Quartic Differential Systems with a Center-focus Critical Point and an Affine Invariant Straight Line of Maximal Multiplicity

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Abstract

In this paper, we show that in the class of quartic differential systems with a center-focus critical point and non-degenerate infinity, the maximal multiplicity of an affine invariant straight line is equal to 6.

Keywords: quartic differential system, invariant line, multiplicity of an invariant line, center-focus critical point.

We consider the real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, and the vector field $\mathbb{X} = p\,(x, y)\,\frac{\partial}{\partial x} + q\,(x, y)\,\frac{\partial}{\partial y}$ are associated to system (1). Denote $n = max\{deg(p), deg(q)\}$. If $n = 4$, then the system (1) is called quartic.

Via an affine transformation of coordinates and time rescaling, each non-degenerate quartic system with a center-focus critical point, i.e., a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x, y) + p_3(x, y) + p_4(x, y) \equiv p(x, y), \\ \dot{y} = -(x + q_2(x, y) + q_3(x, y) + q_4(x, y)) \equiv q(x, y), \end{cases} \tag{2}$$

where $p_i(x, y) = \sum_{j=0}^{i} a_{i-j,j} x^{i-j} y^j$, $q_i(x, y) = \sum_{j=0}^{i} b_{i-j,j} x^{i-j} y^j$, $i = 2, 3, 4$ are homogeneous polynomials in x and y of degree i with real coefficients.

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Suppose that

$$yp_4(x,y) - xq_4(x,y) \not\equiv 0, gcd(p,q) = 1, \tag{3}$$

i.e., at infinity, the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greater than 0.

The straight line $l \equiv \alpha x+\beta y+\gamma = 0,\ \alpha,\beta,\gamma \in \mathbb{C}$ is called invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x,y]$ such that the identity $\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_l(x,y),\ \ (x,y) \in \mathbb{R}^2$ holds.

The invariant straight line $\alpha x + \beta y + \gamma = 0$ has *multiplicity* m if m is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^m$ divides $\mathbb{E}(\mathbb{X}) = p \cdot \mathbb{X}(q) - q \cdot \mathbb{X}(p)$ [1].

The maximal multiplicity of the line at infinify for quartic differential systems of the form (2) was examined in [2]. In this paper, the conditions when an affine invariant straight line for the system (2) has maximal multiplicity are determined.

Let the quartic system (2) has an affine real invariant straight line l_1. By a transformation of the form

$$x \to \nu \cdot (x\cos\varphi + y\sin\varphi),\ y \to \nu \cdot (y\cos\varphi - x\sin\varphi),\ \nu \neq 0$$

we can make l_1 to be described by the equation $x - 1 = 0$. Then

$$\begin{array}{l} a_{40} = -(a_{20} + a_{30}),\ \ a_{31} = -(1 + a_{11} + a_{21}), \\ a_{22} = -(a_{02} + a_{12}),\ \ a_{13} = -a_{03},\quad a_{04} = 0, \end{array} \tag{4}$$

and (2) is reduced to the system

$$\left\{ \begin{array}{l} \dot{x} = (1-x)(a_{20}x^2 + (a_{20} + a_{30})x^3 + y + (1 + a_{11})xy + \\ \qquad +(1 + a_{11} + a_{21})x^2y + a_{02}y^2 + (a_{02} + a_{12})xy^2 + a_{03}y^3), \\ \dot{y} = -(x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + \\ \qquad +b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4). \end{array} \right. \tag{5}$$

For (5), we have $\mathbb{E}(\mathbb{X}) = (x-1)\Big(Y_2(y) + Y_3(y)\cdot(x-1) + \cdots + Y_{12}(y)\cdot(x-1)^{10}\Big)$, where $Y_j(y),\ j = 2,...12$, are polynomial in y.

The invariant line $x-1=0$ has multiplicity at least j if the system of identity $\{Y_2(y)\equiv 0,...,Y_j(y)\equiv 0\}$ holds. Solving the identities $\{Y_2(y)\equiv 0,...,Y_6(y)\equiv 0\}$, we obtain the following results.

Theorem 1. *The invariant straight line $x-1=0$ has for quartic system (5) the multiplicity at least six if and only if the coefficients of the system (5) verify one of the following two series of conditions:*

$$\begin{gathered} a_{11}=-3,\ a_{02}=0,\ a_{30}=-2a_{20},\ a_{21}=3,\ a_{12}=0,\ a_{03}=0,\\ b_{11}=2a_{20},\ b_{02}=3/2,\ b_{30}=-2b_{20}-3,\ b_{21}=0,\ b_{12}=-2b_{02},\\ b_{03}=0,\ b_{40}=2a_{20}^2+b_{20}+2,\ b_{31}=-2a_{20},\ b_{22}=b_{02},\\ b_{13}=0,\ b_{04}=0; \end{gathered} \tag{6}$$

$$\begin{gathered} a_{20}=0,\ a_{11}=-3,\ a_{02}=0,\ a_{30}=0,\ a_{21}=3,\ a_{12}=0,\\ a_{03}=0,\ b_{20}=-3,\ b_{02}=3,\ b_{30}=-2b_{20}-3,\ b_{12}=-2b_{02},\\ b_{03}=0,\ b_{40}=b_{20}+2,\ b_{22}=b_{02},\ b_{13}=0,\ b_{04}=0. \end{gathered} \tag{7}$$

Affirmation of Theorem 1 is established by direct calculation.

In the conditions (6), the quartic system (5) takes the form

$$\begin{aligned} \dot{x}&=(x-1)^2(a_{20}x^2+y-xy),\\ \dot{y}&=(-2x-2b_{20}x^2+2(3+2b_{20})x^3-2(2+2a_{20}^2+b_{20})x^4-\\ &\quad -4a_{20}xy+4a_{20}x^3y-3y^2+6xy^2-3x^2y^2)/2,\quad a_{20}\neq 0. \end{aligned} \tag{8}$$

For this quartic system, we have:

$\mathbb{E}(\mathbb{X})=(x-1)^6(-4x^2-12x^3-8b_{20}x^3+12a_{20}^2x^4-8b_{20}x^4-4b_{20}^2x^4+16x^5+16a_{20}^2x^5+16b_{20}x^5+8a_{20}^2b_{20}x^5+4b_{20}^2x^5-8a_{20}x^2y-32a_{20}x^3y-16a_{20}b_{20}x^3y+16a_{20}x^4y+24a_{20}^3x^4y+8a_{20}b_{20}x^4y-4y^2-12xy^2-8b_{20}xy^2+24x^2y^2+12b_{20}x^2y^2-8x^3y^2-24a_{20}^2x^3y^2-4b_{20}x^3y^2-8a_{20}y^3+12a_{20}x^2y^3+3y^4-3xy^4)/4$, and $Y_7(y)=a_{20}(7a_{20}+2a_{20}b_{20}-6y+6a_{20}^2y-2b_{20}y-6a_{20}y^2+y^3)\not\equiv 0$, $a_{20}\neq 0$.

If $a_{20}=0$, the right-hand sides of (5) have the common divisors of degree greater than 0; therefore, the multiplicity of the invariant straight line $x-1=0$ is exactly six.

In the conditions (7), the quartic system (5) takes the form:

$$\begin{aligned} \dot{x}&=-y(x-1)^3,\\ \dot{y}&=-x+3x^2-3x^3+x^4-b_{11}xy-b_{21}x^2y-b_{31}x^3y-\\ &\quad -3y^2+6xy^2-3x^2y^2,\ \ b_{11}^2+b_{21}^2+b_{31}^2\neq 0, \end{aligned} \tag{9}$$

and $\mathbb{E}(\mathbb{X}) = (x-1)^6(-x^2+3x^3-3x^4+x^5-b_{11}x^2y-b_{21}x^3y-b_{31}x^4y-y^2+3xy^2-3x^2y^2+x^3y^2-b_{11}y^3-2b_{21}xy^3-3b_{31}x^2y^3+6y^4-6xy^4)$.

The polynomial $Y_7(y)$ looks as $Y_7(y) = -y(b_{11}+b_{21}+b_{31}+b_{11}y^2+2b_{21}y^2+3b_{31}y^2)$. If the identity $Y_7(y) \equiv 0$ holds, then the right-hand sides of (5) have the common divisors of degree greater than 0; therefore, the multiplicity of the invariant straight line $x-1=0$ is exactly six.

In this way, we have proved the following theorem.

Theorem 2. *In the class of quartic differential systems with a center-focus critical point and non-degenerate infinity, the maximal multiplicity of an affine real invariant straight line is equal to six.*

References

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