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Collocation Method for Solving Singular Integral Equations with Discontinuous Coefficients

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Abstract

The paper presents an efficient method for solving the Cauchy singular integral equation, which is defined on a closed and smooth contour in the complex plane. The coefficients and the right-hand side of the equation are piecewise continuous functions, numerically defined on a finite set of points along the contour.

Keywords: Singular integral equation, piecewise continuous coefficients, collocation method, B-spline functions.

1 Introduction and problem formulation

Let a closed and smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset \mathbb{C}$, with the point $z = 0 \in \Omega^+$. We consider the points on the contour Γ to be defined by the Riemann function $z = \psi(w)$, which maps the circle $\Gamma_0 = \{w \in \mathbb{C} : |w| = 1\}$ onto the contour Γ. Let's consider the Cauchy singular integral equation defined on the contour Γ

$$c(t)\varphi(t) + d(t)(S\varphi)(t) + (K\varphi)(t) = f(t), \quad t \in \Gamma, \tag{1}$$

where $(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau$ is the Cauchy singular integral, and $(K\varphi)(t) = \frac{1}{2\pi i}\int_\Gamma h(t,\tau)\varphi(\tau)\, d\tau$ is the regular part, with the kernel $h \in C(\Gamma \times \Gamma)$. We assume that the coefficients c and d of the equation, the right-hand side f, and the solution φ are piecewise continuous functions.

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Let $\{t_j\}_{j=1}^{n_B}$, $t_j = \psi(w_j)$, $w_j = e^{i\theta_j}$, $\theta_j = 2\pi(j-1)/n_B$, $j = 1, ..., n_B$, be the set of distinct points of the contour Γ, where the values of the right-hand side f and the coefficients c and d are defined. Additionally, we assume that the values of the functions f, c and d are known at the discontinuity points t_r^d, $r = 1, ..., n_{pd}$, on the contour Γ.

We aim to develop a numerical method that determines a sequence of approximations φ_n, which converge uniformly to the solution φ of equation (1) everywhere on Γ, except for a finite number of discontinuity points of φ. To approximate the solution of equation (1), we apply a modification of the collocation method, where the solution is sought as a linear combination of B-spline functions and Heaviside step functions, resulting in a piecewise continuous approximation function.

2 The numerical method

The method we propose for approximating the solution φ of equation (1) uses the concept of B-spline functions of order m, which are defined at the points t_j, $j = 1, ..., n_B$, by the recursive formula $B_{m,j}(t) = \frac{m}{m-1}\left(\frac{t-t_j^B}{t_{j+m}^B - t_j^B} B_{m-1,j}(t) + \frac{t_{j+m}^B - t}{t_{j+m}^B - t_j^B} B_{m-1,j+1}(t)\right)$. The set of nodes $\left\{t_j^B\right\}_{j=1}^{n_B+m}$ satisfies the condition $t_j^B = t_j$, $j = 1, ..., n_B$, $t_{n_B+1}^B, t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$. It is known that any continuous function on Γ can be uniformly approximated by a linear combination of B-spline functions. When linear combinations of B-splines are used to approximate piecewise continuous functions, the approximation error decreases as the number of spline knots increases, even in the vicinity of discontinuity points [1].

With respect to the points of discontinuity t_r^d, $r = 1, ..., n_{pd}$, we define the Heaviside step function H on the contour Γ:

$$H\left(t - t_s^d\right) = \begin{cases} 0 & if\ t \in \left(\Gamma_1 \setminus \left\{t_1^B\right\}\right) \cup ... \cup \Gamma_{q-1} \cup arc\left[t_q^B, t_s^d\right) \\ 1 & if\ t \in arc\left[t_s^d, t_{q+1}^B\right) \cup \Gamma_{q+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

where $\Gamma_q = arc\left[t_q^B, t_{q+1}^B\right]$, $t_s^d \in \Gamma_q$.

Taking into account that the solution φ of equation (1) exhibits jump discontinuities on the contour Γ, we seek to approximate φ in the form [1]

$$\varphi_{n_B}^H(t) = \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t) + \sum_{s=1}^{n_{pd}} \beta_s H\left(t - t_s^d\right),$$

where the coefficients $\alpha_k \in \mathbb{C},\ k = 1, ..., n_B$, and $\beta_s \in \mathbb{C},\ s = 1, ..., n_{pd}$, are determined by imposing $n = n_B + n_{pd}$ interpolation conditions

$$c\left(t_j^C\right)\varphi_{n_B}^H\left(t_j^C\right) + d\left(t_j^C\right)\left(S\varphi_{n_B}^H\right)\left(t_j^C\right) + \left(K\varphi_{n_B}^H\right)\left(t_j^C\right) = f\left(t_j^C\right). \quad (2)$$

In relation (2), the first n_B interpolation points $t_j^C,\ j = 1, ..., n_B$, are the nodes $t_j^B = t_j,\ j = 1, ..., n_B$, and the remaining n_{pd} points $t_j^C,\ j = n_B + 1, ..., n$, are the discontinuity points $t_s^d,\ s = 1, ..., n_{pd}$, of the function f (and also of c and d). If among the interpolation points $t_j^C,\ j = 1, ..., n_B$, there are discontinuity points $t_s^d = \psi\left(e^{i\theta_s^d}\right)$ of the function f on Γ, then instead of them, for sufficiently small $\varepsilon_2 > 0$, we consider points $\tilde{t}_s^d = \psi\left(e^{i(\theta_s^d - \varepsilon_2)}\right)$ such that $f\left(\tilde{t}_s^d\right) = f\left(t_s^d\right)$ (also for values of c and d). The Riemann integrals $\left(K\varphi_{n_B}^H\right)\left(t_j^C\right)$ are aproximated using the generalized trapezoidal rule, which is also applicable to functions with complex values.

As the interpolation points $t_j^C,\ j = 1, ..., n$, form a subdivision covering the contour Γ, if $t_j^C \in arc\left[t_{k+r-1}^B, t_{k+r}^B\right]$, the integral $\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)$ will exhibit a singularity of type $1\Big/\left(\tau - t_j^C\right)$. In such cases, we interpret $\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)$ as the Cauchy principal value and require an efficient method for approximating this integral.

Let θ_j^C be the polar angle corresponding to the point $t_j^C = \psi\left(e^{i\theta_j^C}\right) \in \Gamma$. The singular integral $\left(S\varphi_{n_B}^H\right)\left(t_j^C\right)$ can be expressed as $I = p.v.\frac{1}{\pi i}\int_c^d g(\theta)\, d\theta$, where $g(\theta) = f(\theta)/(\varsigma(\theta) - \varsigma(\theta_0)),\ \theta_0 = \theta_j^C$. Using the notation $a = min\{\theta_0 - c, d - \theta_0\}\ (a > 0)$, the integral I can be represented as $I = (1/\pi i)(I_1 + I_2)$, where $I_1 = p.v.\int_{\theta_0 - a}^{\theta_0 + a} g(\theta)\, d\theta$,

$I_2 = \begin{cases} \int_{\theta_0+a}^{d} g(\theta)\, d\theta \;\; if \;\; a = \theta_0 - c \\ \int_{c}^{\theta_0-a} g(\theta)\, d\theta \;\; if \;\; a = d - \theta_0 \end{cases}$. The value of I_2 represents a Riemann integral, while the integral I_1 is interpreted as a generalization of the Cauchy principal value. The integral I_2 will be computed using the generalized trapezoidal rule, and for the computation of I_1, an analogous procedure to that outlined in [2] can be applied.

3 Conclusion

This paper presents a new method for solving Cauchy singular integral equations with piecewise continuous coefficients defined on a smooth and closed contour in the complex plane. The method constructs a sequence of approximations that converge almost uniformly to the exact solution of the integral equation.

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