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# The Family of Quadratic Systems with Two Invariant Conics: Parabola and Hyperbola

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**Abstract**

We consider the family of quadratic systems $\mathbf{QSPH}_{\eta>0}$ having three real distinct infinite singularities and possessing at least two invariant conics: a parabola and a hyperbola. All the possible configurations of these invariant conics including invariant lines (when they exist) are determined. We describe completely the set of such systems and we prove the existence of exactly 38 distinct configurations which could possess a system in this family.

**Keywords:** quadratic systems, invariant conics, invariant line, configurations of invariant conics, affine transformation.

## 1 Introduction

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where $P$, $Q \in \mathbb{R}[x, y]$, i.e., $P$, $Q$ are polynomials of degree 2 in $x$ and $y$, i.e., $\max(\deg P, \deg Q) = 2$.

An important ingredient in this work is the notion of *configuration of algebraic solutions* of a polynomial differential system. This notion appeared for the first time in [1].

**Definition 1.** Consider a planar polynomial system which has a finite number of algebraic solutions and a finite number of singularities, finite

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or infinite. By *configuration of algebraic solutions* of this system we mean the set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singularities of this system located on these curves, each one of these singularities endowed with its own multiplicity.

We are interested in the family of quadratic systems (1) possessing three real distinct infinite singularities and one invariant parabola. According to [2], any system in this class could be brought via an affine transformation and time rescaling to the form

$$\dot{x} = m+nx-\frac{1}{2}(1+g)y+gx^2+xy, \;\; \dot{y} = 2mx+2ny+(g-1)xy+2y^2 \quad (2)$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

Our goal is to determine when a system belonging to the above family could possess in addition to the invariant parabola an invariant hyperbola. Our main result is the following.

**Theorem 1. ($\mathcal{A}$)** *Assume that for a system* (2) *the condition* $(g-2)(g+3) \neq 0$ *holds. Then this system could possess at least one invariant hyperbola if and only if there exists an affine transformation which could bring this system to form* (2)*, the parameters* $(m,n,g)$ *of which satisfy one of the conditions* $(\mathcal{A}_1)$*–*$(\mathcal{A}_{17})$*. Moreover this system could possess one of the configurations Config.* $\mathcal{PH}$*.1– Config.* $\mathcal{PH}$*.38 (given in Figure 1) if and only if the corresponding conditions presented below are satisfied, respectively:*

$(\mathcal{A}_1)$ $(m,n,g) = \left(\frac{1}{32}(g-1),\ \frac{1}{16}(3-4g),\ g\right) \Rightarrow$

- *Config.* $\mathcal{PH}$*.1 if* $(2g-1)(3+2g) \neq 0$, $U_1 > 0$*;*
- *Config.* $\mathcal{PH}$*.2 if* $(2g-1)(3+2g) \neq 0$, $U_1 < 0$, $g(g+1) < 0$*;*
- *Config.* $\mathcal{PH}$*.3 if* $(2g-1)(3+2g) \neq 0$, $U_1 < 0$ $g(g+1) > 0$*;*
- *Config.* $\mathcal{PH}$*.4 if* $(2g-1)(3+2g) \neq 0$, $U_1 = 0$*;*
- *Config.* $\mathcal{PH}$*.5 if* $(2g-1)(3+2g) = 0$, where $U_1 = 4g(1+g) - 7$.

$(\mathcal{A}_2)$ $(m,n,g)=\Big(\frac{(g-1)(1+g)^2(2g-1)}{8(1+2g)^3},\,-\frac{(1+g)(1-3g+4g^2)}{4(1+2g)^2},\,g\Big)\Rightarrow$

- *Config.* $\mathcal{PH}.6$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.7$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)>0,\ \rho_1<0$;
- *Config.* $\mathcal{PH}.8$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)>0,\ \rho_1>0$;
- *Config.* $\mathcal{PH}.9$ *if* $\kappa_1=0$;
- *Config.* $\mathcal{PH}.10$ *if* $\rho_1=0$;
- *Config.* $\mathcal{PH}.11$ *if* $(2g-1)(3+2g)=0$,

*where* $\rho_1=g^2+g-1;\ \kappa_1=2g^2+2g-1$.

$(\mathcal{A}_3)$ $(m,n,g)=\big(\frac{1}{8}(g-1)(2g-1),\,\frac{1}{4}(2g-1-2g^2),\,g\big)\Rightarrow$

- *Config.* $\mathcal{PH}.12$ *if* $-1<g<0$;
- *Config.* $\mathcal{PH}.13$ *if* $g<-1$;
- *Config.* $\mathcal{PH}.14$ *if* $0<g<2/3$;
- *Config.* $\mathcal{PH}.15$ *if* $2/3<g<2$;
- *Config.* $\mathcal{PH}.16$ *if* $g>2$;
- *Config.* $\mathcal{PH}.17$ *if* $g=2/3$;
- *Config.* $\mathcal{PH}.18$ *if* $g=1/2$.

$(\mathcal{A}_4)$ $(m,n,g)=\big(\frac{1}{24}(g-1)(1+g)^2,\,\frac{1}{36}(1+g)(13g-11),\,g\big)\Rightarrow$

- *Config.* $\mathcal{PH}.19$ *if* $2g-1\neq 0,\ U_2>0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.20$ *if* $2g-1\neq 0,\ U_2>0,\ g<-1$;
- *Config.* $\mathcal{PH}.21$ *if* $2g-1\neq 0,\ U_2>0,\ 0<g<2$;
- *Config.* $\mathcal{PH}.20$ *if* $2g-1\neq 0,\ U_2>0,\ g>2$;
- *Config.* $\mathcal{PH}.22$ *if* $2g-1\neq 0,\ U_2<0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.23$ *if* $2g-1\neq 0,\ U_2<0,\ g(g+1)>0$;

- *Config.* $\mathcal{PH}$*.24 if* $2g-1\neq 0$, $U_2=0$, $g(g+1)<0$;
- *Config.* $\mathcal{PH}$*.25 if* $2g-1\neq 0$, $U_2=0$, $g(g+1)>0$;
- *Config.* $\mathcal{PH}$*.26 if* $2g-1=0$, where $U_2=13g^2-4g-8$.

$(\mathcal{A}_5)$ $(m,n,g)=\left(-\frac{1}{48},\,\frac{5}{48},\,\frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.3*
$(\mathcal{A}_6)$ $(m,n,g)=\left(-\frac{4}{81},\,\frac{20}{81},\,\frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.23*
$(\mathcal{A}_7)$ $(m,n,g)=\left(\frac{4}{375},\,-\frac{4}{75},\,\frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.7*
$(\mathcal{A}_8)$ $(m,n,g)=\left(0,\,-\frac{1}{16},\,1\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.27*
$(\mathcal{A}_9)$ $(m,n,g)=\left(0,\,-\frac{1}{4},\,1\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.28*
$(\mathcal{A}_{10})$ $(m,n,g)=\left(0,\,-\frac{1}{9},\,1\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.29*
$(\mathcal{A}_{11})$ $(m,n,g)=\left(0,\,-\frac{8}{9},\,-\frac{1}{3}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.30*
$(\mathcal{A}_{12})$ $(m,n,g)=\left(-\frac{21}{1000},\,\frac{27}{100},\,-\frac{2}{5}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.31*
$(\mathcal{A}_{13})$ $(m,n,g)=(0,\,-4,\,1)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.32*
$(\mathcal{A}_{14})$ $(m,n,g)=\left(-\frac{3}{64},\,\frac{5}{16},\,-\frac{1}{2}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.33*
$(\mathcal{A}_{15})$ $(m,n,g)=\left(-\frac{1}{32},\,\frac{3}{16},\,0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.34*
$(\mathcal{A}_{16})$ $(m,n,g)=\left(0,\,-\frac{3}{2},\,0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.35*
$(\mathcal{A}_{17})$ $(m,n,g)=\left(-\frac{1}{24},\,\frac{11}{36},\,0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.36*

**(**$\mathcal{B}$**)** *Assume that, for a system* (2)*, the condition* $(g-2)(g+3)=0$ *holds. Then, due to an affine transformation, we may assume* $g=2$ *and, therefore, this system could possess at least one invariant hyperbola if and only if the parameters* $(m,n)$ *satisfy one of the conditions* $(\mathcal{B}_1)$*–*$(\mathcal{B}_5)$*. Moreover, each one of these conditions leads to one of the configurations Config.* $\mathcal{PH}$*.1– Config.* $\mathcal{PH}$*.38 given in Figure 1 as it is presented below, respectively:*

$(\mathcal{B}_1)$ $(m,n)=\left(\frac{1}{32},\,-\frac{5}{16}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.1*
$(\mathcal{B}_2)$ $(m,n)=\left(0,\,-\frac{15}{2}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.13*
$(\mathcal{B}_3)$ $(m,n)=\left(\frac{3}{8},\,-\frac{5}{4}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.37*
$(\mathcal{B}_4)$ $(m,n)=\left(\frac{175}{72},\,-\frac{145}{36}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}$*.38*
$(\mathcal{B}_5)$ $(m,n)=\left(\frac{27}{1000},\,-\frac{33}{100}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}$*.38*

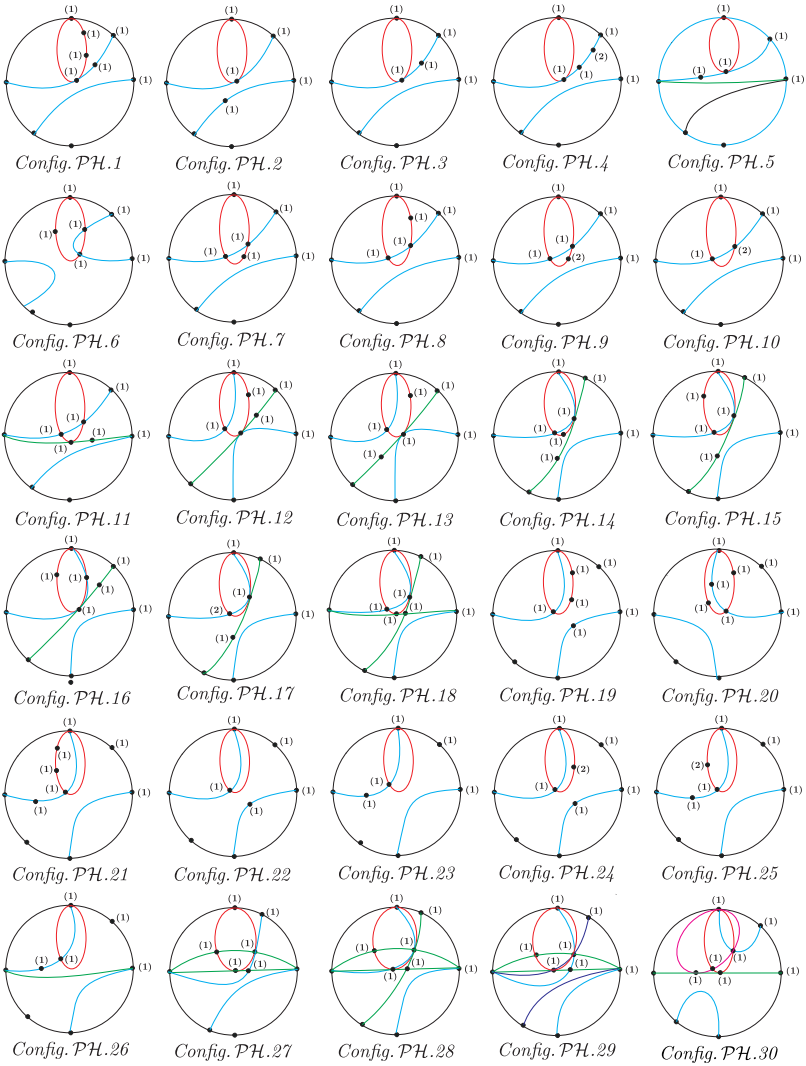



Figure 1. Configurations of systems in $\mathbf{QSPH}_{\eta>0}$

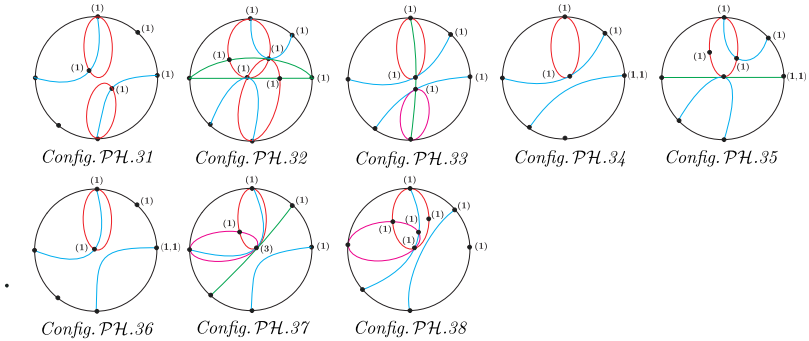



Figure 1 (*cont.*) Configurations of systems in $\mathbf{QSPH}_{\eta>0}$

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