Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

Strong Edge-Colorings of Some Lexicographic Products of Graphs

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Abstract

The edge-coloring of a graph G is strong if edges at distance 0 or 1 receive different colors. The minimum number of colors required for a strong edge-coloring of G is called strong chromatic index of G and denoted by $\chi'_s(G)$. In this paper, we determine the exact value of the strong chromatic index of lexicographic products of graphs G and H when $\Delta(G) \leq 2$ and H is an arbitrary graph, and provide an upper bound on $\chi'_s(G[H])$ when $\Delta(G) = 3$ and H is an arbitrary graph.

Keywords: edge-coloring, strong edge-coloring, strong chromatic index, lexicographic product of graphs.

1 Introduction

We consider only finite and simple graphs. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. The *distance between two edges* $e, e' \in E(G)$ in a graph G is the length of the shortest path between endpoints of e and e'. A strong *edge-coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$, where the edges that are at distance 0 or 1 from each other receive different colors. *The strong chromatic index* of a graph G is the minimum number of colors required for a strong edge-coloring of G, and is denoted with $\chi'_s(G)$. The problem of determining a strong chromatic index of a graph was suggested by Fouquet and Jolivet in 1983 [5].

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In 1985, Erdős and Nešetřil conjectured that for any graph G with maximum degree $\Delta(G)$, the strong chromatic index is at most $\frac{5}{4}\Delta(G)$ (see [1]). Later, Faudree et al. conjectured that for any bipartite graph G with maximum degree $\Delta(G)$, $\chi'_s(G) \leq \Delta(G)^2$ [1]. The conjecture proposed by Erdős and Nešetřil was proved for graphs G with $\Delta(G) = 3$ by Andersen and independently by Horák et al. in [2, 3]. For graphs G with $\Delta(G) = 4$, the currently best known upper bound on $\chi'_s(G)$ is 21 and it was proved in [6]. In [4], Togni investigated strong edge-colorings of Cartesian, Kronecker and strong products of graphs. In this paper, we bound or determine the strong chromatic index of lexicographic products of graphs G and H when $\Delta(G) \leq 3$ and H is an arbitrary graph.

2 Main Results

Let G and H be two graphs.

Definition 1 . *The lexicographic product of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which $x = (u_1, v_1)$ is adjacent to $y = (u_2, v_2)$ whenever u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent to v_2, and is denoted by $G[H]$.*

Definition 2 . *For any $v \in V(G)$ and H, we denote by $v\,H$, the subgraph of $G[H]$ induced by vertices $\{(v, u) \mid u \in V(H)\}$.*

Definition 3 . *For any $v_1, v_2 \in V(G)$ and H, we denote by $[v_1, v_2]H$, the subgraph of $G[H]$, where $V([v_1, v_2]H) = \{(v, u) \mid v \in \{v_1, v_2\}, u \in V(H)\}$ and $E([v_1, v_2]H) = \{(v_1, u_1)(v_2, u_2) \mid u_1, u_2 \in V(H)\}$. By the definition of the lexicographic product of graphs G and H, it follows that $[v_1, v_2]H$ is isomorphic to the complete bipartite graph.*

We begin our considerations with strong edge-colorings of the lexicographic products of graphs G and H, where $\Delta(G) \leq 2$. Since for a graph G, $\Delta(G) \leq 2$, each component of G is either a path or a cycle.

Theorem 1. *For any graph H and $n \geq 4$,*

$$\chi'_s(P_n[H]) = 2\chi'_s(H) + 3|V(H)|^2$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. First, let us consider the edges incident to vertices of subgraphs v_2H and v_3H. All the edges from $[v_1, v_2]H$, $[v_2, v_3]H$, and $[v_3, v_4]H$ subgraphs should receive different colors as they are at distances 0 or 1 from each other. At the same time, edges from v_2H and v_3H subgraphs, are also at distances 0 or 1 from edges in $[v_1, v_2]H$, $[v_2, v_3]H$, $[v_3, v_4]H$, and each other. Using these constraints, we get $\chi'_s(P_n[H]) \geq 2\chi'_s(H)+3|V(H)|^2$. To construct a strong edge-coloring of the graph $P_n[H]$, which uses exactly $2\chi'_s(H) + 3|V(H)|^2$ colors, we will start from the coloring of subgraphs vH. For each i $(1 \leq i \leq n)$, we assign v_iH subgraph, to group one if i is even and to group two if i is odd. Subgraphs in each group can be colored using the same set of colors as the edges between subgraphs that are at distances at least 2 from each other. Using the fact that each subgraph can be colored with $\chi'_s(H)$ colors, we can color all of them using $2\chi'_s(H)$ colors. Now we will color $[v_i, v_{i+1}]H$ subgraphs $(1 \leq i \leq n-1)$ using $3|V(H)|^2$ colors. Clearly, for the coloring of each $[v_i, v_{i+1}]H$ subgraph, we need $|E([v_i, v_{i+1}]H) = |V(H)|^2$ colors as it is isomorphic to the complete bipartite graph. Subgraphs $[v_i, v_{i+1}]H$ preserve the same distance properties as edges of P_n. In other words, subgraphs $[v_i, v_{i+1}]H$, that either share vertices or edges should be colored using different colors. From the fact that $\chi'_s(P_n) = 3$, we get that we need $3|V(H)|^2$ new colors to complete the strong edge-coloring of $P_n[H]$. Thus, our coloring uses $2\chi'_s(H) + 3|V(H)|^2$ colors, which matches with a lower bound. □

Using a more careful analysis, we prove that the following results hold.

Theorem 2. *For any graph H and $n \geq 5$,*

1. *If $n = 5$, then $\chi'_s(C_n(H)) = 5|V(H)|^2$.*
2. *If $n = 6k$, then $\chi'_s(C_n(H)) = 2\chi'_s(H) + 3|V(H)|^2$.*
3. *If $n = 6k + 3$, then $\chi'_s(C_n(H)) \leq 3\chi'_s(H) + 3|V(H)|^2$.*

4. If $n = 6k+1, 6k+2, 6k+4, 6k+5$*, then*

$$\chi'_s(C_n(H)) \leq 2\chi'_s(H) + 4|V(H)|^2.$$

Theorem 3. *For any graph* G *with* $\Delta(G) = 3$ *and any graph* H*,*

$$\chi'_s(G[H]) \leq 5\chi'_s(H) + 10|V(H)|^2.$$

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