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Factor-Group of a Topological Group and Factor-Ring of a Topological Ring and Completeness Issues

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Abstract

The connection between the completeness of topological groups and topological rings and the completeness of their factor groups and factor rings, respectively, are discussed in this article.

Keywords: Separable topological group, bounded set in a topological ring, locally bounded topological ring, Cauchy sequence over a directed set, limit of a sequence over a directed set, complete topological group.

1 Introduction

Preserving the completeness of topological groups and the completeness of topological rings under various constructions is one of the areas of research in topological algebra (see, for example, [1]).

The question of preserving the completeness of topological groups and topological rings when taking factor groups and factor-rings, respectively, is considered in this work.

2 Basic definitions

This section outlines some well-known concepts that are necessary to present the main results.

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Notation 2.1. If (X, τ) is a topological space and $A \subseteq X$, then we denote the restriction of the topology τ to the set SAS by $\tau|_A$.

Definition 2.1. As usual, a partially ordered set $(\Gamma, <)$ is called *a directed set* if for any two elements γ_1 and γ_2 from Γ there exists an element γ_3 such that $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_3$.

Definition 2.2. If $(\Gamma, <)$ is some directed set, then, as usual, a sequence $\{g_\gamma | \gamma \in \Gamma\}$ of elements of a topological Abelian group $(G(+), \tau)$ is called *a Cauchy sequence over a directed set* $(\Gamma, <)$ in a topological group $(G(+), \tau)$ if for any neighborhood V of zero of a topological group $(G(+), \tau)$ there exists an element $\gamma_0 \in \Gamma$ such that $\gamma - \gamma_0 \in V$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$.

Definition 2.3. If $(\Gamma, <)$ is a directed set, then an element g of a topological Abelian group $(G(+), \tau)$ is called the *limit of the Cauchy sequence* $\{g_\gamma | \gamma \in \Gamma\}$ *over this directed set* if for any neighborhood V of zero in the topological group $(G(+), \tau)$ there is an element γ_0 such that $g - g_\gamma \in V$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$.

In this case, we will write that $g = \lim_{\gamma \in \Gamma} |\{g_\gamma | \gamma \in \Gamma\}$.

Definition 2.4. A topological group $(G(+), \tau)$ is called a *complete group* if for any directed set $(\Gamma, <)$, any Cauchy sequence $\{g_\gamma | \gamma \in \Gamma\}$ over this directed set has a limit.

Definition 2.5. A topological group $(G(+), \tau)$ is called *separable* if its topology is Hausdorff.

Definition 2.6. A subset S of a topological ring (R, τ) is called *bounded* if for any neighborhood V of zero of the topological ring (R, τ) there is a neighborhood U of zero of the topological ring (R, τ) such that $S \cdot U \subseteq V$ and $U \cdot S \subseteq V$.

Definition 2.7. A topological ring is called *locally bounded* if it has a neighborhood of zero that is a bounded set.

3 Main results

This section presents results that address this issue; some of them were previously published in other articles (see [1] and [2]) but are included here for completeness.

Theorem 3.1. *For any separable topological Abelian group $(G(+),\tau)$, there exist a complete separable topological Abelian group $(\widetilde{G}(+),\widetilde{\tau})$ and a closed subgroup N in the topological group $(\widetilde{G}(+),\widetilde{\tau})$, such that the factor group $(\widetilde{G}(+),\widetilde{\tau})/N$ and the topological group $(G(+),\tau)$ are isomorphic.*

Remark 3.1. Unfortunately, there is an error in the proof of the analogue of Theorem 3.1 for topological rings in the monograph [1]. The main difficulty is to construct the corresponding complete topological ring. This error for locally bounded topological rings is corrected by the following theorem.

Theorem 3.2. For any separable locally bounded topological ring (R,τ), there exist a complete separable topological ring $(\widetilde{R},\widetilde{\tau})$ and a closed ideal N in the topological ring $(\widetilde{R},\widetilde{\tau})$ such that the factor ring $(\widetilde{R},\widetilde{\tau})/N$ and the ring (R,τ) are isomorphic.

Theorem 3.3. *Let N be a closed subgroup of a commutative group $G(+)$ and let τ be a separable group topology on a group $G(+)$. If the topological group $(N,\tau|_N)$ and the factor group $(G(+),\tau)/N$ are complete groups, then the topological group $(G(+),\tau)$ is also a complete group.*

Theorem 3.4. *If a separable, complete topological commutative group (G,τ) satisfies the first axiom of countability (i.e., it has a countable basis of the filter of neighborhoods of zero), then the factor group $(G,\tau)/N$ is a complete group for any closed subgroup N of the topological group (G,τ).*

Remark 3.2. Since the completeness of a topological ring is determined by the completeness of its additive group, then for topological rings analogues of Theorems 3.3, 3.4, and 3.5 are true.

Remark 3.3. Since from Theorems 2.1 and 2.2 it follows that the factor group of a complete topological group and the factor ring of a complete topological ring may not be complete. Therefore, it is interesting to obtain conditions when the completeness of the factor group and the factor ring is preserved. The next theorem addresses this issue.

Theorem 3.5. *If M is a subgroup of an Abelian group $G(+)$ and τ is such a group topology that (G,τ) is a complete separable topological group and M is a discrete subgroup of the topological group (G,τ), then the factor group $(G,\tau)/M$ is a complete group.*

Corollary. *If I is an ideal of a ring R and τ is a ring topology on the ring R such that it is a complete separable topological ring and $(I,\tau|_I)$ is a discrete ideal of the topological ring (R,τ), then the factor ring $(R,\tau)/I$ is a complete topological ring.*

References

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