Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

Polynomial Differential Cubic Systems with Invariant Straight Lines of Total Multiplicity Seven and Four Distinct Infinite Singularities

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Abstract

The family of cubic differential systems possessing invariant straight lines was considered by many authors. But a strong classification of this family according to the configurations of invariant lines began in 2006 [11], where the existence of the maximum number of invariant lines (nine) is required. Here we continue the investigation of the subfamily of cubic systems possessing invariant lines of total multiplicity seven and four distinct infinite singularities. We prove that this subfamily could have a total of 166 distinct configurations of invariant straight lines.

Keywords: cubic system, affine transformation, invariant straight line, infinite and finite singularities, multiplicity of an invariant line and singularity, configuration of invariant straight lines.

1 Introduction and preliminary results

Consider real polynomial differential cubic systems

$$\frac{dx}{\partial t} = P(x, y), \quad \frac{dy}{\partial t} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, i.e., are polynomials in x, y with real coefficients and $\max(\deg P, \deg Q) = 3$.

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Systems (1) depend on 20 parameters and their study is difficult in general, even impossible at this moment. For this reason, people began by studying particular subclasses of cubic systems, especially according to the geometrical proprieties of such systems and using the Qualitative Theory for their study.

Here we consider a particular case of cubic systems which possess invariant straight lines of total multiplicity seven. A line $f(x,y) = ux + vy + w = 0$ over C is an *invariant line* if and only if it there exists $K(x,y) \in \mathbb{C}[x,y]$, which satisfies the following identity in $\mathbb{C}[x,y]$: $uP(x,y) + vQ(x,y) = (ux + vy + w)K(x,y)$. We point out that if we have an invariant line $f(x;y) = 0$ over $\mathbb{C}[x,y]$, it could happen that multiplying the equation by a number $\lambda \in \mathbb{C}^*$, the coefficients of the new equation become real, i.e., $(u\lambda; v\lambda; w\lambda) \in \mathbb{R}^3$. In this case, along with the line $f(x,y) = 0$ sitting in $\mathbb{C}^2$, we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x,y) = 0$. Note that, since a system (1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux + vy + w = 0$, then its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$ is also invariant.

It is well known that the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [11], the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their configurations of invariant lines, where the authors detected 23 configurations. In [1], the author detected another class of cubic system whose configuration of invariant lines was not detected in [11]. Systems possessing 8 invariant straight lines, where the line at infinity is considered, have been made in [2], [3], [4], [5] and 51 distinct configurations have been detected. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [12].

Definition 1 [12]. *Consider a real planar polynomial differential system* (1)*. We call configuration of invariant straight lines of this system, the set of (complex) invariant straight lines (which may have real coefficients) including the line at infinity of the system, each endowed*

with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

We continue our investigation of the family of cubic systems with invariant lines of total multiplicity seven considering the line at infinity (CSL_7). Moreover, we focus on such systems which have at infinity four singularities, all distinct.

It is well known that the infinite singularities (real or complex) of a system (1) are determined by the linear factors of the factorization of the cubic polynomial $C_3(x,y) = yp_3(x,y) - xq_3(x,y)$, where p_3 and q_3 are cubic homogeneities of systems (1). Clearly, for ex., in the case of four distinct infinite singularities, the polynomial $C_3(x,y)$ has four distinct linear factors, which could be all real (denoted in this case as $4r\infty$), two real and two complex ($2r2c\infty$), or all complex ($4c\infty$).

As we have four infinite singularities and the total multiplicity of invariant lines (including the line at infinity) must be 7, the systems (1) could only have one of the following five types of configurations of invariant lines – τ: $(3,3); (3,2,1); (3,1,1,1); (2,2,2)$ and $(2,2,1,1)$. The notion of the type of configuration was introduced in [8].

In [7, 6, 9, 10], there are determined all possible configurations for the systems (1) belonging to the classes $CSL_\tau^{4r\infty}$ and $CSL_\tau^{2r2c\infty}$. Additionally, all the orbit representatives of the systems in $CSL_\tau^{4r\infty}$ and $CSL_\tau^{2r2c\infty}$ with respect to affine group of transformations and a time rescaling are obtained.

2 Main results

Notation 1 *We shall denote by* $\boldsymbol{CSL_7^{4s.p.\infty}}$ *the subfamily of cubic systems with invariant straight lines of total multiplicity 7 which have four distinct infinite singularities.*

We set the following notations related to configurations of invariant lines for a cubic system: $\mathcal{N}$ = the number of all distinct invariant straight lines (including the line at infinity); $\mathcal{M}_L$=the maximum mul-

tiplicity of an invariant line; $\mathcal{M}_s$=the maximum multiplicity of a real singularity located on the configuration. We apply the parameter $\mathcal{N}$ to split the set of all configurations according to the theorem given below.

According to [7, 6, 9, 10], the next result is obtained.

Theorem 1. *The subfamily* $\boldsymbol{CSL}_7^{4s.p.\infty}$ *has a total of 166 distinct configurations of invariant straight lines. More exactly:*

- *For* $\mathcal{N} = 7$*, we have* 109 *configurations among which there exist* 52 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_s = 2$*.*
- *For* $\mathcal{N} = 6$*, we have* 25 *configurations among which there exist* 11 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_L = 2$ *and* $\mathcal{M}_s = 6$*.*
- *For* $\mathcal{N} = 5$*, we have* 30 *configurations among which there exist* 13 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_L = 3$ *and* $\mathcal{M}_s = 9$*.*
- *For* $\mathcal{N} = 4$*, we have 2 configurations with only real invariant lines: one triple, one double, and two simple. In this class, we have* $\mathcal{M}_s = 6$*.*

Acknowledgments. The work of the first and the second authors was supported by the Program SATGED 011303, Moldova State University. Moreover the second author is partially supported by the grant number 21.70105.31 S.D.

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