On the Volumes of Polyhedra which Tile the Hyperbolic Space Face-to-Face and Solid-Transitively

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Abstract

We constructed five countable series of bounded and unbounded polyhedra of the same volume in H^3.

Keywords: Polyhedra, volume, hyperbolic space, synthetic geometry.

In [1], Milnor introduced the function $\Lambda(\alpha) = -\int_0^\alpha \ln|2\sin u|du$ which he used for calculating the volumes of hyperbolic polyhedra. He showed that the volume of a simplex with all infinitely remote vertices is equal to $V(\Delta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$, where α, β, γ are dihedral angles of the simplex which satisfy the equality $\alpha + \beta + \gamma = \pi$.

Later on, in many works, the questions on the volumes of fundamental polyhedra of discrete motion groups of the hyperbolic space arose.

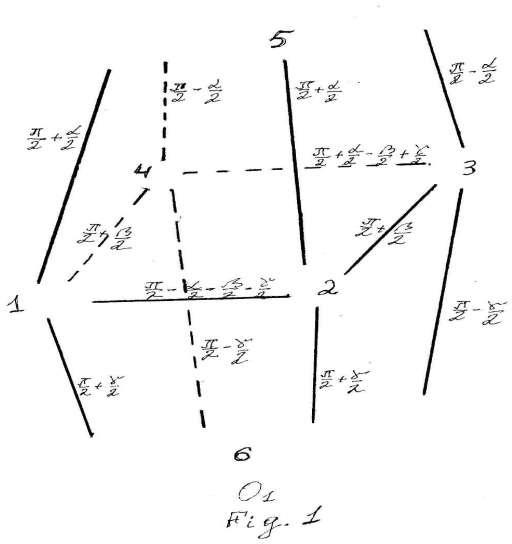
In the present communication, we state the following.

Theorem 1. *In the hyperbolic space H^3, there exist five countable series of bounded and unbounded polyhedra of the same volume which tile the space face-to-face and solid-transitively and possess the following properties: dihedral angles of polyhedra depend on three integer parameters; for a fixed choice of parameters, there exist at least five polyhedra with the same volume such that one polyhedron is bounded, three polyhedra are unbounded with one infinitely remote vertex but have different dihedral angles, and one polyhedron has three infinitely remote vertices.*

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We prove this theorem using synthetic geometry of hyperbolic space H^3; moreover, we construct the mentioned polyhedra. The construction is done for continuous parameters (i.e., angles of polyhedra); however, we are interested only in polyhedra that tile H^3 face-to-face and solid-transitively.

Let the angles α, β, γ satisfy the conditions: $\alpha+\beta+\gamma < \pi, \alpha > \beta > \gamma$. Consider octahedron O_1 with all infinitely remote vertices depicted in Fig. 1.



O_1
Fig. 1

We denote its infinitely remote vertices by numbers $1, 2, 3, 4, 5, 6$. It is easy to show that such an octahedron exists and to obtain its partition into four simplices with all infinitely remote vertices. To the octahedron O_1 we "glue" along face $(1, 2, 5)$ a simplex with all infinitely remote vertices and dihedral angles α, $\pi\backslash 2$ - $\alpha\backslash 2$, $\pi\backslash 2 - \alpha\backslash 2$, along faces $(1, 4, 6)$ and $(2, 3, 6)$ quadrangular pyramids with all infinitely remote vertices. Dihedral angles at the bases of pyramids are equal to: $\beta\backslash 2 + \gamma\backslash 2$ at $(1, 7)$; $(2, 9)$, $\beta\backslash 2$ - $\gamma\backslash 2$ at $(4, 8)$, $(3, 10)$ and $\pi\backslash 2$ - $\beta\backslash 2$ at $(1, 4)$ and $(7, 8)$. So, we obtain polyhedron R_1 with all infinitely remote

vertices depicted in Fig. 2.

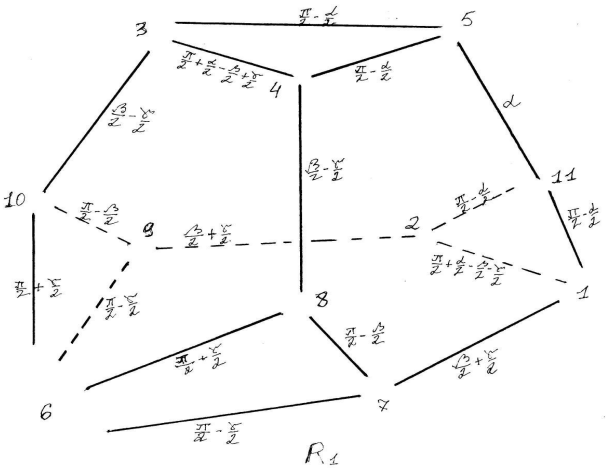


Fig. 2

Edges $(3, 10)$, $(4, 8)$, and $(5, 11)$ of this polyhedron form a hyperbolic bundle of straight lines; therefore, there exists a plane orthogonal to these edges. An orthogonal plane exists analogously for edges $(1, 7)$, $(2, 9)$, and $(11, 5)$. Also, for quadrangular pyramids with infinitely remote vertices $(6, 7, 1, 4, 8)$ and $(6, 10, 3, 2, 9)$ there exist symmetry planes which are orthogonal to edges $(1, 7)$ and $(4, 8)$, respectively, edges $(3, 10)$ and $(2, 9)$. After cutting from polyhedron R_1 some parts with above planes, we get polyhedron K_1 depicted in Fig. 3.

This polyhedron has only one infinitely remote vertex, the other vertices are proper. All the dihedral angles which are not indicated in Fig. 3 are right, i.e., are equal to $\pi\backslash 2$.

Using the same reasoning, we get two more polyhedra K_2 and K_3 which have analogous combinatorial type. But dihedral angles at analogous edges are equal to $(\beta, \alpha\backslash 2 - \gamma\backslash 2, \alpha\backslash 2 + \gamma\backslash 2)$ in one case and to $(\gamma, \alpha\backslash 2 - \beta\backslash 2, \alpha\backslash 2 + \beta\backslash 2)$ in other case. All the rest angles are right.

By the same way, we get two more polyhedra, one of them is

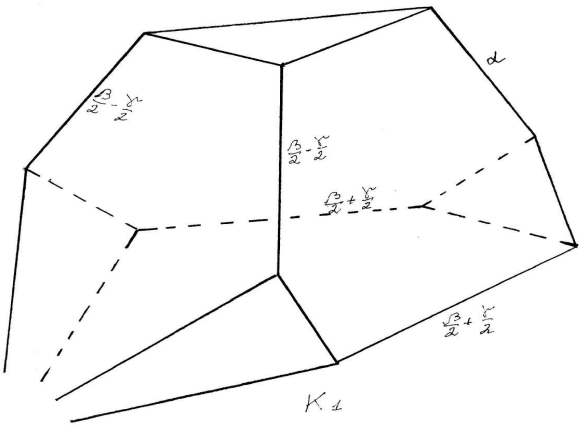


Fig. 3

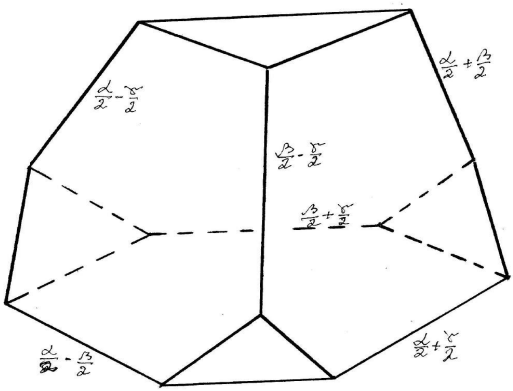
bounded (see Fig.4) and the other is unbounded with three infinitely remote vertices (see Fig.5).

All the dihedral angles without labels of polyhedra K_4 and K_5 are equal to $\pi\backslash 2$.

The volumes of all polyhedra K_1, K_2, K_3, K_4, and K_5 are equal to:
$V(K_i) = \Lambda(\pi\backslash 4-\alpha\backslash 4-\beta\backslash 4-\gamma\backslash 4) + \Lambda(\pi\backslash 4+\alpha\backslash 4+\beta\backslash 4+\gamma\backslash 4) +$
$+ \Lambda(\pi\backslash 4-\alpha\backslash 4-\beta\backslash 4+\gamma\backslash 4) + \Lambda(\pi\backslash 4+\alpha\backslash 4+\beta\backslash 4-\gamma\backslash 4) +$
$+\Lambda(\pi\backslash 4-\alpha\backslash 4+\beta\backslash 4-\gamma\backslash 4) + \Lambda(\pi\backslash 4+\alpha\backslash 4-\beta\backslash 4+\gamma\backslash 4) +$
$+ \Lambda(\pi\backslash 4+\alpha\backslash 4-\beta\backslash 4-\gamma\backslash 4) + \Lambda(\pi\backslash 4-\alpha\backslash 4+\beta\backslash 4+\gamma\backslash 4)$

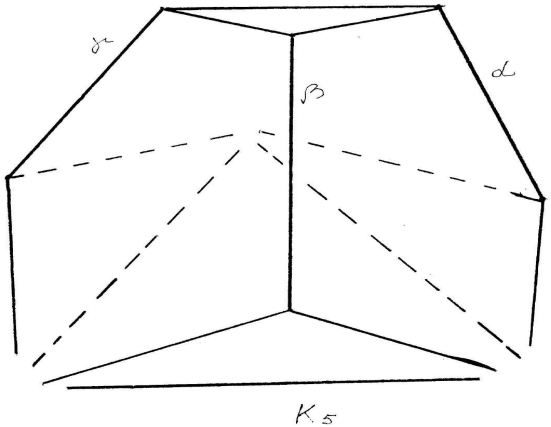
In order that constructed polyhedra would tile H_3 face-to-face and solid-transitively, it is sufficient to make them Coxeter.

Let $k, n, m \in N$ and $k > n > m$, we take angles:
$\alpha = \pi\backslash(m(m+n)(n-m)(m+k)(k-m)(n+k)(k-n))$
$\beta = \pi\backslash(n(m+n)(n-m)(m+k)(k-m)(n+k)(k-n))$
$\gamma = \pi\backslash(k(m+n)(n-m)(m+k)(k-m)(n+k)(k-n)),$
and then any polyhedron K_1, K_2, K_3, K_4, K_5 tiles H_3 face-to-face and solid-transitively. Therefore, we have a countable series of such poly-



K_4

Fig. 4



K_5

Fig. 5

hedra which depends on three integer parameters.

Acknowledgments. Institutional Research Program of the Moldova State University for 2024-2027, subprogram 011303, has supported part of the research for this paper.

References

[1] J. Milnor. *Hyperbolic geometry: the first 150 years.* Bull. Amer. Math. Soc., 1982, vol. 6, pp. 9–24.

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