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Two-sided Remotely Almost Periodic Solutions of Ordinary Differential Equations in Banach Spaces

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Abstract

The aim of this paper is studying the two-sided remotely almost periodic solutions of ordinary differential equations in Banach spaces of the form $x' = A(t)x + f(t) + F(t,x)$ with two-sided remotely almost periodic coefficients if the linear equation $x' = A(t)x$ satisfies the condition of exponential trichotomy and nonlinearity F is sufficiently "small".

Keywords: Remotely Almost Periodic Solution; Non-autonomous Dynamical Systems; Cocycles.

1 Introduction

This study continues the author's work devoted to the study of remotely almost periodic (RAP) motions of dynamical systems and solutions of differential equations [2].

The notion of remotely almost periodicity on the real axis $\mathbb{R}$ for the scalar functions was introduced and studied by D. Sarason [7]. Remotely almost periodic functions on the semi-axis $\mathbb{R}_+$ with the values in the Banach space were introduced and studied by W. M. Ruess and W. H. Summers [6] (see also [2]). Remotely almost periodic functions on the real axis with the values in Banach spaces were introduced and studied by A. G. Baskakov [1]. He calls theses functions "almost periodic at infinity". Remotely almost periodic (on the real-axis $\mathbb{R}$) solutions of ordinary differential equations with remotely almost periodic

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coefficient were studied by C. Maulen, S. Castillo, M. Kostic, and M. Pinto [4].

2 Two-sided RAP Solutions of Linear Differential Equations

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $\mathbb{R} := (-\infty, +\infty)$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. Denote by $[\mathfrak{B}]$ the Banach space of any linear bounded operators A.

Consider a linear differential equation

$$x' = A(t)x, \tag{1}$$

where $A \in C(\mathbb{R}, [\mathfrak{B}])$, and denote by $U(t, A)$ the Cauchy operator of the equation (1).

Definition 1. A linear differential equation (1) is said [3] to have an exponential trichotomy (on $\mathbb{R}$) if there exist linear projections P, Q such that $PQ = QP$, $P + Q - PQ = I$, and constants $K \geq 1$ and $\alpha > 0$ such that

$$\begin{gathered}
\|U(t, A)PU^{-1}(\tau, A)\| \leq Ke^{-\alpha(t-\tau)} \quad \text{for} \quad 0 \leq \tau \leq t, \\
\|U(t, A)(I - P)U^{-1}(\tau, A)\| \leq Ke^{-\alpha(\tau-t)} \quad \text{for} \quad t \leq \tau, \tau \geq 0 \\
\|U(t, A)QU^{-1}(\tau, A)\| \leq Ke^{-\alpha(\tau-t)} \quad \text{for} \quad t \leq \tau \leq 0 \\
\|U(t, A)(I - Q)U^{-1}(\tau, A)\| \leq Ke^{-\alpha(t-\tau)} \quad \text{for} \quad \tau \leq t, \ \tau \leq 0.
\end{gathered}$$

Definition 2. If the equation (1) has the exponential trichotomy on $\mathbb{R}$, then for (1) can be constructed a Green function [3, 5] by the equality

$$G_A(t, \tau) = \begin{cases}
U(t, A)PU^{-1}(\tau, A), & (0 \leq \tau < t) \\
-U(t, A)(I - P)U^{-1}(\tau, A), & (t \leq \tau, \ \tau \geq 0) \\
-U(t, A)QU^{-1}(\tau, A), & (t \leq \tau \leq 0) \\
U(t, A)(I - Q)U^{-1}(\tau, A), & (\tau \leq t, \ \tau \leq 0).
\end{cases} \tag{2}$$

Denote by $C(\mathbb{R}, \mathfrak{B})$ the space of all continuous functions $\varphi : \mathbb{R} \to \mathfrak{B}$ equipped with the compact-open topology.

Definition 3. A subset $A \subseteq \mathbb{R}$ is called relatively dense if there exists a positive number l such that $A \bigcap [a, a + l] \neq \emptyset$ for any $a \in \mathbb{R}$.

Definition 4. A function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ is said to be:

1. two-sided remotely almost periodic (respectively, remotely ω-periodic or remotely stationary) if for arbitrary $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{F}(\varepsilon, \varphi)$ (respectively, $\omega\mathbb{Z} \subseteq \mathcal{F}(\varepsilon, \varphi)$ or $\mathbb{R} = \mathcal{F}(\varepsilon, \varphi)$) such that, for any $\tau \in \mathcal{F}(\varepsilon, \varphi)$, we can find a positive number $L(\varepsilon, \tau, \varphi)$ with the property $|\varphi(t + \tau) - \varphi(t)| < \varepsilon$ for any $|t| \geq L(\varepsilon, \tau, \varphi)$;

2. Lagrange stable if the subset $\{\varphi^h |\ h \in \mathbb{R}\}$ of $C(\mathbb{R}, \mathfrak{B})$ is precompact.

Theorem 5. Assume that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ and the following conditions are fulfilled:

1. (A, f) is Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

2. the equation (1) has an exponential trichotomy on $\mathbb{R}$.

Then

1. the equation
$$u' = A(t)u + f(t) \tag{3}$$
has a unique two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution φ_0 with $R\varphi_0(0) = 0$, where $R := PQ$;

2. the solution φ_0 is defined by $\varphi_0(t) = \int_{-\infty}^{+\infty} G_A(t, \tau) f(\tau) d\tau$, where $G_A(t, \tau)$ is the Green function of the equation (1) defined by (2);

3. every bounded on $\mathbb{R}$ solution of the equation (3) is two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary).

3 Two-sided RAP Solutions of Perturbed Linear Differential Equations

Let $W \subseteq X$ be a bounded (respectively, compact) subset of X. Denote by $C(\mathbb{R} \times W, X)$ the space of all continuous functions $f : \mathbb{R} \times W \to X$ equipped with the compact-open topology.

Definition 6. A function $F \in C(\mathbb{R} \times W, X)$ is said to be Lagrange stable if the set $\Sigma_F := \{\sigma(\tau, F) | \ \tau \in \mathbb{R}\}$ is pre-compact in $C(\mathbb{R} \times W, X)$.

Let us consider a differential equation

$$u' = A(t)u + f(t) + F(t, u), \tag{4}$$

where $F \in C(\mathbb{R} \times \mathfrak{B}, \mathfrak{B})$ and $F(t, 0) = 0$ for any $t \in \mathbb{R}$.

Assume that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is Lagrange stable and φ_0 is a unique remotely compatible solution of the equation (3) with $R\varphi_0(0) = 0$.

Theorem 7. Let $f \in C(\mathbb{R}, \mathfrak{B})$, $F \in C(\mathbb{R} \times \mathfrak{B}, \mathfrak{B})$ and $A \in (\mathbb{R}, [\mathfrak{B}])$. Assume that the following conditions are fulfilled:

1) the linear equation (1) has the exponential trichotomy on $\mathbb{R}$;

2) the functions A, f, and F are Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

3) F satisfies the condition of Lipschitz with respect to $x \in \mathfrak{B}$ with the constant of Lipschitz $L < \frac{\nu}{2N}$.

Then

1. the equation (4) has at least one Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution φ defined by the equality

$$\varphi(t) = \int_{-\infty}^{+\infty} G_A(t, \tau)(f(\tau) + F(\tau, \varphi(\tau)))d\tau$$

and

2. $\|\varphi - \varphi_0\| \leq r$, where $r := \frac{4\mathcal{N}^2 L\|f\|}{\nu(\nu - 2\mathcal{N}L)}$.

Theorem 8. Let $A \in C(\mathbb{R}, [\mathfrak{B}])$, $f \in C(\mathbb{R}, \mathfrak{B})$ and $F \in C(\mathbb{R}_+ \times \mathfrak{B}, \mathfrak{B})$.

Assume that the following conditions are fulfilled:

1) the equation (1) has an exponential trichotomy;

2) the functions A, f, and F are Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

3) F is Lipschitzian with respect to the second variable with the constant $L > 0$.

Then the following statements hold:

1. there exists a number $\varepsilon_0 > 0$ such that, for every $|\varepsilon| \leq \varepsilon_0$, the equation
$$\frac{dx}{dt} = A(t)x + f(t) + \varepsilon F(t, x)$$
has a two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution $\varphi_\varepsilon \in C_b(\mathbb{R}, \mathfrak{B})$ with $R\varphi_\varepsilon(0) = 0$;

2. $\{\varphi_\varepsilon\}$ converges to φ_0 as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}$.

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