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Finding the set of all Nash equilibria of a polymatrix mixed-strategy game

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Abstract

The method of intersection of best response mapping graphs is applied to determine the Nash equilibrium set of a finite mixed-strategy game. Results of a Wolfram language implementation of the method are presented. Appeared issues are highlighted, too.
Keywords: noncooperative game, polymatrix game, mixed strategy, Nash equilibrium set, best response mapping.

1 Introduction

The problem of all Nash equilibria finding in bimatrix game was considered earlier by Vorob'ev (1958) and Kuhn (1961), but as it is stressed by different researchers (see e.g. Raghavan (2002)), these results have only been of theoretical interest. They where rarely used practically to compute Nash equilibria as well as the results of Mills (1960), Mangasarian (1964), Winkels (1979), Yanovskaya (1968), Howson (1972), Eaves (1973), Mukhamediev (1978), Savani (2006), and Shokrollahi (2017). The first practical algorithm for Nash equilibrium computing was the algorithm proposed by Lemke and Howson (1964). Unfortunately, it doesn't compute Nash equilibrium sets. There are algorithms for polymatrix mixed strategy games too [4, 1].

Currently, the number of publications devoted to the problem of finding the Nash equilibrium set is increasing, see, e.g., bibliography surveys in [2, 3].

In this work, we present the results of a Wolfram Language implementation of the method of intersection of best response mapping graphs in polymatrix mixed-strategy games.

2 Problem formulation

The Nash equilibrium set is determined as the intersection of best response mapping graphs [4, 5]. This idea yields a natural method for Nash equilibrium set computing in mixed extensions of two-player $m \times n$ games and n-player $m_1 \times m_2 \times \cdots \times m_n$ games.

Consider a noncooperative finite strategic game:

$$\Gamma = \langle N, \{S_p\}_{p \in N}, \{a^p_{\mathbf{s}} = a^p_{s_1 s_2 \dots s_n}\}_{p \in N} \rangle,$$

where

- $N = \{1, 2, ..., n\} \subset \mathbb{N}$ is a set of players,
- $S_p = \{1, 2, \dots, m_p\} \subset \mathbb{N}$ is a set of (pure) strategies of the player $p \in N$,
- $\# S_p = m_p < +\infty$, $p \in N$,
- $a^p_{\mathbf{s}} = a^p_{s_1 s_2 \dots s_n} : S \to \mathbb{R}$ is a player's $p \in N$ payoff function,
- $S = \underset{p \in N}{\times} S_p$ is the set of profiles.

A mixed extension of Γ or a mixed-strategy game $\widetilde{\Gamma}$ is

$$\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N \rangle,$$

where

- $$\begin{aligned} f_p(\mathbf{x}) &= \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{s_1 s_2 \dots s_n} x^1_{s_1} x^2_{s_2} \dots x^n_{s_n} \\ &= \sum_{s_1=1}^{m_1} \sum_{s_2=1}^{m_2} \cdots \sum_{s_n=1}^{m_n} a^p_{\mathbf{s}} \prod_{p=1}^{n} x^p_{s_p} \end{aligned}$$

  is the payoff function of the p^{th} player;

- $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n) \in X = \underset{p \in N}{\times} X_p \subset \mathbb{R}^m$ is a global profile;
- $m = m_1 + m_2 + \cdots + m_n$ is the profile space dimension;
- $X_p = \left\{ \mathbf{x}^p = (x_1^p, \ldots, x_{m_p}^p) : \begin{array}{c} x_1^p + \cdots + x_{m_p}^p = 1, \\ x_1^p \geq 0, \ldots, x_{m_p}^p \geq 0 \end{array} \right\}$ is the set of mixed strategies of the player $p \in N$.

The problem of finding all Nash equilibria in $\widetilde{\Gamma}$ is considered.

3 Best response mapping graphs intersection

Consider the n-player mixed strategy game $\widetilde{\Gamma} = \langle X_p, f_p(\mathbf{x}), p \in N \rangle$. The payoff function of the player p is linear if the strategies of the others are fixed, i.e. the player p has to solve a linear parametric problem

$$f_p\left(\mathbf{x}^p, \mathbf{x}^{-p}\right) \to \max, \quad \mathbf{x}^p \in X_p, \quad p = 1, \ldots, n,$$

with the parameter vector $x^{-p} \in X^{-p}$.

Theorem 3.1. *The set of Nash equilibria in polymatrix mixed-strategy game is equal to*

$$NES(\widetilde{\Gamma}) = \bigcup_{\substack{i_1 \in U_1,\ I_1 \in \mathcal{P}(U_1 \setminus \{i_1\}) \\ \cdots \\ i_n \in U_n,\ I_n \in \mathcal{P}(U_n \setminus \{i_n\})}} X\left(i_1 I_1 \ldots i_n I_n\right).$$

The proof of the theorem has a constructive nature. It permits to develop on its basis both a general method for Nash equilibrium set computing, and different algorithms based on the method.

The components $X\left(i_1 I_1 \ldots i_n I_n\right)$ are solution sets of systems of multi-linear simultaneous equations. Their solving needs special conceptual and methodological approaches both from the perspectives of multi-linear algebra and algorithmic theory.

The Wolfram Programming Language, which has a symbolic nature by its origin, is a valuable practical tool for the set of Nash equilibria computing and representation.

4 Conclusion

The symbolic and numerical strength of the Mathematica System and the Wolfram Language permits to construct a package oriented on finding the set of all Nash equilibria in polymatrix mixed-strategy games.

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