

d) A company's profit function is given by $P(x, y) = 2x^2 + 3xy - y^2$, where *x* represents the number of units produced and *y* represents the price per unit. Compute the directional derivative of profit at the point *(3, 4)* in the direction of the vector $\boldsymbol{u} = (-1, 2)$.

e) The height of a hill is described by the function $h(x, y) = 200 - 5x^2 - 4y^2$, where *h* represents the height in meters and *x, y* represent distance in meters. Determine the slope of the hill at the point *(2, 3)* in the direction of the vector $\boldsymbol{u} = (4, -1)$.

f): A firm's cost function is given by $C(x, y) = 1000x + 500y^2$, where *C* represents the total cost and *x, y* represent input quantities. Find the rate of change of cost at the point *(5, 2)* in the direction of the vector $\boldsymbol{u} = (2, 3)$.

g) The concentration of a pollutant in a lake is described by the function $C(x, y) = 10xy + 5x^2$, where *C* represents the concentration in parts per million and *x, y* represent spatial coordinates. Calculate the rate of change of concentration at the point *(1, 2)* in the direction of the vector $\boldsymbol{u} = (3, 4)$.

h) A company's revenue function is given by $R(x, y) = 500x^2 - 200xy$, where *R* represents the total revenue and *x, y* represent the quantity sold and price per unit, respectively. Determine the slope of revenue at the point *(2, 3)* in the direction of the vector $\boldsymbol{u} = (-1, 1)$.

i) The temperature distribution in a room is given by $T(x, y) = 30 + 3x - 2y$, where *T* represents the temperature in degrees Celsius and *x, y* represent spatial coordinates. Compute the directional derivative of temperature at the point *(1, 2)* in the direction of the vector $\boldsymbol{u} = (2, 1)$.

j) A company's profit function is given by $P(x, y) = 1000x - 500xy$, where *P* represents the total profit and *x, y* represent the number of units produced and price per unit, respectively. Find the rate of change of profit at the point *(4, 5)* in the direction of the vector $\boldsymbol{u} = (3, -2)$.

4.4.5 In each exercise, calculate the directional derivative of the given function of three variables, at the specified point in the given direction. Interpret the results:

a) Consider the function $f(x, y, z) = 2x^2 + 3y - z^3$. Calculate the directional derivative of f at the point *(1, 2, -1)* in the direction of the vector $\boldsymbol{v} = (1, 1, 1)$. Interpret the result in the context of the function.

b) Given the function $f(x, y, z) = x^3 + 2y^2 - 3z$, find the directional derivative at the point *(2, -1, 3)* in the direction of the unit vector $\boldsymbol{u} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Interpret the meaning of the result.

c) For the function $f(x, y, z) = e^{xy} + 2z - x^2y$, determine the directional derivative at the point *(0, 1, 2)* in the direction of the vector $\boldsymbol{v} = (1, -2, 3)$. Interpret the significance of the obtained value.

d) Consider the function $f(x, y, z) = sin(x) + cos(y) + z^2$. Calculate the directional derivative at the point *(π/4, 0, 1)* in the direction of the unit vector $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Provide an interpretation of the result.

e) Given the function $f(x, y, z) = x^2 + y^2 + z^2$, find the directional derivative at the point *(1, -1, 2)* in the direction of the vector $\boldsymbol{v} = (2, 1, -2)$. Explain the geometric significance of the obtained value.

f) For the function $f(x, y, z) = 3xy + 2yz - xz$, determine the directional derivative at the point *(2, -1, 3)* in the direction of the unit vector $\boldsymbol{u} = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Interpret the meaning of the result.

g) Consider the function $f(x, y, z) = e^{xyz}$. Calculate the directional derivative at the point *(0, 1, -1)* in the direction of the vector $\boldsymbol{v} = (1, -2, 3)$. Explain the physical interpretation of the result.

h) Given the function $f(x, y, z) = x^3 + y^3 + z^3$, find the directional derivative at the point *(-1, 2, 0)* in the direction of the unit vector $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Interpret the significance of the obtained value.

i) For the function $f(x, y, z) = x^2 - y^2 + 2z$, determine the directional derivative at the point *(1, 1, 1)* in the direction of the vector $\boldsymbol{v} = (-1, 1, 2)$. Interpret the meaning of the result.

j) Consider the function $f(x, y, z) = ln(x) + 2y + 3z^2$. Calculate the directional derivative at the point *(2, 1, 0)* in the direction of the unit vector $\boldsymbol{u} = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$. Provide an interpretation of the result.

4.5 OPTIMIZATION PROBLEMS. THE METHOD OF LEAST SQUARES

Keywords*: Optimization Problems, Method of Least Squares, Least Squares, Optimization Techniques, Numerical Optimization, Regression Analysis, Curve Fitting, Mathematical Optimization, Minimization, Data Fitting, Linear Regression, Nonlinear Regression, Residuals, Sum of Squares, Objective Function, Parameter Estimation, Data Modeling, Error Minimization, Overfitting, Underfitting.*

Optimization problems are fundamental in mathematics and have broad applications in various fields, including economics. In optimization, the goal is to find the best solution (maximum or minimum) for a given objective function, subject to certain constraints. In economic mathematics, optimization problems are used to maximize utility, profit, production output, or minimize costs, among other objectives.

Key Concepts:

Objective Function: The objective function is the mathematical representation of the quantity to be optimized. In economics, this could be a utility function, profit function, or production function.

Decision Variables: Decision variables are the variables that can be controlled or changed in the optimization problem. In economics, decision variables could represent the quantities of goods consumed, the level of investment, or the allocation of resources.

Constraints: Constraints are the limitations or conditions that restrict the feasible region where the optimization must take place. In economic applications, constraints could represent budget constraints, resource limitations, or technological restrictions.

Feasible Region: The feasible region is the set of all feasible combinations of decision variables that satisfy the given constraints.

Local vs. Global Optima: Optimization problems can have multiple solutions, each representing a potential optimum. Local optima are points where the objective function has the best value in a particular region, while global optima are the best solutions across the entire feasible region.

Analytical and Numerical Methods: Various methods are used to solve optimization problems, including analytical techniques such as calculus (e.g., derivative-based methods) and numerical methods like linear programming, nonlinear programming, and evolutionary algorithms.

Sensitivity Analysis: Sensitivity analysis involves studying how changes in parameters or constraints affect the optimal solution. In economics, sensitivity analysis helps to assess the robustness of the optimal solution under different scenarios.

Optimization problems find wide applications in economics:

Utility Maximization: Consumers aim to maximize their utility subject to budget constraints, determining the optimal allocation of goods and services.

Profit Maximization: Firms aim to maximize their profits by optimizing production levels, considering input costs and market demand.

Cost Minimization: Firms seek to minimize their production costs by determining the optimal combination of inputs.

Production Efficiency: Optimization is used to find the efficient production frontier where output is maximized for a given set of inputs.

Investment Allocation: Investors optimize their portfolio allocation to maximize returns while managing risk.

✍ *Optimization problems are prevalent in economic mathematics, and economists use a variety of mathematical techniques to find optimal solutions. These techniques help economists make informed decisions, understand economic behavior, and formulate effective economic policies. The choice of the optimization method depends on the specific problem structure, complexity, and the available computational resources.*

The Method of Least Squares: The method of least squares is a statistical technique widely used in mathematical economics for estimating the parameters of a linear regression model and minimizing the sum of the squared differences between observed data points and the predicted values from the model. It plays a crucial role in econometrics, which is the application of statistical methods to economic data.

Summary of the *Method of Least Squares* in Mathematical Economics:

Purpose: The main objective of the method of least squares is to find the best-fitting line or model that explains the relationship between two or more variables. In mathematical economics, this technique is used to estimate the coefficients of a linear regression model.

Linear Regression Model: In mathematical economics, many relationships between economic variables can be approximated by linear equations. The linear regression model takes the form: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon$, where Y is the dependent variable, $X_1, X_2, \ldots, X_n$ are the independent variables, $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are the coefficients to be estimated, and ε is the error term representing the deviation between the observed data and the model predictions.

Minimizing Residuals: The method of least squares aims to minimize the sum of the squared residuals, which are the differences between the observed data points and the corresponding predicted values from the linear regression model. By minimizing the squared residuals, the method finds the "best-fitting" line or model that represents the data as closely as possible.

Ordinary Least Squares (OLS): The most common form of the method of Least Squares is Ordinary Least Squares (OLS). OLS estimates the coefficients by finding the values that minimize the sum of squared residuals. It is computationally efficient and widely used in economic modeling.

Applications: The Method of Least Squares is used in various economic applications, such as demand estimation, cost analysis, production function estimation, time series forecasting, and econometric modeling. It helps economists understand the relationships between variables, make predictions, and analyze economic phenomena.

Assumptions: The Method of Least Squares assumes that the error term ε is normally distributed with a mean of zero and constant variance (homoscedasticity). Additionally, it assumes that the independent variables are not perfectly correlated (no multicollinearity) and that the error terms are not correlated (no autocorrelation).

Interpretation: After applying the Method of Least Squares, economists interpret the estimated coefficients to understand the magnitude and direction of the relationship between the dependent and

independent variables. The coefficients represent the change in the dependent variable for a one-unit change in the corresponding independent variable, holding other variables constant.

Minimize SSR (Sum of Squared Residuals) in the context of the Method of Least Squares: To minimize SSR (Sum of Squared Residuals) in the context of the Method of Least Squares, we aim to find the optimal values for the coefficient estimates (β_0 and $\beta_1$in the case of a simple linear regression) that provide the best-fitting line or model for the given data points. Minimizing SSR means finding the line that best represents the relationship between the dependent variable and the independent variable(s) based on the observed data. Here's the step-by-step process to minimize SSR:

- Formulate the Regression Model*:* Start by formulating the regression model that represents the relationship between the dependent variable (Y) and the independent variable(s) (X). In a simple linear regression, the model takes the form: $\hat{Y} = \beta_0 + \beta_0 * X$, where β_0 and β_1 are the coefficients to be estimated as follow

$$\beta_0 = \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum (X_i * Y_i)}{n * \sum X_i^2 - (\sum X_i)^2} \quad \text{and} \quad \beta_1 = \frac{n * \sum (X_i * Y_i) - \sum X_i * \sum Y_i}{n * \sum X_i^2 - (\sum X_i)^2}$$

- Define the Objective Function (SSR): The objective function, SSR, is the sum of the squared differences between the observed data points (Y) and the corresponding predicted values $(\hat{Y})$ from the regression model. SSR is given by: $SSR = \Sigma(Y - \hat{Y})^2$, where Σ represents the sum over all data points.

- Calculate the Predicted Values ($\hat{Y}$): Calculate the predicted values $\hat{Y}$ using the initial coefficient estimates β_0 and β_1. Plug each value of X into the regression model to obtain the corresponding predicted value $\hat{Y}$.

- Compute the Residuals (ε): The residuals (ε) are the differences between the observed data points (Y) and the predicted values $(\hat{Y})$ from the model. Residuals are calculated as: $\varepsilon = Y - \hat{Y}$.

- Square and Sum the Residuals: Square each residual to eliminate the negative signs and then sum the squared residuals to obtain SSR: $SSR = \Sigma\varepsilon^2$, where Σ represents the sum over all data points.

- Minimize SSR by Adjusting Coefficients: To minimize SSR, adjust the coefficient estimates (β_0 and β_1). This is achieved by finding the values of β_0 and β_1 that make SSR as small as possible. The optimal estimates for β_0 and β_1 represent the coefficients of the best-fitting line that approximates the relationship between the dependent and independent variables.

- Calculating Optimal Coefficient Estimates: Calculate the partial derivatives of SSR with respect to $\widehat{\beta_0}$ and $\widehat{\beta_1}$ and set them equal to zero:

$$\begin{cases} \frac{\partial(SSR)}{\partial\beta_0} = 0 \\ \frac{\partial(SSR)}{\partial\beta_1} = 0 \end{cases} \Rightarrow \begin{cases} \widehat{\beta_0} = \bar{Y} - \widehat{\beta_1} \\ \widehat{\beta_1} = \frac{\sum (X_i * Y_i) - \bar{Y} * \sum X_i}{\sum X_i^2 - \bar{X} * \sum X_i} \end{cases}$$

and solve the system of equations to find the values of $\widehat{\beta_0}$ and $\widehat{\beta_1}$ that minimize SSR.

➢ Interpretation: Interpret the optimal estimates of $\widehat{\beta_0}$ and $\widehat{\beta_1}$ in the context of the problem. The estimated regression equation, $Y = \widehat{\beta_0} + \widehat{\beta_1} * X$, provides the best linear representation of the relationship between the variables based on the observed data.

✍ *In summary, minimizing SSR involves finding the optimal coefficient estimates ($\widehat{\beta_0}$ and $\widehat{\beta_1}$) that result in the smallest sum of squared differences between the observed data points and the predicted values from the regression model. This process is commonly achieved through the Ordinary Least Squares (OLS) method, which is widely used for linear regression analysis.*

Example 4.4.d: Let's consider an example of estimating a linear demand function in mathematical economics using the method of least squares. Suppose we have data on the quantity demanded (Q) and the price (P) of a product for several periods. We want to estimate the linear demand function:

$$Q = \beta_0 + \beta_1 * P$$

where Q is the quantity demanded, P is the price, and $\beta_0 = \frac{\sum P_i^2 \sum Q_i - \sum P_i \sum (P_i * Q_i)}{n * \sum P_i^2 - (\sum P_i)^2}$ and $\beta_1 = \frac{n * \sum (P_i * Q_i) - \sum P_i * \sum Q_i}{n * \sum P_i^2 - (\sum P_i)^2}$ are the coefficients to be estimated.

We have the following data:

P: $[10, 15, 20, 25, 30]$

Q: $[100, 80, 70, 60, 50]$

Step 1: Calculate $\hat{Q}$ (predicted values) using the initial coefficient estimates $\beta_0 = 120$ and $\beta_1 = -2.4$:

$$\hat{Q} = 120 - 2.4 * P$$

Step 2: Calculate the squared residuals (ε^2) for each data point:

$$\varepsilon^2 = (Q - \hat{Q})^2$$

Step 3: Sum the squared residuals to get the total sum of squared residuals (SSR):

$$SSR = \Sigma \varepsilon^2 = 40$$

Step 4: Minimize SSR by adjusting the coefficient estimates β_0 and β_1: The values that minimize SSR are the final estimates for β_0 and β_1, which represent the coefficients of the best-fitting demand function.

$$\begin{cases} \frac{\partial (SSR)}{\partial \beta_0} = 0 \\ \frac{\partial (SSR)}{\partial \beta_1} = 0 \end{cases} \Rightarrow \begin{cases} \widehat{\beta_0} = \bar{Q} - \widehat{\beta_1} \\ \widehat{\beta_1} = \frac{\sum (P_i * Q_i) - \bar{Q} * \sum P_i}{\sum P_i^2 - \bar{P} * \sum P_i} \end{cases}$$

By applying the method of least squares, we obtain the estimates $\widehat{\beta_0} \approx 120$ and $\widehat{\beta_1} \approx -2.4$. The estimated demand function is:

$$Q \approx 120 - 2.4 * P$$

⭘ *This linear demand function represents the best-fitting line that approximates the relationship between quantity demanded and price based on the given data points. It can be used for demand forecasting and pricing strategies in economic decision-making.*

✍ *In conclusion, the Method of Least Squares is a fundamental statistical technique in mathematical economics used to estimate the parameters of linear regression models and explain the*

relationships between economic variables. It is a powerful tool for economists to analyze and model economic data, make predictions, and derive valuable insights to inform economic decision-making and policy analysis.

Questions you should be able to answer:

What is the Method of Least Squares in the context of economics?

How is the Method of Least Squares used to solve optimization problems in economics?

What are the main steps involved in applying the Method of Least Squares?

How does the Method of Least Squares relate to regression analysis in economics?

What are the advantages and limitations of using the Method of Least Squares in economic analysis?

How is the Method of Least Squares used in economic modeling and forecasting?

Can you explain the concept of "residuals" in the context of the Method of Least Squares and its economic implications?

How does the Method of Least Squares help in estimating parameters in economic models?

Provide an example of an economic problem that can be solved using the Method of Least Squares.

What are some alternative optimization techniques used in economics, and how do they compare to the Method of Least Squares?

In what situations is the Method of Least Squares not appropriate to use in economic analysis?

How does the Method of Least Squares contribute to decision-making and policy analysis in economics?

Remember, the field of economics is vast and diverse, and the Method of Least Squares is just one of the many tools used for optimization and data analysis in this domain. Understanding this method will help you in various economic applications, such as demand estimation, cost function analysis, production function estimation, and more.

EXERCISES

4.5.1 Fill-in-the-blank exercises related to linear functions and the method of least squares in the context of economic mathematics:

a) The method of least squares is a technique used to find the best-fitting _____ line to a set of data points.

b) In simple linear regression, the equation of the regression line is $\hat{Y} = a + bX$, where "a" represents the _____ and "b" represents the slope.

c) The goal of the method of least squares is to minimize the sum of the squared _____ between the observed Y values and the predicted Y values from the regression line.

d) The _____ of the regression line represents the value of the dependent variable (Y) when the independent variable (X) is zero.

e) The _____ of the regression line represents the change in the dependent variable (Y) for a one-unit change in the independent variable (X).

f) The method of least squares aims to find the values of "a" and "b" that make the sum of the squared _____ as small as possible.

g) The method of least squares is commonly used in economics to estimate the parameters of _____ relationships between variables.

h) The method of least squares provides a way to fit a _____ line to a scatter plot of data.

4.5.2 In each exercise, use the method of least squares to find the equation of the line of best fit for the given data. The equation should be in the form $Y = a + bX$, where "a" is the y-intercept and "b" is the slope of the line. Provide the equations for each scenario and interpret the coefficients "a" and "b" in the context of the problem:

a) Sales Revenue Analysis: Consider the following data representing the relationship between a firm's advertising expenditure (X) and its monthly sales revenue (Y):

X: $[6, 8, 10, 12, 14]$ and Y: $[15, 18, 22, 24, 28]$

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

b) Demand Analysis: A firm wants to analyze the relationship between the price of its product (X) and the monthly demand (Y). The data is as follows:

X: $[10, 15, 20, 25, 30]$ and Y: $[80, 60, 50, 40, 30]$

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

c) Cost Analysis: A firm wants to analyze the relationship between its production quantity (X) and the monthly total cost (Y). The data is as follows:

X: $[50, 100, 150, 200, 250]$ and Y: $[3000, 4000, 5000, 6000, 7000]$

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

d) Investment Analysis: A company wants to analyze the relationship between its investment expenditure (X) and the monthly return on investment (Y). The data is as follows:

X: $[1000, 2000, 3000, 4000, 5000]$ and Y: $[150, 250, 320, 380, 420]$

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

e) Cost Analysis: A firm wants to analyze the relationship between its production quantity (X) and the monthly total cost (Y). The data is as follows:

X: $[10, 15, 20, 25, 30]$ and Y: $[500, 600, 700, 800, 900]$

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

f) Time Series Forecasting: Suppose you have monthly sales data for a product for the last 12 months:

Month: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]

Sales: [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]

find the equation of the line of best fit ($Y = a + bX$) using the method of least squares.

4.5.3 For each example, follow the steps given in the previous example (Simple Linear Regression) to perform the calculations, interpret the regression coefficients, and estimate the linear economic function:

a) Sales Revenue Analysis: Consider the following data representing the relationship between a firm's advertising expenditure (X) and its monthly sales revenue (Y):

X: $[5, 7, 10, 15, 20]$, and Y: $[12, 15, 18, 22, 25]$

a.1) Plot the data points on a scatter plot. a.2) Find the equation of the line of best fit ($Y = a + bX$) using the method of Least Squares. a.3) Interpret the values of *a* and *b* in the context of the problem. a.4) Estimate the linear sales revenue function.

b) Labor Market Analysis: Suppose a company wants to analyze the relationship between the number of employees (X) and its monthly production output (Y). The data is as follows:

X: $[10, 15, 20, 25, 30]$ and Y: $[2000, 3000, 4000, 5000, 6000]$

b.1) Plot the data points on a scatter plot. b.2) Find the equation of the line of best fit ($Y = a + bX$) using the method of Least Squares. b.3) Interpret the values of *a* and *b* in the context of the problem. b.4) Estimate the linear production function.

c) Demand Analysis: A firm wants to analyze the relationship between the price of its product (X) and the monthly demand (Y). The data is as follows:

X: $[5, 7, 9, 11, 13]$ and Y: $[100, 80, 70, 60, 50]$

c.1) Plot the data points on a scatter plot. c.2) Find the equation of the line of best fit ($Y = a + bX$) using the method of Least Squares. c.3) Interpret the values of a and b in the context of the problem. c.4) Estimate the linear demand function.

d) Investment Analysis: A company wants to analyze the relationship between its investment expenditure (X) and the monthly return on investment (Y). The data is as follows:

X: $[500, 1000, 1500, 2000, 2500]$ and Y: $[30, 60, 90, 120, 150]$

d.1) Plot the data points on a scatter plot. d.2) Find the equation of the line of best fit ($Y = a + bX$) using the method of Least Squares. d.3) Interpret the values of a and b in the context of the problem. d.4) Estimate the linear return on investment function.

e) Cost Analysis: A firm wants to analyze the relationship between its production quantity (X) and the monthly total cost (Y). The data is as follows:

X: $[100, 150, 200, 250, 300]$ and Y: $[5000, 6000, 7000, 8000, 9000]$

e.1) Plot the data points on a scatter plot. e.2) Find the equation of the line of best fit ($Y = a + bX$) using the method of Least Squares. e.3) Interpret the values of a and b in the context of the problem. e.4) Estimate the linear total cost function.

f) Time Series Forecasting: Suppose you have monthly sales data for a product for the last 12 months:

Month: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]

Sales: [120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230]

f.1) Plot the time series data on a line chart. f.2) Fit the data to a linear equation ($Sales = a + b * Month$) using the method of least squares. f.3) Estimate the linear total cost function.). f.4) Use the regression equation to forecast the sales for the next 3 months (13, 14, and 15).

5 DIFFERENTIAL EQUATIONS

CONTENT UNITS: The study of differential equations in economic mathematics covers several important content units. These units are essential for understanding and modeling dynamic economic processes, such as economic growth, investment decisions, population dynamics, and more. Here are some of the key content units that can be covered in the study of differential equations in economic mathematics:

First-Order Ordinary Differential Equations (ODEs): Separable ODEs, Linear ODEs, Exact ODEs, Integrating Factor Method.

Second-Order Ordinary Differential Equations: Homogeneous and Non-homogeneous ODEs, Constant Coefficients, Method of Undetermined Coefficients, Variation of Parameters.

Economic Applications of Differential Equations: Economic Growth Models (e.g., Solow-Swan Model), Investment and Savings Models, Population Dynamics Models (e.g., Logistic Growth Model), Capital Accumulation Models, Dynamic Input-Output Models.

Partial Differential Equations (PDEs): Economic Applications of PDEs (e.g., Black-Scholes Equation in Finance), Difference Equations, Discrete-Time Dynamic Models, Economic Applications (e.g., Recursive Macroeconomic Models).

✍ *The study of differential equations in economic mathematics is fundamental for analyzing and modeling the dynamic behavior of economic systems over time. It provides valuable tools and techniques for economists to make predictions, analyze economic trends, and optimize economic decision-making. Differential equations play a critical role in understanding various economic phenomena, and their applications extend across various economic subfields, including macroeconomics, microeconomics, finance, and econometrics.*

KEYWORDS: *Differential Equation, Ordinary Differential Equation (ODE), Partial Differential Equation (PDE), First-Order Differential Equation, Second-Order Differential Equation, Initial Value Problem (IVP), Boundary Value Problem (BVP), Separable Differential Equation, Linear Differential Equation, Homogeneous Differential Equation, Inhomogeneous Differential Equation, Exact Differential Equation, Integrating Factor, Method of Undetermined Coefficients, Economic Applications of Differential Equations, Dynamic Input-Output Models, Solow-Swan Model, Logistic Growth Model, Capital Accumulation Models, Investment and Savings Models, Optimal Resource Management.*

LEARNING OBJECTIVES: Learning objectives for studying Differential Equations in economic mathematics can be designed to help students acquire the necessary skills and knowledge to apply differential equations in modeling and analyzing dynamic economic systems. Here are some specific learning objectives that can be set:

Understand the concept of Differential Equations: Define what a differential equation is and distinguish between ordinary and partial differential equations. Explain the role of differential equations in modeling dynamic economic phenomena.

Solve First-Order Ordinary Differential Equations: Solve separable differential equations using appropriate integration techniques. Apply the method of integrating factors to solve linear differential equations. Solve exact differential equations and identify integrating factors.

Solve Second-Order Ordinary Differential Equations: Solve homogeneous and non-homogeneous second-order linear differential equations with constant coefficients. Use the method of undetermined coefficients and variation of parameters to find particular solutions. Solve higher-order linear differential equations.

Analyze Systems of Ordinary Differential Equations: Set up and solve first-order linear systems of differential equations. Analyze the stability and behavior of equilibrium points in systems of differential equations. Solve second-order linear systems of differential equations.

Apply Differential Equations in Economic Models: Formulate and analyze economic growth models using differential equations (e.g., Solow-Swan model). Model investment and savings decisions using differential equations. Apply differential equations to population dynamics models.

Understand Partial Differential Equations and their Applications: Define and solve partial differential equations. Apply partial differential equations in economic applications, such as option pricing in finance (e.g., Black-Scholes equation).

Apply Differential Equations to Optimal Control Theory: Apply optimal control theory to economic models to optimize resource allocation and decision-making.

Apply Differential Equations in Dynamic Input-Output Models: Use differential equations to model dynamic input-output relationships in economics.

✍ *By setting these learning objectives, students will develop a strong foundation in using differential equations to analyze and model dynamic economic systems. These objectives will equip them with valuable skills applicable to various economic scenarios, supporting their understanding of economic dynamics and enhancing their ability to make informed economic decisions.*

5.1 FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODES)

Keywords*: Differential Equation, Ordinary Differential Equation (ODE), First-Order ODE, Separable Differential Equation, Linear Differential Equation, Exact Differential Equation, Integrating Factor, Method of Integrating Factors, Initial Value Problem (IVP).*

First-order Ordinary Differential Equations (ODEs) are fundamental in various scientific and engineering disciplines and are extensively used in modeling dynamic processes and systems. They describe the rate of change of a function with respect to the independent variable, and the solutions to these equations represent curves or trajectories in the (x, y) plane.

Solving first-order ODEs involves finding the function $y(x)$ that satisfies the given equation for a specified range of x values, along with initial conditions if it is an initial value problem (IVP). There are various methods to solve first-order ODEs, depending on their form and properties, including separation of variables, integrating factors, exact equations, and numerical techniques.

In economic mathematics, first-order ODEs are employed to model economic growth, investment decisions, population dynamics, capital accumulation, and other dynamic economic processes. By analyzing the solutions to these ODEs, economists can make predictions, analyze the stability of economic systems, and optimize economic decision-making in various contexts.

In conomic mathematics, first-order ordinary differential equations (ODEs) play a significant role in modeling and analyzing various economic phenomena. They are used to study dynamic processes where the rate of change of a variable depends on its current value and time.

Here are some key applications of first-order ODEs in economic mathematics:

Economic Growth Models: First-order ODEs are used to model economic growth and development. The Solow-Swan model is a classic example where a first-order ODE represents the growth rate of an economy's capital stock over time, influenced by factors like investment, depreciation, and technological progress.

Investment and Savings Models: First-order ODEs are employed to analyze investment and savings decisions. These models help understand how individuals and firms allocate resources between consumption and investment, affecting long-term economic growth.

Population Dynamics: First-order ODEs are used in population models to study the growth and decline of populations. The logistic growth model, based on a first-order ODE, is commonly used to model population growth with limited resources or carrying capacity.

Economic Equilibrium: First-order ODEs are used to study economic equilibria, where the rate of change of a variable becomes zero. In macroeconomics, this is used to analyze steady states and long-run equilibrium of economic models.

Dynamic Input-Output Models: First-order ODEs are applied to model dynamic input-output relationships in economics. These models help analyze the interdependence between different sectors of the economy and how changes in one sector affect others over time.

Optimal Control Theory: First-order ODEs are utilized in optimal control theory to determine the optimal path of economic variables to maximize a certain objective function. This helps economists study optimal resource allocation and decision-making.

Comparative Statics: First-order ODEs are used to perform comparative static analysis, which involves studying how economic variables change in response to exogenous shocks or changes in parameters.

Time Series Analysis: First-order ODEs can be used to model and forecast time series data, such as economic indicators, stock prices, or exchange rates.

Economic Policy Analysis: First-order ODEs provide a valuable tool for studying the effects of economic policies over time. They help economists assess the long-term impact of policy changes on economic variables.

Neoclassical Growth Model: In economics, the neoclassical growth model uses first-order ODEs to study the interplay between savings, investment, and technological progress in determining long-term economic growth.

✍ *Overall, first-order ordinary differential equations are a fundamental tool in economic mathematics, allowing economists to mathematically model and analyze dynamic economic systems. They provide insights into economic behavior, growth, and stability, and are widely used in economic research and policymaking.*

The concept of a first-order Ordinary Differential Equation (ODEs)*:* A first-order ordinary differential equation (ODE) is a type of differential equation that involves the first derivative

of an unknown function with respect to a single independent variable. In mathematical notation, a first-order ODE can be expressed as:

$$\frac{dy}{dx} = f(x, y)$$

where: y is the unknown function of x (dependent variable), x the independent variable, $\frac{dy}{dx}$ the first derivative of y with respect to x, $f(x, y)$ a given function that relates the variables x and y.

Solving a first-order ODEs: The goal in solving a first-order ODE is to find the function $y(x)$ that satisfies the equation for a specified range of x values, along with appropriate initial conditions if it is an initial value problem (IVP).

Example 5.1.a: *L*et's consider an example of a first-order ordinary differential equation in the context of economic mathematics: Investment Model.

Suppose we have an investment model that describes the relationship between the rate of investment (I) and the level of capital stock (K) in an economy. The model assumes that the rate of investment is proportional to the difference between a fixed target capital stock $(K*)$ and the current capital stock, with a constant proportionality factor (r).

The differential equation representing this investment model is:

$$\frac{dK}{dt} = r * (K^* - K)$$

where: *K* is the capital stock in the economy at time *t*, $\frac{dK}{dt}$ the rate of change of capital stock over time, r the proportionality factor representing the speed of adjustment, K^* the target level of capital stock that the economy aims to achieve.

In this example, the first-order ODE describes the dynamics of capital accumulation over time. The equation states that the rate of change of the capital stock is equal to the rate at which the economy adjusts towards its target capital stock level, which is determined by the difference between the target level (K^*) and the current level (K) multiplied by the proportionality factor (r).

Solving this first-order ODE involves finding the function $K(t)$ that satisfies the equation for a given initial condition (e.g., the capital stock at a specific time t_0). By solving this equation, economists can analyze how the economy's capital stock changes over time, how quickly it converges to the target level, and the impact of different values of r and K^* on the investment dynamics.

There economic models and scenarios that can be represented and analyzed using first-order ODEs: This investment model is just one example of how first-order ODEs can be applied in economic mathematics to study dynamic economic processes, such as investment decisions and capital accumulation. There are numerous other economic models and scenarios that can be represented and analyzed using first-order ODEs, making them a powerful tool in economic modeling and analysis. The follow examples are the economic models and scenarios, along with the corresponding first-order ordinary differential equations (ODEs) that can represent and analyze them.

Example 5.1.b: Savings and Consumption Model:

$$ODE: \quad \frac{ds}{dt} = \alpha(Y - c)$$

where s is the savings rate, t is time, Y is disposable income, c is the consumption rate, and α is the proportionality factor.

Example 5.1.c: Population Growth and Migration Model:

$$ODE: \quad \frac{dp}{dt} = \beta p - \delta p + \mu$$

where p is the population size, t is time, β is the birth rate, δ is the death rate, and μ is the migration rate.

Example 5.1.d: Inflation and Price Adjustment Model:

$$ODE: \quad \frac{dp}{dt} = \pi e - \pi$$

where p is the price level, t is time, $\pi * e$ is expected inflation, and π is actual inflation.

Example 5.1.e: Asset Pricing Model:

$$ODE: \quad \frac{dr}{dt} = i - \pi - \theta$$

where r is the nominal interest rate, t is time, i is the real interest rate, π is inflation, and θ is the risk premium.

Example 5.1.f: Demand and Supply Adjustment Model:

$$ODE: \quad \frac{dp}{dt} = \alpha(Q_d - Q_s)$$

where p is the price of the good, t is time, Q_d is the quantity demanded, Q_s is the quantity supplied, and α is the price adjustment coefficient.

Example 5.1.g: Economic Growth with Depreciation:

$$ODE: \quad \frac{dK}{dt} = s - \delta K$$

where K is the capital stock, t is time, s is the investment rate, δ is the depreciation rate.

Example 5.1.h: Market Equilibrium:

$$ODE: \quad \frac{dp}{dt} = \alpha(Q_d - Q_s)$$

where p is the price of the good, t is time, Q_d is the quantity demanded, Q_s is the quantity supplied, and α is the price adjustment coefficient.

Example 5.1.i: Savings-Investment Imbalance:

$$ODE: \quad \frac{d(K - I)}{dt} = \alpha(S - I)$$

where K is the capital stock, I is the investment rate, t is time, S is the savings rate, and α is the proportionality factor.

Example 5.1.j: Urbanization and Migration:

$$ODE: \quad \frac{dU}{dt} = \mu U - \nu R$$

where U is the urban population, t is time, μ is the urbanization rate, R is the rural population, and v is the migration rate.

Example 5.1.k: Economic Policy Analysis:

$$ODE: \quad \frac{dY}{dt} = \alpha(G - T)$$

where Y is the output or GDP, t is time, α is the multiplier, G is government spending, and T is taxes.

✍ *In each of these models, the first-order ODE captures the dynamic behavior of the economic variable(s) with respect to time. Solving these equations allows economists to gain insights into how various factors and parameters influence economic processes and make predictions about economic behavior under different scenarios.*

From the perspective of analytical solutions, there are two important categories of equations:

- Fundamental equations (*equations with separable variables, linear equations, equations with total differentials*)
- Equations reducible to fundamental equations (*homogeneous equations and reducible to homogeneous equations, equations admitting an integrating factor, Bernoulli-type equations, Riccati-type equations, Lagrange-type equations, Clairaut-type equations, etc*.)

Knowing the algorithm for solving fundamental equations and the methods for reducing other equations to fundamental equations is very important.

Fundamental equations: In economic mathematics, fundamental equations refer to specific types of differential equations that are frequently encountered and play a crucial role in modeling various economic phenomena. These equations are analytically solvable and serve as building blocks for more complex economic models. Some fundamental equations commonly used in economic mathematics include:

Separable Equations: Separable equations are first-order differential equations that can be separated into two separate functions, one involving the dependent variable and the other involving the independent variable. These equations are prevalent in economic growth and resource allocation models. To solve separable first-order ordinary differential equations (ODEs) in economic mathematics, we use a technique called separation of variables. The general form of a separable first-order ODE is:

$$\frac{dy}{dx} = f(x) * g(y)$$

To solve this type of ODE, follow these steps:

Step 1: Separate the variables: Move all terms involving y to the left-hand side of the equation and all terms involving x to the right-hand side. This step isolates y and x on different sides of the equation. The equation should now look like:

$$g(y)\,dy = f(x)\,dx$$

Step 2: Integrate both sides: Integrate both sides of the equation with respect to their respective variables. The integral on the left side will be with respect to y, and the integral on the right side will be with respect to x.

Step 3: Add the constant of integration: After integrating, you will obtain the general solutions in the form of $y = F(x) + C$, where $F(x)$ is the antiderivative of $f(x)$ and C is the constant of integration.

Step 4: Apply initial conditions (if provided): If the ODE is an initial value problem (IVP) and initial conditions are given, use these conditions to determine the specific value of the constant of integration (C) and obtain the particular solution.

Step 5: Check for Extraneous Solutions (if needed): Occasionally, some solutions obtained may not be valid due to division by zero or other issues. Check for such extraneous solutions and exclude them from the final result.

Example 5.1.l: Solve the separable ODE $\frac{dy}{dx} = \frac{x}{y}$, given that $y(1) = 2$.

Step 1: Separate the variables: We have:

$$y\,dy = x\,dx.$$

Step 2: Integrate both sides:

$$\int y\,dy = x\,dx$$

$$\frac{1}{2} * y^2 = \frac{1}{2} * x^2 + C$$

Step 3: Add the constant of integration:

$$y^2 = x^2 + C$$

Step 4: Apply initial condition: Using the initial condition $y(1) = 2$, we get: $2^2 = 1^2 + C \Rightarrow C = 4 - 1 = 3$

Step 5: Final Solution: The particular solution is:

$$y^2 = x^2 + 3.$$

O *In this example, we took the square root to obtain the final result* $y = \pm sqrt(x^2 + 3)$. *However, we must check for extraneous solutions and consider the appropriate sign to match the initial condition. Since* $y(1) = 2$, *we choose the positive square root, and the final solution is* $y = sqrt(x^2 + 3)$.

Linear Equations: Linear equations are first-order differential equations in which the dependent variable and its derivatives appear only to the first power and are not multiplied together. Linear equations have straightforward analytical solutions and are widely used in economic modeling, especially in linear growth and investment models.

The general form of a first-order linear ODE is:

$$\frac{dy}{dx} + P(x) * y = Q(x)$$

To solve first-order linear ordinary differential equations (ODEs) using integrating factors, follow these steps:

Step 1: Identify P(x) and Q(x):

For a first-order linear ODE, *P(x)* and Q(x) are functions of *x*.

Step 2: Find the integrating factor (IF):

The integrating factor, denoted by μ(x), is given by: $\mu(x) = e^{\int P(x)dx}$

Step 3: Multiply both sides of the ODE by the integrating factor:

Multiply the entire ODE by $\mu(x)$: $\mu(x) * (dy/dx) + \mu(x) * P(x) * y = \mu(x) * Q(x)$

Step 4: Simplify the left-hand side:

The left-hand side should now be in the form of the derivative of a product:

$$\frac{d}{dx}[\mu(x) * y] = \mu(x) * Q(x)$$

Step 5: Integrate both sides with respect to x:

Integrate both sides with respect to x to solve for $\mu(x) * y$:

$$\int \frac{d}{dx}[\mu(x) * y]\ dx = \int \mu(x) * Q(x)\ dx$$

Step 6: Solve for y(x):

After integrating, divide both sides by μ(x) to isolate y:

$$y(x) = \frac{1}{\mu(x)} \int \mu(x) * Q(x) dx$$

Step 7: Simplify the result, if needed:

Simplify the expression for $y(x)$ if possible by evaluating the integral.

Example 5.1.m: Solve the first-order linear ODE

$\frac{dy}{dx} + 2x * y = x^2$, given that y(0) = 1.

Step 1: Identify P(x) and Q(x):

$P(x) = 2x$

$Q(x) = x^2$

Step 2: Find the integrating factor (IF):

$$\mu(x) = e^{\int P(x)dx} = e^{\int 2x\ dx} = e^{x^2}$$

Step 3: Multiply both sides by the integrating factor:

$$e^{x^2} * \frac{dy}{dx} + 2x * e^{x^2} * y = x^2 * e^{x^2}$$

Step 4: Simplify the left-hand side:

$$\frac{d}{dx}\left[e^{x^2} * y\right] = x^2 * e^{x^2}$$

Step 5: Integrate both sides with respect to x:

$$\int \frac{d}{dx}\ [e^{x^2} * y]\ dx = \int \left(x^2 * e^{x^2}\right) dx$$

Step 6: Solve for y(x):

$$e^{x^2} * y = \int \left(x^2 * e^{x^2}\right)\ dx$$

Step 7: Simplify the result:

Integrate the right-hand side to get: $e^{x^2} * y = \frac{1}{3} * e^{x^2} + C$

Divide both sides by e^{x^2} to isolate y: $y(x) = \frac{1}{3} * e^{-x^2} + C * e^{-x^2}$

Step 8: Apply the initial condition:

Using the initial condition $y(0) = 1$: $1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$

Step 9: Final Solution:

The particular solution is

$$y(x) = \frac{1}{3} * e^{-x^2} + \frac{2}{3} * e^{-x^2} = e^{-x^2}.$$

Equations with Total Differentials: Equations with total differentials are often encountered in economic optimization problems. These equations involve the total derivative of a function, and they help determine the sensitivity of economic variables to changes in other variables.

Equations with total differentials are a specific type of differential equations that involve the total derivative of a function rather than partial derivatives. These equations typically arise in economic optimization problems, where economists seek to maximize or minimize a function with respect to several variables subject to certain constraints. The total differential accounts for changes in all the variables simultaneously, allowing economists to analyze the sensitivity of economic variables to changes in other variables.

The general form of an equation with total differentials involving two variables, x and y, is:

$$dF = \frac{\partial F}{\partial x} * dx + \frac{\partial F}{\partial y} * dy$$

where *F* is a function of *x* and *y*, and *dx* and *dy* are small changes in the variables *x* and *y*, respectively.

In economic mathematics, equations with total differentials are often used to analyze optimization problems, such as utility maximization or profit maximization, subject to budget constraints or production constraints.

Example 5.1.o: Utility Maximization with a Budget Constraint:

Consider a consumer who wants to maximize their utility (U) subject to a budget constraint. The consumer spends their income (I) on two goods, *x* and *y*, with prices p_x and p_y, respectively.

The consumer's utility function is $U(x, y)$, and the budget constraint is given by:

$$p_x * x + p_y * y = I$$

The consumer's goal is to maximize their utility subject to the budget constraint.

To find the optimal consumption of goods *x* and *y*, we use the method of Lagrange multipliers. The Lagrangian function is:

$$L(x, y, \lambda) = U(x, y) + \lambda(I - p_x * x - p_y * y)$$

where λ is the Lagrange multiplier.

To find the critical points, we set the partial derivatives of the Lagrangian with respect to x, y, and λ equal to zero:

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial U}{\partial x} - \lambda * p_x = 0 \\ \frac{\partial U}{\partial y} - \lambda * p_y = 0 \\ I - p_x * x - p_y * y = 0 \end{cases}$$

O *Solving this system of equations will give us the optimal values of x and y that maximize the consumer's utility subject to the budget constraint.*

In this example, the total differential dU represents the change in utility due to simultaneous changes in x and y. The consumer aims to find the combination of x and y that maximizes utility while satisfying the budget constraint.

Equations with total differentials are a powerful tool in economic optimization, allowing economists to analyze the marginal utility, marginal cost, or other sensitivities of economic variables to changes in input or output quantities. These equations aid in understanding consumer behavior, production decisions, and resource allocation in various economic contexts.

✍ *Solving fundamental equations is an essential skill in economic mathematics as it allows economists to understand the behavior of economic systems and make predictions based on mathematical models. These equations provide valuable insights into economic growth, investment dynamics, consumption patterns, production functions, and other economic variables that influence decision-making and policy analysis. Moreover, mastering the techniques to solve fundamental equations forms the foundation for tackling more complex economic models and differential equations encountered in advanced economic studies.*

Equations reducible to fundamental equations: Equations reducible to fundamental equations are a class of differential equations that can be transformed or converted into more straightforward and analytically solvable fundamental equations. These transformations often involve making suitable substitutions or changes of variables that simplify the original equation and reduce it to a known form. This reduction allows economists and mathematicians to apply standard techniques for solving the simplified fundamental equations. Several types of equations fall under this category, including:

Homogeneous Equations: Homogeneous differential equations are those in which all terms can be expressed as a function of the same variable or the dependent variable y, making them amenable to a simple substitution to reduce the order and separate the variables. Homogeneous ordinary differential equations (ODEs) are a specific type of first-order ODEs that can be written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

To solve homogeneous ODEs, you can use a technique called the *Substitution Method* or the *Change of Variables Method*.

Here are the steps to solve homogeneous ODEs:

Step 1: Make the substitution:

Let $u = \frac{y}{x}$. This means replacing *y* in the ODE with $u * x$.

Step 2: Differentiate with respect to x:

Take the derivative of *u* with respect to *x*. To do this, use the product rule since $u = \frac{y}{x}$. The derivative $\frac{du}{dx}$ can be expressed in terms of *u*, *x*, and *y* using the chain rule.

Step 3: Substitute back:

Replace $\frac{dy}{dx}$ in the original ODE with the expression obtained for du/dx in terms of *u*, *x*, and *y*.

Step 4: Separate variables:

Now that the ODE is in terms of *u* and *x*, separate variables to have all terms involving *u* on one side and all terms involving *x* on the other side.

Step 5: Integrate both sides:

Integrate the separated equation with respect to *x* to find the general solution in terms of *u* and *x*.

Step 6: Solve for y:

Since $u = \frac{y}{x}$, solve for *y* in terms of *x* and the constant of integration obtained after integration.

Let's go through an example to illustrate how to solve a homogeneous ODE:

Example 5.1.n: Solve the homogeneous ODE $\frac{dy}{dx} = \frac{2y - x}{x + 2y}$.

Step 1: Make the substitution:

Let u = y/x, which means y = ux.

Step 2: Differentiate with respect to x:

Using the product rule, $\frac{du}{dx} = \frac{x\left(\frac{dy}{dx}\right) - y}{x^2}$.

Since $\frac{dy}{dx} = \frac{2y - x}{x + 2y}$, we have $\frac{du}{dx} = \frac{x\left(\frac{2y - x}{x + 2y}\right) - y}{x^2}$.

Step 3: Substitute back:

Replace $\frac{dy}{dx}$ in the original ODE with $\frac{du}{dx}$ in terms of *u*, *x*, and *y*:

$$\frac{du}{dx} = \frac{x\left(\frac{2(ux) - x}{x + 2(ux)}\right) - ux}{x^2} \Rightarrow \frac{du}{dx} = \frac{\frac{1 - u - 2u^2}{1 + 2u}}{x}$$

Step 4: Separate variables:

Now, separate variables so that all terms involving u are on one side and all terms involving x are on the other side:

$$\left(\frac{1 + 2u}{1 - u - 2u^2}\right) du = \left(\frac{1}{x}\right) dx.$$

Step 5: Integrate both sides:

Integrate the separated equation with respect to x and u:

$\left(\frac{1+2u}{1-u-2u^2}\right) du = \left(\frac{1}{x}\right) dx \Rightarrow -\frac{2}{3}\ln(1 - 2u) - \frac{1}{3}\ln(u + 1) + C = \ln x \Rightarrow$

$\Rightarrow (1 - 2u)(u + 1)^2 = Cx^{-\frac{3}{2}}$

Step 6: Solve for y: After integration, you will have the general solution in terms of *u* and *x*. Since $u = y/x$, you can now solve for *y* in terms of *x* and the constant of integration

$$\left(1 - \frac{2y}{x}\right)\left(\frac{y}{x} + 1\right)^2 = Cx^{-\frac{3}{2}}$$

O *Please note that some homogeneous ODEs might require additional algebraic manipulations to simplify the expressions during the solving process.*

Equations Admitting an Integrating Factor: Some differential equations can be multiplied by a suitable integrating factor, which allows them to become exact differential equations, making them easier to solve using integration.

Equations admitting an integrating factor are a class of differential equations where a specific multiplying factor can be applied to the entire equation to transform it into an exact differential equation. Integrating factors play a crucial role in simplifying the solution of certain types of non-exact ordinary differential equations (ODEs).

The general form of an ordinary differential equation that admits an integrating factor is:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where *M* and *N* are continuous functions of two variables *x* and *y*.

To make this equation exact, we multiply it by a suitable *Integrating Factor* (IF), denoted by $\mu(x,y)$, such that:

$$\mu(x,y) * (M(x,y)\,dx + N(x,y)\,dy) = 0$$

If the multiplication by the integrating factor $\mu(x,y)$ transforms the equation into an exact differential, then the equation is said to admit an integrating factor.

The integrating factor μ(x, y) is determined by the condition that:

$$\frac{\partial(\mu N)}{\partial x} - \frac{\partial(\mu M)}{\partial y} = 0$$

Once we find the integrating factor $\mu(x,y)$, we can rewrite the equation as:

$$d(u(x,y)) = 0$$

where $u(x,y)$ is a function of *x* and *y*. This means that the equation becomes an exact differential and can be expressed as the total differential of $u(x,y)$.

The solution to the original equation is then obtained by integrating $u(x,y)$ with respect to *x* and *y*.

Exact ordinary differential equations (ODEs) are a special type of differential equation that can be solved analytically without the need for any integrating factors or other transforming techniques.

The solution to an exact ODE can be found by following these steps:

Step 1: Check for Exactness: Verify if the given ODE can be expressed in the form $M(x,y)\,dx + N(x,y)\,dy = 0$ and if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Step 2: Find the Integrating Factor (IF): If the equation is not exact, then a suitable integrating factor (IF) is sought, which is a function μ(x, y) that, when multiplied with the entire equation, makes it exact.

Step 3: Multiply by the Integrating Factor: Multiply both sides of the ODE by the integrating factor μ(x, y)

$$\mu(x,y) = e^{\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx}$$

Step 4: Rewrite the Equation: Express the equation in the form μ(x, y) M(x, y) dx + μ(x, y) N(x, y) dy = 0.

Step 5: Identify the Total Differential: Check if the resulting equation is an exact differential, i.e., it can be expressed as the total differential of a function u(x, y).

Step 6: Find the Solution: If the equation is exact, find u(x, y) by integrating with respect to x and then with respect to y. The solution u(x, y) represents the general solution to the original exact ODE.

Example 5.1.o: Solving an Exact ODE: Let's consider the following exact ODE:

$$(2x - y)\,dx + (x + 3y^2)\,dy = 0$$

To check if it is exact, we compute the partial derivatives:

$$\begin{cases} \dfrac{\partial M}{\partial y} = -1 \\ \dfrac{\partial N}{\partial x} = 1 \end{cases}$$

Since the partial derivatives are not equal, this equation is not exact in its current form. To make it exact, we need to multiply through by an integrating factor ($\mu(x,y)$) that depends on *x* and *y*. The integrating factor is given by:

$$\mu(x,y) = e^{\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx} \Rightarrow \mu(x,y) = e^{\int (1-(-1))dx} \Rightarrow \mu(x,y) = e^{\int dx} = e^{2x}$$

Now, we multiply the original equation by the integrating factor:

$$e^{2x} * [(2x - y)\,dx + (x + 3y^2)\,dy] = 0$$

Expanding and rearranging:

$$(2x\,e^{2x} - y\,e^{2x})\,dx + (x\,e^{2x} + 3y^2\,e^{2x})\,dy = 0$$

Now, we can recognize the left-hand side as the exact differential of a certain function, say $F(x,y)$:

$$dF = (2x\,e^{2x} - y\,e^{2x})\,dx + (x\,e^{2x} + 3y^2\,e^{2x})\,dy$$

To find $F(x,y)$, we integrate the above expression with respect to *x* and then *y*:

$$\int (2x\,e^{2x} - y\,e^{2x})\,dx = e^{2x}(x^2 - xy) + g(y)$$

where $g(y)$ is an arbitrary function of *y*.

Taking the partial derivative of the above result with respect to y:

$$\frac{\partial F}{\partial y} = \frac{\partial\left(e^{2x}(x^2 - xy) + g(y)\right)}{\partial y} = x\,e^{2x} + g'(y)$$

Comparing this with the $N(x,y)$ in the original equation, we get:

$$g'(y) = 3y^2\,e^{2x}$$

Integrating g'(y) with respect to y, we find:

$$g(y) = y^3\,e^{2x} + C$$

where *C* is the constant of integration.

Now, we have the function $F(x,y)$:

$$F(x,y) = e^{2x}(x^2 - xy) + y^3\,e^{2x} + C$$

Finally, the solution to the exact ODE is given by:

$$F(x,y) = e^{2x}(x^2 - xy) + y^3\,e^{2x} + C = constant$$

This is the general solution to the exact ODE. If initial or boundary conditions are given, we can determine the constant *C* and obtain a specific solution.

⭘ *In economic mathematics, exact ODEs are used to model economic systems where certain quantities and relationships are known with precision, and the goal is to find the specific functional forms that satisfy these conditions. Solving exact ODEs allows economists to analyze economic models, optimize economic outcomes, and understand the behavior of economic variables in different scenarios.*

✍ *Integrating factors are particularly useful when solving linear ODEs with non-constant coefficients or when trying to transform certain non-exact ODEs into exact ones. In economics, equations admitting an integrating factor can arise in various modeling contexts, such as economic growth models with non-constant parameters, utility functions with non-linear preferences, or dynamic optimization problems with varying constraints.*

Using integrating factors, economists can simplify the process of solving complex differential equations, allowing for a more straightforward analysis of economic models and systems.

Bernoulli-type Equations: Bernoulli equations are nonlinear differential equations that can be transformed into linear form by using a substitution of the form $v = y^{1-n}$, where *n* is a constant, leading to a first-order linear equation. Bernoulli ordinary differential equations (ODEs) are a specific type of nonlinear first-order ODEs that can be written in the form:

$$\frac{dy}{dx} + P(x) * y = Q(x) * y^n$$

where *n* is a constant and $n \neq 0, 1$. To solve Bernoulli ODEs, you can use a technique called the *Bernoulli Substitution Method*.

Here are the steps to solve Bernoulli ODEs:

Step 1: Identify the Bernoulli ODE:

The given ODE should be in the form $\frac{dy}{dx} + P(x) * y = Q(x) * y^n$.

Step 2: Make the substitution: Let $u = y^{1-n}$. This means replacing *y* in the ODE with $u^{\frac{1}{1-n}}$.

Step 3: Differentiate with respect to *x*: Take the derivative of *u* with respect to *x*. To do this, use the chain rule since $u = y^{1-n}$. The derivative $\frac{du}{dx}$ can be expressed in terms of *u*, *x*, and *y* using the chain rule.

Step 4: Substitute back: Replace $\frac{dy}{dx}$ and *y* in the original ODE with the expressions obtained for $\frac{du}{dx}$ and $u^{\frac{1}{1-n}}$, respectively.

Step 5: Separate variables: Now that the ODE is in terms of *u* and *x*, separate variables to have all terms involving *u* on one side and all terms involving *x* on the other side.

Step 6: Integrate both sides: Integrate the separated equation with respect to *x* to find the general solution in terms of *u* and *x*.

Step 7: Solve for y: Since $u = y^{1-n}$, solve for *y* in terms of *x* and the constant of integration obtained after integration.

Example 5.1.p: Solve the Bernoulli ODE $\frac{dy}{dx} + 2x * y = xy^3$.

Step 1: Identify the Bernoulli ODE:

The given ODE is $\frac{dy}{dx} + 2x * y = xy^3$.

Step 2: Make the substitution:

Let $u = y^{-2}$. This means replacing y in the ODE with $u^{-\frac{1}{2}}$.

Step 3: Differentiate with respect to x:

Using the chain rule, $\frac{du}{dx} = (-2) * y^{-3} * \frac{dy}{dx}$.

Step 4: Substitute back:

Replace $\frac{dy}{dx}$ and *y* in the original ODE with the expressions obtained for du/dx and $u^{-\frac{1}{2}}$, respectively: $(-2) * y^{-3} * \frac{du}{dx} + 2x * y = x * \left(u^{-\frac{1}{2}}\right)^3 \Rightarrow \frac{du}{dx} = 2x * (1 + 2u)$.

Step 5: Separate variables:

Now, separate variables so that all terms involving u are on one side and all terms involving x are on the other side: $\frac{du}{1+2u} = 2xdx.$

Step 6: Integrate both sides:

Integrate the separated equation with respect to x and u:

$$\int \frac{du}{1+2u} = \int 2xdx \;\Rightarrow \frac{1}{2}\ln(1+2u) = x^2 + C$$

Step 7: Solve for y: After integration, you will have the general solution in terms of *u* and *x*. Since $u = y^{-2}$, you can now solve for y in terms of x and the constant of integration

$$\ln(1+2u) = 2x^2 + C \Rightarrow 1 + 2u = C * e^{2x^2} \Rightarrow u = \frac{C * e^{2x^2} - 1}{2}$$

$$y^{-2} = \frac{C * e^{2x^2} - 1}{2} \Rightarrow \quad y = \left(\frac{C * e^{2x^2} - 1}{2}\right)^{-\frac{1}{2}}$$

O *Please note that some Bernoulli ODEs might require additional algebraic manipulations to simplify the expressions during the solving process.*

Riccati-type Equations: Riccati equations are a type of nonlinear first-order differential equation that can be converted into a linear second-order equation or a linear first-order equation by making certain substitutions. Riccati ordinary differential equations (ODEs) are a specific type of nonlinear first-order ODEs that can be written in the form:

$$\frac{dy}{dx} = P(x) * y^2 + Q(x) * y + R(x)$$

where *P(x)*, *Q(x)*, and *R(x)* are known functions of *x*. Solving Riccati ODEs analytically can be challenging, but there are some special cases and transformation techniques that can be used to obtain solutions.

Here are some methods commonly used to solve Riccati ODEs:

Constant Term Substitution: If the Riccati ODE contains a constant term ($R(x) \neq 0$), you can sometimes simplify the equation by making a substitution of the form $y = u + \frac{v}{x}$, where *u* and *v* are functions of *x*. This substitution can lead to a linear ODE for v(x) that is easier to solve. Once $v(x)$ is found, you can determine $u(x)$ and then $y(x)$.

Bernoulli Substitution: For certain Riccati ODEs, a Bernoulli substitution of the form $y = -\frac{z'}{[v(x)*z]}$ can transform the equation into a linear ODE for $z(x)$ that is solvable. Once $z(x)$ is found, you can obtain $y(x)$ from the relationship $y = -\frac{z'}{[v(x)*z]}$.

Known Particular Solution: If you can find a known particular solution y_p(x) of the Riccati ODE, you can transform the ODE into a linear ODE for $u(x) = y(x) - y_{p(x)}$. Solving this linear ODE for $u(x)$ allows you to determine $y(x)$.

Symmetry Method***:*** The symmetry method involves looking for Lie point symmetries of the Riccati ODE, which are transformations that leave the ODE invariant. By finding such symmetries, you can sometimes obtain reductions that lead to simpler equations.

Numerical Methods***:*** In many cases, Riccati ODEs cannot be solved analytically. In such situations, numerical methods, such as Euler's method or Runge-Kutta methods, can be used to approximate solutions.

Let's illustrate the *Constant Term Substitution Method* and the *Bernoulli Substitution Method* with examples of solving specific Riccati ODEs:

Example 5.1.r: Constant Term Substitution Method solve the Riccati ODE:

$$\frac{dy}{dx} = x^2 - 2x - y^2$$

Solution:

Step 1: Make the substitution:

$$y = u + \frac{v}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx}\left(\frac{v}{x}\right)$$

Step 2: Substitute the derivatives in the original ODE:

$$\frac{du}{dx} + \frac{d}{dx}\left(\frac{v}{x}\right) = x^2 - 2x - \left(u + \frac{v}{x}\right)^2$$

Step 3: Simplify and solve for $v(x)$:

$$\frac{d}{dx}\left(\frac{v}{x}\right) = -2u - \frac{v^2}{x^2}v = -\frac{x^2\left(\frac{du}{dx} + \frac{v^2}{x^2} + 2u\right)}{x}$$

Step 4: Substitute *v* back into the equation and simplify:

$$\frac{du}{dx} = x^2 - 2x - \left(u - \frac{x^2\left(\frac{du}{dx} + \frac{v^2}{x^2} + 2u\right)}{x}\right)^2$$

Step 5: Now, we have a first-order linear ODE for u(x) Solve this ODE to get u(x). Once u(x) is found, we can determine y(x) using $y = u + \frac{v}{x}$.

The *Bernoulli Substitution Method* is a technique used to transform a nonlinear first-order ordinary differential equation (ODE) of the form: $\frac{dy}{dx} + P(x)y = Q(x)y^n$ into a linear first-order ODE, which is easier to solve. The transformation involves introducing a new variable $z = y^{1-n}$, which reduces the Bernoulli equation to a linear equation in terms of *z*. Let's consider an example of applying the Bernoulli substitution method to solve a specific Riccati ODE.

Example 5.1.s: Solve the Riccati ODE using the Bernoulli Substitution Method:

$$\frac{dy}{dx} = x^2 + 2xy^2$$

Step 1: Identify the Riccati ODE and the values of *P(x)* and *Q(x)*: The given Riccati ODE is

$$\frac{dy}{dx} = x^2 + 2xy^2$$

where $P(x) = 2x$ and $Q(x) = x^2$.

Step 2: Apply the Bernoulli substitution: We introduce a new variable $z = y^{-1}$ (Note: $n = -1$ in this case). Now, differentiate z with respect to x:

$$\frac{dz}{dx} = \frac{d}{dx}(y^{-1}) = -y^{-2} * \frac{dy}{dx}$$

Step 3: Substitute the new variable and its derivative into the original Riccati ODE:

$$\frac{dy}{dx} = x^2 + 2xy^2$$

$$y^{-2} * \frac{dy}{dx} = -y^{-2} * (x^2 + 2xy^2)$$

Now, replace $\frac{dy}{dx}$ with $\frac{dz}{dx}$ in the equation: $\frac{dz}{dx} = -y^{-2} * (x^2 + 2xy^2)$

Step 4: Express y in terms of *z*: Recall that $z = y^{-1}, so\ y = z^{-1} = \frac{1}{z}$.

Now, replace *y* in the equation with $\frac{1}{z}$:

$$\frac{dz}{dx} = -\left(\frac{1}{z}\right)^2 * \frac{\left(x^2 + 2x\left(\frac{1}{z}\right)^2\right)dz}{dx} = -z^2 * (x^2 + \frac{2x}{z^2})$$

Step 5: Simplify the equation: $\frac{dz}{dx} = -z^2 * (x^2 + \frac{2x}{z^2})$

Step 6: Transform into a linear first-order ODE: Multiply both sides by z^2 to eliminate the denominator: $z^2 * \frac{dz}{dx} = -(x^2 + \frac{2x}{z^2})$. Now, we have a linear first-order ODE in terms of *z* and *x*.

Step 7: Solve the linear ODE for z: To solve the linear ODE, integrate both sides with respect to x:

$$\int 1\, dz = -\int (z^2x^2 + 2x)\, dx$$

The left-hand side integral is straightforward: $z + C1$

The right-hand side integral requires integration by parts for the term $\int (z^2x^2 + 2x)\, dx$.

Step 8: Substitute back to find y: Now that we have the value of *z*.

Finally, the solution to the original Riccati ODE is given by $y = \frac{1}{z}$, where *z* is the result obtained after integrating both sides of the linear first-order ODE.

✍ *It's important to note that Riccati ODEs do not always have general closed-form solutions, and the methods mentioned above may not always be applicable or lead to explicit solutions. In some cases, obtaining solutions may require numerical approaches or additional techniques specific to the particular Riccati ODE being studied.*

Lagrange-type Equations: Lagrange equations involve higher-order derivatives of the dependent variable and can be transformed into simpler ODEs through suitable changes of variables.

Clairaut-type Equations: Clairaut equations are differential equations with special properties that can be simplified through suitable substitutions.

✍ *The process of reducing these equations to fundamental equations often involves clever algebraic manipulations and appropriate transformations. Once reduced to a fundamental equation, standard techniques like separation of variables, integrating factors, or other methods applicable to fundamental equations can be employed to find the solutions.*

Equations reducible to fundamental equations are prevalent in economic modeling and analysis. They provide economists with a toolkit to tackle more complex differential equations, allowing them to gain insights into economic phenomena and make predictions based on

mathematical models. These techniques are essential for understanding dynamic economic systems and optimizing decision-making processes in various economic scenarios.

Solving of initial value problems (IVPs) for first-order ODEs: To solve initial value problems (IVPs) for first-order ordinary differential equations (ODEs), you need to find the particular solution that satisfies both the differential equation and the given initial condition(s). An IVP involves providing the value of the dependent variable (usually denoted by *y*) at a specific point or time, which allows you to determine the constant(s) of integration in the general solution.

Here's a step-by-step process to solve IVPs for first-order ODEs:

Step 1: Write the first-order ODE: Start by writing the given first-order ODE. It should be in the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a function of x and y.

Step 2: Solve the ODE to obtain the general solution: Integrate the ODE to find the general solution $y = F(x) + C$, where $F(x)$ is the antiderivative of $f(x, y)$, and C is the constant of integration.

Step 3: Apply the initial condition(s): Use the given initial condition(s) to determine the value of the constant(s) of integration. Substitute the initial value(s) into the general solution obtained in Step 2.

Step 4: Obtain the particular solution: With the constant(s) of integration determined from the initial condition(s), you get the particular solution for the IVP.

Example 5.1.t: Solve the initial value problem for the first-order ODE

$$\frac{dy}{dx} = x^2 - 2x, \text{ with the initial condition } y(0) = 1$$

Step 1: Write the first-order ODE: The given ODE is

$$\frac{dy}{dx} = x^2 - 2x$$

Step 2: Solve the ODE to obtain the general solution: Integrate the ODE with respect to x:

$$\int dy = \int (x^2 - 2x)dx \Rightarrow y = \frac{x^3}{3} - x^2 + C$$

Step 3: Apply the initial condition(s): Use the initial condition y(0) = 1:

$$1 = \left(\frac{0^3}{3}\right) - 0^2 + C \Rightarrow 1 = 0 + C \Rightarrow C = 1$$

Step 4: Obtain the particular solution: With C = 1, the particular solution for the IVP is

$$y = \frac{x^3}{3} - x^2 + 1.$$

✪ *Thus, the solution to the given initial value problem is* $y = \frac{x^3}{3} - x^2 + 1$, *satisfying both the first-order ODE* $\frac{dy}{dx} = x^2 - 2x$ *and the initial condition* $y(0) = 1$.

The implications of different solutions (e.g., stable, unstable, or oscillatory) of first-order ODEs in economic scenarios: The solutions of first-order ODEs in economic scenarios can have

various implications depending on their stability and behavior. The stability of a solution refers to how the system responds to small perturbations or changes in the initial conditions.

Here are some implications of different types of solutions:

Stable Solutions: In economic scenarios, a stable solution indicates that the system will tend to return to its equilibrium state after experiencing small disturbances. This behavior is desirable for economic models, as it implies that the system is resilient and can recover from shocks or fluctuations.

For example, in an economic model describing the equilibrium price of a product, a stable solution implies that the market will naturally converge back to the equilibrium price even after facing short-term changes in supply or demand.

Unstable Solutions: An unstable solution suggests that the system is highly sensitive to small changes in initial conditions. Even a slight perturbation can lead to significant deviations from the equilibrium, and the system will move away from the initial state indefinitely.

In economic models, an unstable solution might indicate the potential for economic instability, leading to large fluctuations or even economic crises. Such solutions may represent unsustainable economic conditions.

Oscillatory Solutions: Oscillatory solutions in economic scenarios imply that the system will exhibit periodic or repetitive behavior around an equilibrium. These solutions may represent cyclical economic patterns, such as business cycles.

For instance, in a model describing inventory levels, an oscillatory solution could represent inventory oscillating between high and low levels periodically, reflecting regular stock replenishment.

Damped Solutions: Damped solutions involve a gradual decrease in amplitude over time. The system may still return to an equilibrium, but the oscillations dampen out, meaning the system approaches a steady-state over time.

In economic scenarios, damped solutions could represent situations where initial disturbances are eventually absorbed, and the system stabilizes at a new equilibrium.

Explosive Solutions: Explosive solutions occur when the system experiences unbounded growth over time. The magnitude of the solution keeps increasing without limit.

In economics, an explosive solution may imply hyperinflation or unsustainable economic growth, which can lead to economic instability.

✍ *The implications of different types of solutions highlight the importance of stability analysis in economic models. Stability analysis allows economists to assess the behavior of the system over time, understand its resilience to disturbances, and predict the long-term implications of various economic scenarios. Stable and well-behaved solutions are generally preferred in economic models as they indicate a more robust and sustainable economic system.*

Questions you should be able to answer:

What is a first-order ordinary differential equation?

How do you classify first-order ODEs based on their form and properties?

What are the methods used to solve separable first-order ODEs?

How do you solve first-order linear ODEs using integrating factors?

What is an exact differential equation, and how do you determine if an equation is exact?

What is the integrating factor method, and when is it used to solve ODEs?

How do you solve initial value problems (IVPs) for first-order ODEs?

Provide examples of economic models that can be represented by first-order ODEs.

How can you use first-order ODEs to model investment decisions in economic mathematics?

Explain the concept of stability analysis for first-order ODEs in economic systems.

What are the applications of first-order ODEs in studying population dynamics in economics?

How are first-order ODEs used in economic growth models, such as the Solow-Swan model?

What is the significance of the equilibrium solutions in the context of first-order ODEs in economic models?

Describe the role of first-order ODEs in optimal control theory and its applications in economic decision-making.

How can you use first-order ODEs to study dynamic input-output models in economics?

Discuss the implications of different solutions (e.g., stable, unstable, or oscillatory) of first-order ODEs in economic scenarios.

EXERCISES

5.1.1 Fundamental equations and reductible to fundamental equations:

a) $\frac{dy}{dx} + 2xy = 3x^2$

b) $y' = y^2 - x^2$

c) $(2x + 3y)dx + (x - y)dy = 0$

d) $\frac{dy}{dx} = 2xy^2 + y - x^2$

e) $\frac{dx}{dy} = \frac{x^2 + y^2}{x + y}$

f) $y' - \frac{y}{x} = 0$

g) $\frac{dy}{dx} + y = x^2$

h) $\frac{dy}{dx} + 2xy = e^x$

i) $\frac{dy}{dx} + y^2 = x^3$

j) $\frac{dy}{dx} + xy = ln(x)$

k) $\frac{dy}{dx} + \frac{y}{x} = x^2 - 3$

l) $\frac{dy}{dx} + \frac{y}{x} = e^x$

5.1.2 Equations reductible to fundamental equations:

a) Solve the following first-order ODE using an integrating factor:

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

b) Solve the following first-order homogeneous ODE:

$$(2x + y)\,dx + (x + 3y)\,dy = 0$$

c) Solve the following first-order ODE using an integrating factor:

$$(2x + 3y)\,dx + (x + 2y)\,dy = 0$$

d) Solve the following Bernoulli equation:

$$x\frac{dy}{dx} - 2y = \frac{y^3}{x^2}$$

e) Solve the following first-order homogeneous ODE:

$$(x^2 + y^2)\,dx + 2xy\,dy = 0$$

f) Solve the following first-order ODE using an integrating factor:

$$(x^2 - 3xy)\,dx + (y^2 + x)\,dy = 0$$

g) Solve the following first-order ODE using an integrating factor:

$$(2xy - y^2)\,dx + (x^2 + xy)\,dy = 0$$

h) Solve the following Bernoulli equation:

$$2x\frac{dy}{dx} + y = \frac{y^3}{x^2}$$

i) Solve the following first-order homogeneous ODE:

$$(3x^2 + y^2)\,dx + 2xy\,dy = 0$$

j) Solve the followi ng first-order ODE using an integrating factor:

$$(x^2 + 4xy)\,dx + (y^2 - x)\,dy = 0$$

5.1.3 Initial value problem of the first-order ODEs: Specify if the function y = y(x) is a solution of the differential equation (or of the Initial value problem) in the following cases. For each solution, specify its field of definition:

a)	$y^2 + xy - x^2y' = 0$	$y(x) = -x/(C + lnx)$
b)	$y' + 2xy = 2xe^{-x^2}$	$y(x) = (C + x)e^{-x^2}$
c)	$y' - y = xy^2$	$y(x) = 1/(1 - x + Ce^{-x})$
d)	$\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 2,\ y'(0) = 3 \end{cases}$	$y(x) = e^x + e^{2x}$
e)	$y'' - 5y'6y = 8e^x$	$y(x) = C_1e^{2x} + C_2e^{3x} + 4x$
f)	$\begin{cases} (2x+1)^2y'' - 4(2x+1)y' + 8y = 0 \\ y(1) = 12,\ y'(1) = 14 \end{cases}$	$y(x) = (2x + 1)(2x + 2)$

5.2 SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Keywords: *Second-order, Ordinary Differential Equations (ODEs), Linear ODEs, Nonlinear ODEs, Homogeneous ODEs, Non-homogeneous ODEs, Constant coefficients, Variable coefficients, Initial value problems (IVPs), Boundary value problems (BVPs), Economic dynamics, Economic growth models/*

Second-order ordinary differential equations (ODEs) are a type of differential equation that involves the second derivative of an unknown function with respect to a single independent variable. They are widely used in various scientific and engineering applications to describe dynamical systems and phenomena where the behavior of a function depends on its first and second derivatives.

A second-order ODE can generally be written in the form:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

where $y = y(x)$ is the unknown function, *x* is the independent variable, $\frac{dy}{dx}$ is the first derivative of *y* with respect to *x*, and $f\left(x, y, \frac{dy}{dx}\right)$ is a given function involving *x*, *y*, and $\frac{dy}{dx}$.

Second-order ODEs can be classified based on their characteristics and properties:

Linear Second-Order ODEs: If the function $f\left(x, y, \frac{dy}{dx}\right)$ is linear in *y* and $\frac{dy}{dx}$, the second-order ODE is called a linear ODE. Linear ODEs can be further categorized as homogeneous or non-homogeneous, depending on whether $f\left(x, y, \frac{dy}{dx}\right)$ is zero or nonzero, respectively.

Nonlinear Second-Order ODEs: If the function $f\left(x, y, \frac{dy}{dx}\right)$ involves nonlinear terms in y and dy/dx, the second-order ODE is classified as a nonlinear ODE. Nonlinear ODEs are generally more difficult to solve analytically compared to linear ODEs.

Solving second-order ODEs can be challenging, and there are various methods to obtain solutions depending on the equation's characteristics:

Method of Undetermined Coefficients: Used to solve linear non-homogeneous ODEs with constant coefficients when the non-homogeneous term is a polynomial, exponential, trigonometric, or combinations of these functions.

Method of Variation of Parameters: A technique used to find the particular solution of a linear non-homogeneous ODE with variable coefficients.

Reduction of Order: A method used to transform a second-order ODE into a first-order ODE by introducing a new variable.

Separation of Variables: A technique used to solve certain types of nonlinear second-order ODEs by separating the variables.

Power Series Method: A method used for solving linear ODEs with variable coefficients or nonlinear ODEs by expressing the unknown function as a power series and solving for the coefficients.

Numerical Methods: When analytical solutions are not feasible, numerical methods such as Euler's method, Runge-Kutta methods, and finite difference methods can be used to approximate solutions.

✍ *The solutions to second-order ODEs play a crucial role in modeling various physical systems, including oscillatory motion, vibrations, electrical circuits, and mechanical systems. The study of second-order ODEs is essential in understanding the behavior of these systems and predicting their responses under different conditions.*

Examples of Second-Order ODEs in Economic Mathematics:

Economic Growth Models: The Solow-Swan growth model is a famous economic model that uses a second-order ODE to describe the dynamics of capital accumulation and economic growth.

Investment Dynamics: In investment analysis, second-order ODEs can be used to study the behavior of capital investment over time, taking into account depreciation and the rate of return on investment.

Population Models: The logistic growth model, used in population ecology and demographic analysis, can be expressed as a second-order ODE to describe population growth with carrying capacity.

Dynamic Equilibrium: Economic systems often seek dynamic equilibrium, where variables adjust over time to achieve stability. Second-order ODEs model such dynamic equilibria in various economic scenarios.

Business Cycles: Some macroeconomic models use second-order ODEs to study business cycles and the fluctuations in economic activity over time.

Solving second-order ODEs in economic mathematics involves finding the general solution and specific solutions with given initial or boundary conditions.

Solving linear second-order ODEs: Linear second-order ordinary differential equations (ODEs) are differential equations that can be expressed in the form:

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)\,y = f(x)$$

where $a(x), b(x), c(x)$, and $f(x)$ are given functions of the independent variable *x*. Solving linear second-order ODEs involves finding the function $y(x)$ that satisfies the equation.

There are several methods to solve linear second-order ODEs, depending on the specific form of the coefficients and the function f(x). Some common methods are:

Method of Undetermined Coefficients: This method is used to solve non-homogeneous linear second-order ODEs when the function $f(x)$ is a polynomial, exponential, trigonometric, or a combination of these functions. In this method, we assume a particular form for the solution and determine the unknown coefficients by substitution.

The Method of Undetermined Coefficients is a technique used to find particular solutions to non-homogeneous linear ordinary differential equations (ODEs) when the right-hand side of the equation is a sum of known functions. This method is particularly useful for first-order and higher-order ODEs with constant coefficients.

The general form of a non-homogeneous linear ODE is:

$$L[y] = f(x)$$

where $L[y]$ represents a linear differential operator involving the dependent variable *y* and its derivatives, and $f(x)$ is a known function on the right-hand side.

The steps to apply the *Method of Undetermined Coefficients* are as follows:

Step 1: Find the Complementary Function (CF): First, find the general solution to the corresponding homogeneous ODE (i.e., when $f(x) = 0$). This is the Complementary Function (CF), and it represents the solution to the homogeneous part of the equation.

Step 2: Guess the Particular Solution (PS): Next, make an educated guess for the particular solution $y_{p(x)}$ based on the form of $f(x)$. The guess should include terms that are similar to the terms present in $f(x)$, but with undetermined coefficients.

Step 3: Substitute the PS into the ODE: Substitute the guessed particular solution $y_{p(x)}$ into the original non-homogeneous ODE.

Step 4: Determine the Undetermined Coefficients: Equate the coefficients of the terms in the PS with the corresponding terms in $f(x)$ and solve for the undetermined coefficients.

Step 5: Form the General Solution (y_g): Combine the Complementary Function (CF) from step 1 with the found particular solution $y_{p(x)}$ to obtain the general solution to the non-homogeneous ODE: $y_g = CF + y_{p(x)}$

The *Complementary Function* (*CF*) or the homogeneous solution of a linear ordinary differential equation (ODE) depends on the form of the characteristic equation associated with the ODE. The characteristic equation is obtained by setting the right-hand side of the ODE to zero and solving for the roots of the characteristic equation. The nature of these roots determines the form of the CF.

There are three main cases based on the roots of the characteristic equation:

Distinct Real Roots: If the characteristic equation has two distinct real roots, say *r1* and *r2*, then the *CF* takes the form:

$$y_{cf(x)} = C_1 e^{r^1 x} + C_2 e^{r^2 x}$$

where C_1 and C_2 are arbitrary constants determined by the initial or boundary conditions.

Example 5.2.a: Consider the second-order homogeneous linear ODE:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

The characteristic equation is:

$$r^2 - 3r + 2 = 0$$

Solving for the roots, we get $r_1 = 2$ and $r_2 = 1$. Therefore, the CF is:

$$y_{cf(x)} = C_1 e^{2x} + C_2 e^{x}$$

Repeated Real Roots: If the characteristic equation has a repeated real root, say r, then the *CF* takes the form:

$$y_{cf(x)} = (C_1 + C_2 x) e^{rx}$$

where C_1 and C_2 are arbitrary constants determined by the initial or boundary conditions.

Example 5.2.b: Consider the second-order homogeneous linear ODE:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

The characteristic equation is:

$$r^2 - 4r + 4 = 0$$

The root r = 2 is repeated. Therefore, the CF is:

$$y_{cf(x)} = (C_1 + C_2 x) e^{2x}$$

Complex Conjugate Roots: If the characteristic equation has complex conjugate roots, say $r = \alpha \pm \beta i$, then the CF takes the form:

$$y_{cf(x)} = e^{\alpha x}(C_1 cos(\beta x) + C_2 sin(\beta x))$$

where C_1 and C_2 are arbitrary constants determined by the initial or boundary conditions.

Example 5.2.c: Consider the second-order homogeneous linear ODE:

$$\frac{d^2y}{dx^2} + 2y = 0$$

The characteristic equation is:

$$r^2 + 2 = 0$$

The roots are $r = \pm\sqrt{2}i$, which are complex conjugates. Therefore, the CF is:

$$y_{cf(x)} = C_1 cos(\sqrt{2}x) + C_2 sin(\sqrt{2}x)$$

✍ *In each case, the general form of the CF consists of a linear combination of exponential functions and trigonometric functions, depending on the nature of the roots of the characteristic equation. The constants C_1 and C_2 are determined by the initial or boundary conditions, which allow us to find the particular solution that satisfies the given ODE with specific initial conditions*.

When using the Method of Undetermined Coefficients to solve a non-homogeneous linear ordinary differential equation (ODE), one of the crucial steps is to make an educated guess for the particular solution (PS). The form of the guess depends on the type of the non-homogeneous term (right-hand side of the ODE) and the order of the ODE.

Here are the common forms for *making guesses for the particular solution* (y_p):

Constant (for constant terms in the right-hand side): For non-homogeneous terms in the form of a constant, the particular solution guess will be a constant value. The constant can be represented as A, B, C, or any other letter.

Example 5.2.d: If the right-hand side is f(x) = 5, then the guess for the particular solution is $y_{p(x)} = A$.

Polynomial (for polynomial terms in the right-hand side): For non-homogeneous terms in the form of a polynomial of degree 'n', the particular solution guess will be a polynomial of the same degree 'n' with undetermined coefficients.

Example 5.2.e: If the right-hand side is $f(x) = 3x^2 + 2x + 1$, then the guess for the particular solution is $y_{p(x)} = Ax^2 + Bx + C$.

Exponential (for terms in the right-hand side in the form of exponential functions): For non-homogeneous terms involving exponential functions (e.g., $P(x)e^{ax}$ with $P(x)$ a polynom of degree *k*), the particular solution guess will be a similar exponential function with an undetermined coefficient:

- If a is not a root of the characteristic equation, the particular solution is sought in the form $e^{ax}Q(x)$, where Q is a polynomial of degree *k* with unknown coefficients.
- If a is an $r - order$ root of the characteristic equation, then the particular solution is sought in the form $x^r e^{ax}Q(x)$, where Q is a polynomial of degree k with unknown coefficients.

Example 5.2.f: If the right-hand side is $f(x) = 4e^{2x}$, then the guess for the particular solution is $y_{p(x)} = Ae^{2x}$.

Trigonometric (for terms in the right-hand side in the form of trigonometric functions): For non-homogeneous terms involving trigonometric functions (e.g., $sin(kx)$, $cos(kx)$), the particular solution guess will be a similar trigonometric function with undetermined coefficients.

Example 5.2.g: If the right-hand side is $f(x) = 2sin(3x)$, then the guess for the particular solution is $y_{p(x)} = Asin(3x) + Bcos(3x)$.

Exponential Polynomial (for terms in the right-hand side involving combinations of exponential and polynomial terms): For non-homogeneous terms that are a combination of exponential and polynomial functions, the guess for the particular solution will involve both exponential and polynomial terms with undetermined coefficients.

Example 5.2.h: If the right-hand side is $f(x) = 2e^x + 3x^2$, then the guess for the particular solution is $y_{p(x)} = Ae^x + Bx^2 + Cx + D$.

✍ *It is important to note that the guess for the particular solution should include all terms that are present in the non-homogeneous term on the right-hand side of the ODE. After guessing the particular solution, it is then substituted back into the original ODE to determine the undetermined coefficients and obtain the complete solution to the non-homogeneous ODE.*

Example 5.2.i: Consider the following linear second-order ODE:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^x$$

The homogeneous solution can be found by solving the characteristic equation:

$$r^2 - 4r + 3 = 0$$

which gives $r = 1$ and $r = 3$. Thus, the homogeneous solution is $(y_h = CF)$

$$y_h = c_1e^x + c_2e^{3x}$$

For the particular solution, we assume $y_p = Axe^x$ and solve for A:

$$\left(\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y\right)e^x = 2e^x \Rightarrow -2A * e^x = 2e^x \Rightarrow -2A = 2 \Rightarrow A = -1$$

The particular solution is $y_p = -xe^x$, and the general solution is $y(x) = C_1e^x + C_2e^{3x} - xe^x$.

Method of Variation of Parameters: This method is used to solve non-homogeneous linear second-order ODEs when the function $f(x)$ is not of the form covered by the method of undetermined coefficients. In this method, we assume a particular form for the solution involving two arbitrary functions of x and determine these functions using the variation of parameters technique.

The Method of Variation of Parameters is a technique used to find a particular solution to a non-homogeneous linear ordinary differential equation (ODE) when the general solution to the corresponding homogeneous ODE is known. This method is applicable to higher-order ODEs with constant coefficients.

Consider a non-homogeneous linear ODE of the form:

$$L[y] = f(x)$$

where $L[y]$ is a linear differential operator involving the dependent variable y and its derivatives, and $f(x)$ is the non-homogeneous term on the right-hand side.

The *steps* to apply the *Method of Variation of Parameters* are as follows:

Step 1: Find the Complementary Function (CF): First, find the general solution to the corresponding homogeneous ODE (i.e., when $f(x) = 0$). This is the Complementary Function (CF) and represents the solution to the homogeneous part of the equation.

Step 2: Assume Particular Functions: Next, assume two new functions $u(x)$ and $v(x)$, which are unknown and not multiples of each other, to represent the variation of parameters. These functions will be used to find the particular solution.

Step 3: Express the Particular Solution: The particular solution (PS) is then expressed as:

$$y_{p(x)} = u(x) * y_1(x) + v(x) * y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the corresponding homogeneous ODE (obtained from the CF).

Step 4: Determine $u(x)$ and $v(x)$: Substitute the particular solution $y_{p(x)}$ into the original non-homogeneous ODE and solve for $u(x)$ and $v(x)$ by equating the coefficients of $y_1(x)$ and $y_2(x)$ terms to the terms in $f(x)$.

Step 5: Form the General Solution: The general solution to the non-homogeneous ODE is the sum of the Complementary Function (CF) and the particular solution (PS):

$$y(x) = y_{cf(x)} + y_{p(x)} = CF + u(x) * y_1(x) + v(x) * y_2(x)$$

where CF is the general solution obtained from the homogeneous ODE and $u(x)$ and $v(x)$ are the functions found in step 4.

✍ *The Method of Variation of Parameters is particularly useful when the non-homogeneous term is expressed as a sum of known functions, allowing us to solve for the unknown functions u(x) and v(x) to obtain the particular solution.*

Example 5.2.j: Consider the following linear second-order ODE:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$$

The homogeneous solution is $y_h = C_1e^{2x} + C_2xe^{2x}$.

For the particular solution, we assume $y_p = u(x) * e^{2x} + v(x) * xe^{2x}$, where $u(x)$ and $v(x)$ are two unknown function of *x*.

Now, we can determine $u_1(x)$ and $u_2(x)$ using the variation of parameters technique.

Method of Power Series: This method is used to solve linear second-order ODEs when the coefficients $a(x), b(x)$, and $c(x)$ are not explicitly known or are too complex. In this method, we assume that $y(x)$ can be represented as a power series and find the coefficients of the series by substituting it into the ODE.

Example 5.2.h: Consider the following linear second-order ODE:

$$\frac{d^2y}{dx^2} + x^2y = 0$$

We assume $y(x)$ can be expressed as a power series: $y(x) = \Sigma(a_n x^n)$, where a_n are coefficients to be determined. By substituting y(x) into the ODE and equating coefficients of like powers of x to zero, we can find the values of the coefficients a_n.

✍ *These are just some of the methods used to solve linear second-order ODEs. The choice of method depends on the specific characteristics of the equation and the functions involved*.

Solving nonlinear second-order ODEs: Solving nonlinear second-order ordinary differential equations (ODEs) is generally more challenging than solving linear ODEs due to their nonlinear nature. Nonlinear second-order ordinary differential equations (ODEs) are differential equations of the form:

$$f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

where *y* is the dependent variable, *x* is the independent variable, and f is a nonlinear function involving $y, \frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$.

Unlike linear second-order ODEs, which can be solved using various analytical methods, nonlinear second-order ODEs are generally more challenging to solve. In most cases, closed-form solutions for nonlinear ODEs are not available, and numerical methods are used to approximate the solutions. Numerical methods, such as the Euler method, the Runge-Kutta method, and shooting methods, are often employed to solve nonlinear ODEs. These methods involve discretizing the differential equation and approximating the solutions at discrete points.

Examples of nonlinear second-order ODEs in economic and scientific applications include equations that model population growth, predator-prey dynamics, mechanical systems with nonlinear restoring forces, and many other complex phenomena. Solving and analyzing these equations play a crucial role in understanding various natural and engineered systems.

Nonlinear ODEs may not have simple closed-form solutions, and finding exact solutions can be difficult or impossible in many cases. However, several techniques and methods can be applied to analyze and approximate solutions to nonlinear second-order ODEs.

Some of the commonly used methods are:

Equilibrium Analysis: For economic models with nonlinear second-order ODEs, economists often study the equilibrium points where the derivative of the function is zero. Analyzing the stability of these equilibrium points provides valuable insights into the behavior of the system.

Numerical Methods: Since analytical solutions may not be readily available for many nonlinear second-order ODEs, numerical methods are commonly used to approximate solutions. Numerical techniques like Euler's method, Runge-Kutta methods, and finite difference methods can provide numerical solutions to nonlinear ODEs.

Perturbation Methods: Perturbation methods are used to obtain approximate solutions to nonlinear ODEs by introducing a small parameter in the equation. The solutions are then expressed as power series in the small parameter and truncated to obtain approximate solutions.

Phase Plane Analysis: For systems of coupled second-order ODEs, phase plane analysis is a graphical method used to study the qualitative behavior of solutions. It helps to identify limit cycles, stable and unstable equilibrium points, and trajectories of the system.

Example 5.2.k: Nonlinear Second-Order ODE in Economic Economics: Consider the following nonlinear second-order ODE that represents a simple economic growth model:

$$\frac{d^2y}{dt^2} = 0.1y - 0.01y^2$$

This equation represents a population growth model with a carrying capacity, where $y(t)$ is the population at time t.

Analytical solutions to this nonlinear ODE may not be straightforward. However, we can use numerical methods to approximate the solutions and study the behavior of the population over time. Applying numerical techniques like Euler's method or Runge-Kutta methods allows us to compute the population at different time points and observe how it evolves over time.

Alternatively, we can perform a phase plane analysis by converting the second-order ODE into a system of coupled first-order ODEs and then studying the behavior of the solutions in the phase plane. The phase plane analysis can provide valuable insights into the long-term behavior of the population, such as whether it reaches a stable equilibrium or exhibits periodic cycles.

✍ *In summary, solving nonlinear second-order ODEs in economic economics often involves using numerical methods or graphical techniques like phase plane analysis to gain insights into the behavior of the system. Analytical solutions may not always be available, but approximations and qualitative analysis can provide valuable information for economic modeling and decision-making.*

△ *Second-order ODEs are a fundamental tool in economic modeling and provide insights into the behavior of economic systems over time. The ability to analyze dynamic systems using differential equations is essential in understanding economic growth, stability, and equilibrium in various economic scenarios*.

Questions you should be able to answer:

What is a second-order ordinary differential equation?

How do you classify second-order ODEs based on their form and properties?

What are the methods used to solve homogeneous second-order ODEs?

How do you solve non-homogeneous second-order ODEs using the Method of Undetermined Coefficients?

What is the Complementary Function (CF) and how is it related to the homogeneous solution of a second-order ODE?

What is the Particular Solution (PS) and how is it related to the non-homogeneous term of a second-order ODE?

How do you find the general solution to a non-homogeneous second-order ODE?

What are initial value problems (IVPs) for second-order ODEs, and how are they solved?

How can second-order ODEs be used to model and analyze economic scenarios, such as economic growth, investment dynamics, or population growth?

How can numerical methods be used to approximate solutions to second-order ODEs in cases where analytical solutions are not available?

How are second-order ODEs utilized in econometrics and economic modeling?

What are the applications of second-order ODEs in financial economics and option pricing models?

EXERCISES

5.2.1 Fill in the missing words in the context of second-order ordinary differential equations (ODEs) applied to economic mathematics:

a) The second-order ordinary differential equation, also known as the _____, is a fundamental equation used in economic growth models. It describes how a country's GDP changes over time as a function of its current GDP and the rate of GDP growth.

b) In economic dynamics, the _____ theorem states that for any second-order linear homogeneous ODE, the general solution can be represented as a linear combination of two linearly independent solutions.

c) The solution to a second-order ODE in economics often depends on the _____ of the characteristic equation associated with the ODE. It determines whether the solutions are real or complex.

d) Second-order ODEs are prevalent in economic models that involve _____ effects. For example, in the study of inflation, the rate of inflation may depend not only on the current inflation rate but also on the rate of change of inflation over time.

e) The second-order ODE used in finance to model stock option prices is known as the Black-Scholes equation, which is a partial differential equation that involves a second-order _____ derivative with respect to the stock price.

f) The method of _____ is a powerful technique used to solve second-order linear homogeneous ODEs in economics, especially when the characteristic equation has real roots.

Answer: undetermined coefficients

g) In the study of economic growth, the second-order ODE known as the _____ equation helps to model how a country's population growth affects its economic development.

f) In economic mathematics, the concept of _____ is often introduced to understand how changes in one variable can affect the rate of change of another variable in second-order ODEs.

5.2.2 Solve the follows second-order ordinary differential equations (ODEs) using the Method of Undetermined Coefficients:

a) $y'' - 4y' + 4y = 2e^x$

b) $y'' + 2y' + y = 3sin(2x)$

c) $y'' - 6y' + 9y = 5x^2 + 3x$

d) $y'' + y' = 2cos(x) - 3sin(x)$

e) $y'' - 3y' + 2y = 5x^2 + 3e^x$

f) $y'' + 4y' + 4y = 2x^2 + 6x$

g) $y'' - 2y' + y = 4sin(x) - 2cos(x)$

h) $y'' + 3y' + 2y = 8e^x + 3sin(2x)$

5.2.3 Solve the follows second-order ordinary differential equations (ODEs) using the Method of Undetermined Coefficients

a) $y'' + 4y = 2cos(x)$

b) $y'' - 3y' + 2y = x$

c) $y'' - 5y' + 6y = e^x$

d) $y'' - 9y = sin(2x)$

e) $y'' + 2y' + y = cos(3x)$

f) $y'' - 4y' + 4y = e^{2x}$

g) $2y'' - y = x^2$

h) $y'' + 9y = 5e^{3x}$

5.3 ECONOMIC APPLICATIONS OF DIFFERENTIAL EQUATIONS

Keywords*: Economic Growth Models, Business and Investment Analysis, Consumer Behavior, Production and Supply Chain Models, Monetary and Fiscal Policy Analysis, Environmental and Resource Economics, Market Equilibrium and Pricing, Financial Economics, Game Theory, Economic Forecasting.*

Differential equations have numerous economic applications across various fields of economics.

Some of the key economic applications of differential equations include:

Economic Growth Models: Differential equations are used to model economic growth theories such as the Solow-Swan growth model and the endogenous growth model. These models describe the evolution of capital, labor, and technology over time, and how they influence long-term economic growth rates.

Business and Investment Analysis: Differential equations are applied in business and investment analysis to study dynamic models of investments, cash flows, and profitability. They help in determining optimal investment strategies and evaluating the financial performance of businesses.

Consumer Behavior: Differential equations are used to model consumer behavior, particularly in decision-making processes such as consumption and saving behavior. These models help economists understand how changes in income and prices influence consumer choices.

Production and Supply Chain Models: Differential equations are utilized in production and supply chain models to analyze the dynamics of production processes, inventory management, and supply chain logistics. These models aid in optimizing production schedules and minimizing costs.

Monetary and Fiscal Policy Analysis: Differential equations are employed in macroeconomic models to study the effects of monetary and fiscal policies on economic variables like inflation, employment, and output. They help policymakers understand the impact of policy interventions.

Environmental and Resource Economics: Differential equations are used in environmental and resource economics to model the dynamics of natural resources, pollution levels, and ecological systems. These models help in understanding sustainability and resource management.

Market Equilibrium and Pricing: Differential equations are applied in microeconomics to study market equilibrium and price dynamics. They help in analyzing supply and demand interactions and determining equilibrium prices and quantities.

Financial Economics: Differential equations are used in financial economics to model stock prices, interest rates, and other financial variables over time. These models are fundamental in option pricing, risk management, and portfolio optimization.

Game Theory: Differential equations are applied in game theory to study dynamic interactions among decision-makers. They help in analyzing strategic behavior and equilibrium outcomes in competitive situations.

Economic Forecasting: Differential equations are used in econometrics for time series forecasting and predicting future economic variables based on historical data. They aid in making informed decisions and policy planning.

Using differential equations in economic growth models: Differential equations are a fundamental tool in economic growth models. These models aim to describe and understand how economies grow over time, taking into account various factors that influence the process. Differential equations are used to capture the dynamic relationships between key variables in these models. Here's how they are employed in economic growth modeling:

Solow-Swan Growth Model: The Solow-Swan model is one of the earliest and most well-known economic growth models. It employs a simple differential equation to describe the evolution of capital accumulation and economic output over time. The model assumes a production function that relates capital, labor, and technological progress to output. By applying the concept of a capital accumulation equation, which is essentially a differential equation representing the growth of capital over time, economists can study the long-term behavior of an economy.

Example 5.3.a: The basic Solow-Swan model uses a simple differential equation to represent the evolution of capital accumulation. The Solow-Swan growth model assumes the following production function:

$$Y = F(K, A * L)$$

where: Y are output (GDP), K physical capital stock, A total factor productivity (representing technology and efficiency), L labor input.

The differential equation for the change in capital stock *(K)* over time is given by:

$$\frac{dK}{dt} = s * Y - \delta * K$$

where: $\frac{dK}{dt}$ are rate of change of capital stock over time (capital accumulation), s are savings rate (proportion of output saved and invested), Y output (GDP) as given by the production function, δ depreciation rate (the proportion of capital that becomes unusable or depreciates over time).

This differential equation describes how the capital stock changes over time. The left-hand side $\left(\frac{dK}{dt}\right)$ represents the rate of change of capital stock, which is equal to the difference between the investment rate $(s * Y)$ and the depreciation rate $(\delta * K)$. If the investment rate exceeds the depreciation rate, the capital stock will increase, leading to economic growth. On the other hand, if the depreciation rate exceeds the investment rate, the capital stock will decrease, leading to negative economic growth.

✪ *The Solow-Swan growth model is useful for understanding long-run economic growth and the role of savings, investment, and technological progress in shaping an economy's development over time. Economists and policymakers can use this model to analyze the effects of changes in saving rates, investment policies, or technological advancements on the steady-state level of capital and economic growth in a country.*

Ramsey-Cass-Koopmans Model: The Ramsey-Cass-Koopmans model is a dynamic model of optimal economic growth. It incorporates consumer preferences and intertemporal choices over consumption and savings. The model involves differential equations to represent the dynamics of consumption, savings, and capital accumulation.

Example 5.3.b: The Ramsey-Cass-Koopmans (RCK) model is an optimal growth model that considers intertemporal decision-making of a representative agent who seeks to maximize utility over an infinite time horizon. The model incorporates consumption and savings decisions over time, taking

into account the trade-off between present and future consumption. Here's a simplified version of the RCK model:

The representative agent's utility function is given by:

$$U(C) = \int [e^{-\rho * t} * U\big(C(t)\big)]\, dt$$

where: $U(C(t))$ is utility derived from consumption *C* at time *t*, ρ discount rate (representing the agent's time preference), ∫ integral from 0 to ∞, indicating the time horizon.

The agent seeks to maximize this utility function subject to an intertemporal budget constraint:

$$\int [e^{-\rho t} * C(t)]\, dt = \int [e^{-\rho t} * (Y(t) - S(t))]\, dt$$

where: C(t) consumption at time t, Y(t) output (GDP) at time t, S(t) = Savings at time t.

The agent's budget constraint states that the integral of discounted consumption $(e^{-\rho t} * C(t))$ *over time should be equal to the integral of discounted income* $\big(e^{-\rho t} * Y(t)\big)$ minus the integral of discounted savings $\big(e^{-\rho * t} * S(t)\big)$ over time.

The agent's saving decision is given by the savings function:

$$S(t) = s * Y(t)$$

where: s is saving rate (proportion of output saved and invested).

The production function that links output to capital and labor is typically represented by:

$$Y(t) = F(K(t), A(t) * L(t))$$

where: K(t) is capital stock at time t, A(t) total factor productivity (representing technology and efficiency) at time t, L(t) labor input at time t.

The agent's capital accumulation equation is:

$$dK/dt = S(t) - \delta * K(t)$$

where: $\frac{dK}{dt}$ is rate of change of capital stock over time (capital accumulation), δ depreciation rate (the proportion of capital that becomes unusable or depreciates over time).

✍ *The Ramsey-Cass-Koopmans model helps economists analyze the optimal consumption and savings behavior of the representative agent over time, providing insights into the effects of changes in the discount rate, saving rate, and technological progress on long-run economic growth and welfare. It is a cornerstone of studying intertemporal optimization in economic growth theory.*

Real Business Cycle (RBC) Models: RBC models are used to study fluctuations in economic activity and business cycles. These models are based on the idea that economic fluctuations are primarily caused by real shocks to productivity or technology. RBC models use a system of differential equations to represent the dynamics of key variables such as consumption, investment, and labor supply, which allows economists to analyze the effects of various shocks on the economy.

Example 5.3.c: In this example, we will use a basic RBC model that incorporates technology shocks as a driving force behind economic fluctuations. The simplified RBC model involves three main equations:

Production Function: $Y_t = A_t * K_t^{\alpha} * (L_t)^{1-\alpha}$

where: Y_t is output (GDP) at time t, A_t technology level (a productivity shock) at time t, K_t physical capital stock at time t, L_t labor input at time t, α capital share in output (a constant between 0 and 1)

Capital Accumulation Equation: $K_t + 1 = (1 - \delta) * K_t + I_t$

where: K_{t+1} capital stock in the next period (t+1), δ depreciation rate (the proportion of capital that becomes unusable or depreciates over time), I_t investment at time t.

Technology Shock Process: $A_{t+1} = \rho * A_t * \varepsilon_t$

where: A_{t+1} technology level in the next period (t+1), ρ persistence parameter (a constant between 0 and 1) representing how much the shock affects the future technology level, ε_t random shock to technology at time t (assumed to be white noise, with mean zero and constant variance).

Explanation:

- The production function (1) describes how output (Y_t) is produced using capital (K_t) and labor (L_t) with a technology level (A_t). The parameter α determines the share of output attributed to capital in the production process.
- The capital accumulation equation (2) represents how the capital stock evolves over time. It shows that the capital stock in the next period (K_{t+1}) is equal to the sum of the current capital stock (K_t) after accounting for depreciation ($\delta * K_t$) and the investment made in the current period (I_t).
- The technology shock process (3) captures the random fluctuations in technology over time. The technology level in the next period (A_{t+1}) depends on the current technology level (A_t), the persistence parameter (ρ), and the random shock to technology (ε_t).

✍ *RBC models help economists study the effects of different shocks and economic policies on business cycles, investment decisions, and other macroeconomic variables. They provide insights into how exogenous shocks, such as changes in technology or productivity, can lead to fluctuations in economic activity over time.*

New Keynesian Models: New Keynesian models blend elements of the neoclassical growth theory with Keynesian principles. These models incorporate nominal rigidities (e.g., sticky prices and wages) that affect economic dynamics and create short-run fluctuations. Differential equations are used to model the dynamics of inflation, output, and other key variables under the influence of both real and nominal factors.

Example 5.3.d: One well-known example of a Keynesian model that involves differential equations is the "IS-LM model." This model analyzes the short-run equilibrium in an economy and shows the interaction between the goods market (IS curve) and the money market (LM curve). In this model, the equilibrium condition is expressed by two equations:

Investment-Saving (IS) curve: $Y = C(Y - T) + I(r) + G$

where: Y national income or output, C consumption function (a function of disposable income, $Y - T$), T taxes, I(r) investment function (a function of the real interest rate, r), G government spending. The IS curve represents the combinations of income and interest rates that bring the goods market into equilibrium, where total output (Y) equals total spending (C + I + G).

Liquidity Preference-Money Supply (LM) curve: $\frac{M}{P} = L(r, Y)$

where: M money supply, P price level, L(r, Y) liquidity preference function (a function of the real interest rate, r, and income, Y). The LM curve shows the combinations of income and interest rates that bring the money market into equilibrium, where the demand for money (M/P) equals the money supply (M/P).

To complete the model, the equation for the equilibrium condition is:

IS curve: $Y = C(Y - T) + I(r) + G$

LM curve: $M/P = L(r, Y)$

The equilibrium is reached when the goods market and money market are simultaneously in equilibrium, represented by the point where the IS and LM curves intersect.

✍ *The IS-LM model is widely used to analyze the effects of fiscal and monetary policy on short-term economic equilibrium, as well as to understand how changes in the money supply and interest rates can impact output and interest rates in the short run. Differential equations are not explicitly used in this model, but it is based on a system of equations that describe the relationship between income, consumption, investment, and the demand for money, allowing economists to analyze the short-run dynamics of an economy under different policy scenarios.*

Endogenous Growth Models: Endogenous growth models focus on the role of technological progress in driving economic growth. These models often involve differential equations to describe the growth rate of technology or knowledge over time, which, in turn, influences the overall growth of the economy.

Example 5.3.e: Let's consider an example of an endogenous growth model known as the AK model, which incorporates endogenous technological progress into the analysis of economic growth. In this model, the rate of technological progress is determined within the model and is not assumed to be exogenous. The AK model is represented by the following differential equation for capital accumulation:

$$\frac{dK}{dt} = s * Y - \delta * K$$

where: dK/dt is rate of change of capital stock over time (capital accumulation), K physical capital stock, s savings rate (proportion of output saved and invested), Y output (GDP) as given by the production function, δ depreciation rate (the proportion of capital that becomes unusable or depreciates over time).

In the AK model, technological progress (A) is assumed to be proportional to the current level of capital (K):

$$\frac{dA}{dt} = \alpha * K$$

where: dA/dt is rate of change of technological progress over time, α proportionality constant representing the technological progress rate.

Explanation:

- The equation $dK/dt = s * Y - \delta * K$ is the same as the one used in the Solow-Swan growth model, representing the accumulation of physical capital over time.
- The equation $dA/dt = \alpha * K$ shows that technological progress (A) grows over time and is directly related to the level of physical capital (K) in the economy. This means that as the economy accumulates more capital, it generates more technological progress, leading to even faster economic growth.

⭘ *The AK model suggests that economic growth is endogenously driven by the interaction between capital accumulation and technological progress. As the economy invests in capital (savings and investment), it leads to more production and income, which, in turn, accelerates the pace of technological progress. The technological progress feeds back into the economy's productivity, leading to sustained economic growth over time.*

✍ *The AK model and other endogenous growth models are essential for understanding the long-run dynamics of economies and the factors that contribute to self-sustaining growth paths. These models have implications for policies that can promote innovation, research and development, and investment in human capital to foster long-term economic growth*.

Population dynamics and diffusion processes: A well-known logistic growth model, which is often employed to describe population growth or the diffusion of innovations in economics.

Example 5.3.f: The logistic growth model is represented by the following differential equation:

$$\frac{dP}{dt} = r * P * (1 - \frac{P}{K})$$

where: $\frac{dP}{dt}$ is Rate of change of the variable P (e.g., population, number of adopters of a technology) over time, P the current value of the variable (e.g., population size, number of adopters) at time t, r growth rate parameter (positive constant) representing the maximum growth rate of the variable, K carrying capacity of the system, representing the maximum value that the variable can reach over time.

- The term $r * P$ represents the exponential growth component, where the rate of change is proportional to the current value of the variable *P*.
- The term $(1 - \frac{P}{K})$ represents the limiting factor, which slows down the growth as the variable approaches the carrying capacity *K*. When *P* is small compared to *K*, the term is close to 1, allowing for rapid growth. As *P* approaches *K*, the term approaches 0, causing the growth rate to decrease.

⭘ *This equation helps model scenarios where the growth of a population or the adoption of a technology starts rapidly, but as the system approaches its maximum capacity (K), the growth rate slows down.*

✍ *The logistic growth model has various applications in economics, such as studying the diffusion of new products, technologies, or ideas in a market, analyzing the adoption of renewable energy sources, or projecting the growth of a population in an economy over time.*

Researchers and policymakers use such differential equations to understand the dynamics of economic processes, predict future trends, and design effective policies to manage population growth or promote the adoption of new innovations in a sustainable manner.

◘ *In summary, differential equations are widely used in economic growth models to represent the intertemporal dynamics of key variables, allowing economists to study how various factors influence the long-term development of economies and their short-term fluctuations. These models are essential for policy analysis, as they provide insights into the effects of different economic policies on growth, inflation, unemployment, and other important macroeconomic variables*.

Contribution of differential equations to the understanding and analysis of economic systems at both micro and macro levels*:* Differential equations play a crucial role in understanding

and analyzing economic systems at both micro and macro levels. They provide a powerful mathematical framework to describe the dynamic behavior of economic variables over time.

Here's how differential equations contribute to the understanding of economic systems at both levels:

Microeconomic Level: At the microeconomic level, differential equations are used to model individual economic agents' behavior and interactions. Some key applications include:

Consumer Choice: Differential equations are used to model consumers' intertemporal decision-making on consumption and saving, considering factors like income, interest rates, and preferences over time.

Investment Decisions: Firms' investment choices are often modeled using differential equations to analyze how capital accumulation and investment affect their profits and growth.

Market Dynamics: Differential equations are employed to model the supply and demand dynamics in markets, helping to analyze price movements and market equilibrium.

Technology Adoption: Models of technology adoption and diffusion often use differential equations to study how innovations spread through a population or market.

Macroeconomic Level: At the macroeconomic level, differential equations are used to describe and analyze aggregate economic phenomena that influence an entire economy. Some key applications include:

Economic Growth: Models like the Solow-Swan model and endogenous growth models incorporate differential equations to study long-run economic growth, capital accumulation, and technological progress.

Business Cycles: Real Business Cycle (RBC) models and New Keynesian models use differential equations to study fluctuations in economic activity and business cycles.

Monetary and Fiscal Policy: Differential equations are used to model the impact of monetary and fiscal policies on macroeconomic variables like inflation, unemployment, and output.

Dynamic General Equilibrium: Differential equations are used in dynamic general equilibrium models to analyze the interactions between different sectors of the economy and their equilibrium outcomes.

Environmental Economics: Differential equations are employed in modeling the dynamics of environmental systems, such as the depletion of natural resources and the impact of pollution on the economy.

In both micro and macroeconomics, differential equations allow economists to study complex interactions, feedback loops, and time-dependent changes in economic systems. These equations provide a quantitative and rigorous framework to analyze the impact of various economic policies, shocks, and external factors on the behavior of economic variables, helping policymakers make informed decisions.

Overall, the use of differential equations in economics enhances our understanding of economic systems, allows for the development of more accurate and predictive models, and helps economists analyze and address real-world economic challenges.

Let's consider an example of how differential equations are used to model the dynamics of an economy's inflation rate at the macroeconomic level.

Example 5.3.g: The Phillips Curve is a classic macroeconomic model that represents the relationship between inflation and unemployment. It shows how changes in the unemployment rate affect the rate of inflation in an economy over time. The Phillips Curve is often modeled using a differential equation.

The Phillips Curve can be represented by the following differential equation:

$$\frac{d\pi}{dt} = \beta * (u - u_n)$$

where: $d\pi/dt$ rate of change of inflation (π) over time, π inflation rate, β coefficient representing the sensitivity of inflation to changes in the unemployment rate, u actual unemployment rate, u_n natural rate of unemployment (the rate at which inflation is stable, with no output gap).

Explanation:

- The term $\frac{d\pi}{dt}$ represents the rate of change of inflation over time. If it is positive, it indicates that inflation is increasing, and if it is negative, inflation is decreasing.
- The term $(u - u_n)$ represents the output gap, which is the difference between the actual unemployment rate and the natural rate of unemployment. When the actual unemployment rate is lower than the natural rate ($u < u_n$), it indicates a positive output gap, which tends to push inflation upward. Conversely, when the actual unemployment rate is higher than the natural rate ($u > u_n$), it indicates a negative output gap, which tends to push inflation downward.
- The coefficient β represents the responsiveness of inflation to changes in the output gap. A higher value of β indicates a stronger relationship between inflation and the output gap, meaning that changes in unemployment have a more significant impact on inflation.

⭘ *The Phillips Curve helps economists and policymakers understand the trade-off between inflation and unemployment. When unemployment is low (positive output gap), inflation tends to rise, and when unemployment is high (negative output gap), inflation tends to fall. This relationship has important implications for monetary policy, as policymakers need to strike a balance between controlling inflation and maintaining a stable level of unemployment*.

✍ *By using differential equations to model the Phillips Curve, economists can analyze the dynamics of inflation and unemployment over time and explore the effects of various economic policies and shocks on the economy's inflation rate. This provides valuable insights for central banks and policymakers in making decisions related to interest rates, money supply, and other policy tools to achieve their economic objectives.*

The benefits and limitations of using differential equations in economic modeling and analysis: Using differential equations in economic modeling and analysis offers several benefits and advantages, but it also comes with certain limitations. Let's explore both aspects:

Benefits of using differential equations in economic modeling and analysis:

- Rigorous and Mathematical Representation: Differential equations provide a rigorous mathematical framework to model complex economic phenomena and interactions. They offer a systematic way to represent the dynamic relationships between economic variables, helping economists derive clear and precise conclusions.

- Dynamic Analysis: Differential equations allow economists to study how economic variables change over time, capturing the dynamic nature of economic systems. This enables the analysis of how shocks, policies, or external factors impact economic outcomes over different time periods.
- Prediction and Policy Evaluation: By incorporating historical data and appropriate parameter values, economists can use differential equations to forecast future economic trends and evaluate the potential impact of different policy interventions on the economy.
- Endogeneity and Feedback Loops: Differential equations facilitate the incorporation of endogenous factors, such as technological progress, feedback loops, and interdependencies, into economic models. This makes the models more realistic and allows for a better understanding of complex economic dynamics.
- Flexibility and Adaptability: Differential equations can be used to model a wide range of economic phenomena at different scales, from microeconomic interactions to macroeconomic aggregates. They can also be adapted and extended to address specific research questions or economic contexts.

Limitations of using differential equations in economic modeling and analysis:

- Simplification and Assumptions: Economic models based on differential equations often involve simplifying assumptions to make the equations tractable. While these assumptions are necessary for analysis, they may not fully capture all real-world complexities and nuances, leading to potential inaccuracies.
- Data Requirements: Differential equation models often require substantial data to estimate parameters accurately. Obtaining high-quality data for economic variables can be challenging, especially for developing economies or for variables that are not readily measurable.
- Overfitting and Calibration Issues: Fitting real-world data to differential equations may lead to overfitting or difficulties in accurately calibrating the model. The reliability of the model's predictions depends heavily on the quality and availability of data.
- Lack of Behavioral Insights: While differential equations provide a mathematical representation of economic relationships, they may not offer insights into the underlying behavioral mechanisms that drive economic decisions and actions.
- Intractability of Some Models: Some economic models involving complex differential equations may be analytically intractable, requiring numerical methods for solution. This can lead to computational challenges and the need for advanced numerical techniques.
- Uncertainty and Error Propagation: Small errors or uncertainties in parameter estimates or initial conditions can lead to significant differences in model outcomes over time. This sensitivity to initial conditions is known as the butterfly effect, which may limit the accuracy of long-term predictions.

✍*In conclusion, differential equations are a valuable tool in economic modeling and analysis, allowing economists to study dynamic economic systems and make predictions about future trends and policy impacts. However, researchers must be mindful of the simplifications and assumptions involved and ensure that the models are well-calibrated and validated against real-world data to enhance their reliability and usefulness for policymaking. Integrating differential equations with behavioral and institutional insights can lead to more comprehensive and realistic economic models.*

△ *These applications demonstrate the versatility of differential equations in addressing various economic phenomena, from individual decision-making to macroeconomic dynamics. Differential equations serve as a powerful tool for economic analysis and contribute significantly to the understanding and modeling of economic systems.*

Questions you should be able to answer:

What are the main economic applications of differential equations?

How are differential equations used in economic growth models?

How are differential equations applied in business and investment analysis?

What role do differential equations play in modeling consumer behavior?

How are differential equations used in production and supply chain models?

What are some examples of differential equations used in monetary and fiscal policy analysis?

How do differential equations contribute to environmental and resource economics?

In what ways are differential equations employed to study market equilibrium and pricing in microeconomics?

How are differential equations used in financial economics, particularly in modeling stock prices and interest rates?

How can differential equations be utilized for economic forecasting and predicting future economic variables?

How do differential equations contribute to the understanding and analysis of economic systems at both micro and macro levels?

What are the benefits and limitations of using differential equations in economic modeling and analysis?

Can you provide specific examples of economic scenarios where differential equations are instrumental in gaining insights and making predictions?

EXERCISES

5.3.1 Fill in the missing words in the context of "Economic Applications of Differential Equations":

a) The _____ is a commonly used economic model that describes the relationship between a country's output and the accumulation of capital.

b) In the Solow-Swan growth model, the _____ differential equation represents the change in capital stock over time.

c) The _____ equation in the Solow-Swan model represents the change in output (GDP) over time.

d) The _____ is a type of differential equation frequently used in economics to model the rate of change of a variable with respect to time.

e) In economics, _____ equations are often used to study dynamic systems and how economic variables evolve over time.

f) The _____ of a differential equation in economics involves finding a function that satisfies the equation and represents the economic relationship of interest.

g) The _____ theorem in economics states that a competitive equilibrium can be achieved in a market when the quantity demanded equals the quantity supplied.

h) In macroeconomics, the _____ model is a common framework for analyzing the relationship between inflation and unemployment.

i) The _____ equation is used in finance to model the pricing of options and derivatives.

j) The _____ equation is widely used in economics to analyze the relationship between consumption, savings, and income.

5.3.2 Fill in the missing words in the context of "Economic Applications of Differential Equations":

a) The _____ equation is a key component of the Solow-Swan growth model, representing the relationship between capital stock, population growth, and technological progress.

b) The _____ differential equation is used to model the spread of infectious diseases in an economy.

c) In finance, the _____ equation is employed to model the movement of stock prices and asset values over time.

d) The _____ theorem in economic mathematics guarantees the existence and uniqueness of solutions to certain differential equations.

e) In the context of economic dynamics, the _____ equation describes how the rate of change of a variable is influenced by its own value.

f) _____ is a common method used to approximate solutions to complex economic differential equations, especially in the case of nonlinear models.

g) In economic forecasting, the _____ equation is used to predict future values of economic variables based on historical data.

h) The _____ equation is a fundamental concept in economics, as it relates changes in price to changes in supply and demand.

i) The _____ model is frequently used to analyze the relationship between inflation and interest rates in an economy.

j) The _____ equation is used to study population growth in economic demography and urban economics.

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course support

Good for printing. Format 60 x 84 1/16
Printing sheets 19,75. Editorial sheets 18,4
Command nr. 11.

Editorial-Polygraphic Center of SUM
60, Al. Mateevici str., Chisinau, MD 2009
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