ON SYSTEMS OF NONLINEAR EQUATIONS DEPENDING ON PARAMETERS

CU PRIVIRE LA SISTEME DE ECUAȚII NELINIARE ÎN FUNCȚIE DE PARAMETRI

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Let us consider a system of n order nonlinear equations with respect to the variables $x_1,\ldots,x_n$, and depending on m parameters $y_1,\ldots,y_m$, that is

$$f_1(x_1,\ldots,x_n;y_1,\ldots,y_m)=0,\ \ldots,\ f_n(x_1,\ldots,x_n;y_1,\ldots,y_m)=0. \qquad (1)$$

It is assumed that zero values of variables $x_1,\ldots,x_n$ and parameters $y_1,\ldots,y_m$ satisfy the system of equations (1), i.e.

$$f_1(0,\ldots,0;0,\ldots,0)=0,\quad \ldots,\quad f_n(0,\ldots,0;0,\ldots,0)=0. \qquad (2)$$

In the range of parameter changes $|y_j|<1$ ($j=1,\ldots,m$), it is required to find the solutions $x_1=x_1(y_1,\ldots,y_m)$, …, $x_n=x_n(y_1,\ldots,y_m)$ satisfying conditions (2). Such solutions are called small solutions. Thus, in a small neighborhood of zero values for the variables $x_1,\ldots,x_n$ and parameters $y_1,\ldots,y_m$, the following relations hold

$$|x_i|=|x_i(y_1,\ldots,y_m)|<1\ (i=1,\ldots,n),\ |y_j|<1\ (j=1,\ldots,m).$$

In the form of power series expansion of functions $f_i(x_1,\ldots,x_n;y_1,\ldots,y_m)$ ($i=1,\ldots,n$) in terms of the variables $x_1,\ldots,x_n,y_1,\ldots,y_m$, one can represent the system of nonlinear equations (1) in the matrix form with an explicit separation of linear terms in variables $x_1,\ldots,x_n$ and parameters $y_1,\ldots,y_m$, that is

$$A\vec{x}=-\left(B\vec{y}+\vec{F}(\vec{x},\vec{y})\right), \qquad (3)$$

where

$$A=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix},\ B=\begin{pmatrix}\frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_m}\end{pmatrix},\ \vec{x}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},\ \vec{y}=\begin{pmatrix}y_1\\ \vdots\\ y_m\end{pmatrix},.$$

$$\vec{F}(\vec{x},\vec{y}) = \begin{pmatrix} F_1(\vec{x},\vec{y}) \\ \vdots \\ F_n(\vec{x},\vec{y}) \end{pmatrix}$$

Note that function $\vec{F}(\vec{x},\vec{y})$ in Eq. (3) is determined by the expression

$$F_i(x_1,\ldots,x_n;y_1,\ldots,y_m) \equiv f_i(x_1,\ldots,x_n;y_1,\ldots,y_m) - \sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}x_j - \sum_{j=1}^{m}\frac{\partial f_i}{\partial y_j}y_j \quad (i=1,\ldots,n) \quad (4)$$

and have a higher order of smallness with respect to the linear terms in Eq. (3). The expression for the main part of the asymptotic representation of the dependence (4) is given by

$$F_i^0(x_1,\ldots,x_n;y_1,\ldots,y_m) \approx \frac{1}{2}\sum_{k,s=1}^{n}\frac{\partial^2 f_i}{\partial x_k \partial x_s}x_k x_s + \sum_{k=1}^{n}\sum_{s=1}^{m}\frac{\partial^2 f_i}{\partial x_k \partial y_s}x_k y_s + \frac{1}{2}\sum_{k,s=1}^{m}\frac{\partial^2 f_i}{\partial y_k \partial y_s}y_k y_s \quad (i=1,\ldots,n) \quad (5)$$

In analyzing small solutions of the system of equations (3), it is of interest to elucidate both cases when the rank of the matrix A is equal to the order of the system of equations (3), i.e. $rank(A)=n$, and, respectively, when the rank of this matrix is less than the order of the system of equations (3), that is $rank(A)<n$.

Let us focus now on the construction of small solutions for systems of nonlinear equations depending on parameters in the case of $rank(A)=n$. In this case, i.e. $rank(A)=n$, there is an inverse matrix A^{-1}, and the system of nonlinear equations (3) can be represented in an equivalent form

$$\vec{x} = -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}(\vec{x},\vec{y})\right). \quad (6)$$

We use the iterative procedure to determine solutions of the system of equations (6), that is

$$\vec{x}_{k+1} = -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}(\vec{x}_k,\vec{y})\right),\ k=0,1,2,\ldots \quad (7)$$

One can note that in the small neighborhood of zero values for the variables $x_1,\ldots,x_n$ and parameters $y_1,\ldots,y_m$, the iterative procedure (7) is convergent, i.e. it leads to a precise solution of the system of equations if $k\to\infty$, and thus we have $\vec{x}(\vec{y}) = \lim_{k\to\infty}\vec{x}_k(\vec{y})$.

Let us indeed consider the system of nonlinear equations in the form

$$x_1 = \varphi_1(x_1,\ldots,x_n;y_1,\ldots,y_m),\ \ldots,\ x_n = \varphi_n(x_1,\ldots,x_n;y_1,\ldots,y_m). \quad (8)$$

Note that the system of nonlinear equations of an arbitrary form can be represented in such a way. It is known that the iterative procedure of solving the system of nonlinear equations (8)

$$x_1^{k+1} = \varphi_1(x_1^k,\ldots,x_n^k;y_1,\ldots,y_m),\ \ldots,\ x_n^{k+1} = \varphi_n(x_1^k,\ldots,x_n^k;y_1,\ldots,y_m),\ k=0,1,2,\ldots$$

converges to some solution of the system of equations (8) when satisfying conditions [1]

$$\sum_{k=1}^{n}\left|\frac{\partial\varphi_i}{\partial x_k}\right| < 1,\ i = 1,\ldots,n \tag{9}$$

In this case, the effectiveness of the iterative procedure increases with a decrease in the values $\left|\frac{\partial\varphi_i}{\partial x_k}\right|$. In the case of the iterative procedure (7) for the system of nonlinear equations (6), i.e. $\vec{\varphi}(\vec{x};\vec{y}) = -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}(\vec{x},\vec{y})\right)$, the conditions of convergence (9) of the iterative process, due to the structure of dependences $\vec{F}(\vec{x},\vec{y})$, is carried out fast due to satisfying conditions $\sum_{k=1}^{n}\left|\frac{\partial\varphi_i}{\partial x_k}\right| < 1$ ($i = 1,\ldots,n$), and thus the iterative procedure (7) is effective. In particular, taking the zero values of variables $\vec{x}_0 = 0$ as an initial approximation and using the iterative procedure (7), we have the approximate solutions that correspond to the first two iterations in the form

$$\vec{x}_1(\vec{y}) = -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}(0,\vec{y})\right),\ \vec{x}_2(\vec{y}) = -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}\left(-\left(A^{-1}B\vec{y} + A^{-1}\vec{F}(0,\vec{y})\right)\vec{y}\right)\right). \tag{10}$$

Note that with an increase in the number of iterations, the solution becomes more precise due to the appearance of terms as quantities of higher order of smallness in the expansion. We will now proceed from the expression for $\vec{x}_2(\vec{y})$ in Eq. (10) and the asymptotic representation (5) for dependencies $\vec{F}(\vec{x},\vec{y})$, and, keeping in further analysis the small values $\vec{x}(\vec{y})$ in the asymptotic representation with an assumed degree of accuracy of the second order of smallness, we arrive, as a result, at the desired degree of accuracy in the asymptotic representation for small solutions of the system of nonlinear equations (4) in the form

$$\vec{x}(\vec{y}) \approx -\left(A^{-1}B\vec{y} + A^{-1}\vec{F}_0\left(-A^{-1}B\vec{y},\vec{y}\right)\right). \tag{11}$$

Note that the main part of the asymptotic representation (11) has the form $\vec{x}(\vec{y}) \approx -A^{-1}B\vec{y}$ and corresponds to the exact solution of the system of linear equations $A\vec{x} = -B\vec{y}$, provided that $\det(A) \neq 0$. Construction of small solutions for systems of nonlinear equations (3) in the case when the determinant of matrix A goes to zero, and its rank is $rank(A) = r < n$, and, therefore, we represent the matrices A and B, and vectors $\vec{x}$, $\vec{y}$ and $\vec{F}$ in the block form, is given in Ref. [2].

Literature:

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