On Self-Orthogonal n-ary Quasigroups

Parascovia Syrbu, Tatiana Rotari

Abstract

We consider self-orthogonal finite n-ary quasigroups and give some estimations of their spectrum in the present work. A method of construction of self-orthogonal n-quasigroups, using self-orthogonal quasigroups of lower arity, is given. In particular, it is shown that there exist: 1) self-orthogonal 2^k-ary quasigroups of every order $q > 3, q \neq 6$, where $k \geq 1$; 2) self-orthogonal p^k-quasigroups of prime order p, for every $p \geq 3$ and every $k \geq 1$; 3) self-orthogonal $2n$-quasigroups of order q, for every $q > 3, q \neq 6$ and $n + 1 \not\equiv 0 \pmod{q}$.

Keywords: orthogonal n-operations, parastrophic-orthogonal n-quasigroups, self-orthogonal n-quasigroups.

The Euler conjecture about non-existence of orthogonal latin squares of order $n \equiv 2 \pmod 4$ solved, negatively for $n \geq 10$ of this type, in the middle of 20th century, led to new concepts, types of orthogonality and new problems concerning orthogonality and parastrophic-orthogonality of operations (quasigroups). A system of n-ary operations $A_1, A_2, ..., A_n$, defined on a set Q, is called orthogonal if the set of equations $\{A_i(x_1, x_2, ..., x_n) = a_i\}_{i=\overline{1,n}}$ has a unique solution for every $a_1, a_2, ..., a_n \in Q$. A system of s n-ary operations defined on a set Q, where $s \geq n$, is orthogonal if every n its operations are orthogonal. The orthogonality of n-ary operations (quasigroups, latin hypercubes) was considered from algebraic and/ or combinatorial point of view by many authors [1].

An n-ary groupoid (Q, A) is an n-ary quasigroup if in the equality $A(x_1, x_2, \ldots, x_n) = x_{n+1}$ every element of $\{x_1, x_2, \ldots, x_{n+1}\}$ is uniquely determined by the remaining n. The operation denoted by

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${}^{\sigma}A$, where $\sigma \in S_{n+1}$ and A is an n-ary quasigroup operation, defined by the equivalence

$${}^{\sigma}A(x_{\sigma 1}, x_{\sigma 2}, ..., x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, ..., x_n) = x_{n+1},$$

is called a parastrophe of A. If $\sigma(n+1) = n+1$, than ${}^{\sigma}A$ is called a principal parastrophe. An n-ary quasigroup is parastrophic-orthogonal (self-orthogonal) if there exist n orthogonal its parastrophes (principal parastrophes). It is known that there exist self-orthogonal latin squares of any order $q \neq 1, 2, 3, 6$ [2]. The complete characterization of the spectrum of parastrophic-orthogonal, in particular self-orthogonal quasigroups, in the n-ary case is still opened [3-7]. We consider new constructions of self-orthogonal finite n-ary quasigroups and give some estimations of their spectrum.

Let $A_1, A_2, ..., A_n$ be n-ary operations ($n \geq 2$), defined on a set Q. It follows from the definition that the n-ary operations $A_1, A_2, ..., A_n$ are orthogonal if and only if the mapping $\theta : Q^n \mapsto Q^n, \theta(x_1^n) = (A_1(x_1^n), A_2(x_1^n), ..., A_n(x_1^n))$, is a bijection, where by (x_1^n) we denoted the sequence $(x_1, x_2, ..., x_n)$. If Q is a finite set then the operations $A_1, A_2, ..., A_n$ are orthogonal if and only if the solution of the system of equations $\{A_i(x_1, x_2, ..., x_n) = a_i\}_{i=\overline{1,n}}$ is unique for each $a_1, a_2, ..., a_n \in Q$ (which is equivalent to the injectivity of θ).

Proposition 1. *If (Q, A) is a finite self-orthogonal n-quasigroup and ${}^{\alpha_1}A$, ${}^{\alpha_2}A$, ..., ${}^{\alpha_n}A$ is an orthogonal system of its principal parastrophes then the n^2-ary groupoid (Q, B_1), where $B_1(x_1^{n^2}) = {}^{\alpha_1}A({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2}))$ is a self-orthogonal quasigroup.*

Proof. Let (Q, A) be a finite self-orthogonal n-quasigroup and let ${}^{\alpha_1}A$, ${}^{\alpha_2}A$, ..., ${}^{\alpha_n}A$ be an orthogonal system of its principal parastrophes. We consider the n^2-ary operations $B_1, B_2, \ldots, B_{n^2}$ defined below:

$$\begin{cases} B_1(x_1^{n^2}) = {}^{\alpha_1}A({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2})), \\ \ldots \quad \ldots \quad \ldots \\ B_n(x_1^{n^2}) = {}^{\alpha_n}A({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2})), \\ B_{n+1}(x_1^{n^2}) = {}^{\alpha_1}A({}^{\alpha_2}A(x_1^n), {}^{\alpha_2}A(x_{n+1}^{2n}), ..., {}^{\alpha_2}A(x_{n(n-1)+1}^{n^2})), \\ \ldots \quad \ldots \quad \ldots \\ B_{n^2}(x_1^{n^2}) = {}^{\alpha_n}A({}^{\alpha_n}A(x_1^n), {}^{\alpha_n}A(x_{n+1}^{2n}), ..., {}^{\alpha_n}A(x_{n(n-1)+1}^{n^2})). \end{cases}$$

The operations B_i, $\overline{1, n}$ are principal parastrophes of the quasigroup (Q, B_1). Indeed, as ${}^{\alpha_1}A$, ${}^{\alpha_2}A$, ..., ${}^{\alpha_n}A$ are principal parastrophes of A, we get that each of the operations B_1, B_2, ..., B_{n^2} is a principal parastrophe of the quasigroup operation

$$D(x_1^{n^2}) = A(A(x_1^n),\, A(x_{n+1}^{2n}, ...,\, A(x_{n(n-1)+1}^{n^2}).$$

Using the transitivity of the relation of parastrophy, we obtain that each of the operations B_i, $i = \overline{1, n^2}$ is a principal parastrophe of (Q, B_1).

As Q is a finite set, the system $\{B_1, B_2, ..., B_{n^2}\}$ is orthogonal if the solution of the system of equations

$$B_i(x_1^{n^2}) = a_i,\ i = \overline{1, n^2}.$$

is unique. Let $\{B_i(x_1^{n^2}) = B_i(y_1^{n^2})\}$, $i = \overline{1, n^2}$, then

$$\begin{cases} {}^{\alpha_1}A({}^{\alpha_1}A(x_1^n), ..., {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2})) = {}^{\alpha_1}A({}^{\alpha_1}A(y_1^n), ..., {}^{\alpha_1}A(y_{n(n-1)+1}^{n^2})), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_n}A({}^{\alpha_1}A(x_1^n), ..., {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2})) = {}^{\alpha_n}A({}^{\alpha_1}A(y_1^n), ..., {}^{\alpha_1}A(y_{n(n-1)+1}^{n^2})), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_1}A({}^{\alpha_n}A(x_1^n), ..., {}^{\alpha_n}A(x_{n(n-1)+1}^{n^2})) = {}^{\alpha_1}A({}^{\alpha_n}A(y_1^n), ..., {}^{\alpha_n}A(y_{n(n-1)+1}^{n^2})), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_n}A({}^{\alpha_n}A(x_1^n), ..., {}^{\alpha_n}A(x_{n(n-1)+1}^{n^2})) = {}^{\alpha_n}A({}^{\alpha_n}A(y_1^n), ..., {}^{\alpha_n}A(y_{n(n-1)+1}^{n^2})). \end{cases}$$

As the system of parastrophes $\{{}^{\alpha_1}A, ..., {}^{\alpha_n}A\}$ is orthogonal we obtain

$$\begin{cases} {}^{\alpha_1}A(x_1^n) = {}^{\alpha_1}A(y_1^n), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_n}A(x_1^n) = {}^{\alpha_n}A(y_1^n), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_1}A(x_{n(n-1)+1}^{n^2}) = {}^{\alpha_1}A(y_{n(n-1)+1}^{n^2}), \\ \dots \quad \dots \quad \dots \\ {}^{\alpha_n}A(x_{n(n-1)+1}^{n^2}) = {}^{\alpha_n}A(y_{n(n-1)+1}^{n^2}) \end{cases} \Rightarrow \begin{cases} x_1 = y_1, \\ \dots \quad \dots \quad \dots \\ x_n = y_n, \\ \dots \quad \dots \quad \dots \\ x_{n(n-1)+1} = y_{n(n-1)+1}, \\ \dots \quad \dots \quad \dots \\ x_{n^2} = y_{n^2} \end{cases}$$

hence $\{B_1, B_2, ..., B_{n^2}\}$ is orthogonal, i.e. (Q, B_1) is a n^2-ary self-orthogonal quasigroup. □

Example 1. The ternary quasigroup $(\mathbb{Z}_p, A)$, where $A(x_1, x_2, x_3) = x_1 + x_2 - x_3$ and p is a prime number, is self-orthogonal and A, ${}^{(13)}A$, ${}^{(23)}A$ is

an orthogonal system of its principal parastrophes. According to Proposition 1, the 9-ary quasigroup $(\mathbb{Z}_p, B_1)$, where $B_1 = A(A(x_1^3), A(x_4^6), A(x_7^9))$ is self-orthogonal. An orthogonal system of principal parastrophes of $(\mathbb{Z}_p, B_1)$ is given by B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, where

$$
\begin{array}{l}
B_2(x_1^9) = {}^{(13)}A(A(x_1^3), A(x_4^6), A(x_7^9)) = {}^{(17)(28)(39)}B_1(x_1^9);\\
B_3(x_1^9) = {}^{(23)}A(A(x_1^3), A(x_4^6), A(x_7^9)) = {}^{(47)(58)(69)}B_1(x_1^9);\\
B_4(x_1^9) = A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(13)(46)(79)}B_1(x_1^9);\\
B_5(x_1^9) = {}^{(13)}A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(19)(28)(37)(46)}B_1(x_1^9);\\
B_6(x_1^9) = {}^{(23)}A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(13)(49)(58)(67)}B_1(x_1^9);\\
B_7(x_1^9) = A({}^{(23)}A(x_2^3), {}^{(23)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(23)(56)(89)}B_1(x_1^9);\\
B_8(x_1^9) = {}^{(13)}A({}^{(23)}A(x_1^3), {}^{(23)}A(x_4^6), {}^{(23)}A(x_7^9)) = {}^{(17)(29)(38)(56)}B_1(x_1^9);\\
B_9(x_1^9) = {}^{(23)}A({}^{(23)}A(x_1^3), {}^{(23)}A(x_4^6), {}^{(23)}A(x_7^9)) = {}^{(23)(47)(68)}B_1(x_1^9).
\end{array}
$$

Proposition 2. *Let (Q, A) and (Q, B) be finite n-ary, and respectively m-ary, self-orthogonal quasigroups. If $\{{}^{\alpha_1}A, {}^{\alpha_2}A, ..., {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B, {}^{\beta_2}B, ..., {}^{\beta_m}B\}$ are orthogonal systems of principal parastrophes then (Q, C_1), where*

$$C_1(x_1^{mn}) = {}^{\beta_1}B({}^{\alpha_1}A(x_1^n), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn}))$$

is a self-orthogonal nm-ary quasigroup.

Proof. Let (Q, A) and (Q, B) be finite n-ary, and respectively m-ary, self-orthogonal quasigroups and let $\{{}^{\alpha_1}A, {}^{\alpha_2}A, ..., {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B, {}^{\beta_2}B, ..., {}^{\beta_m}B\}$ be orthogonal systems of their principal parastrophes. We consider the following nm-ary operations:

$$
\left\{
\begin{array}{rcl}
C_1(x_1^{mn}) & = & {}^{\beta_1}B({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})),\\
... \quad ... & & ...\\
C_m(x_1^{mn}) & = & {}^{\beta_m}B({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})),\\
... \quad ... & & ...\\
C_{n(m-1)+1}(x_1^{mn}) & = & {}^{\beta_1}B({}^{\alpha_n}A(x_1^n), {}^{\alpha_n}A(x_{n+1}^{2n}), ..., {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})),\\
... \quad ... & & ...\\
C_{mn}(x_1^{mn}) & = & {}^{\beta_m}B({}^{\alpha_n}A(x_1^n), {}^{\alpha_n}A(x_{n+1}^{2n}), ..., {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})).
\end{array}
\right.
$$

Remark that the system $\{C_1, C_2, ..., C_{mn}\}$ is orthogonal. Indeed, as Q is a finite set, it is sufficient to show that, for each $c_1, c_2, ..., c_{nm} \in Q$, the solution of the system of equations

$$C_i(x_1^{mn}) = c_i,\ i = \overline{1,\ mn}$$

is unique. Let $\{C_i(x_1^{mn}) = C_i(y_1^{mn})\}$, $i = \overline{1,\ mn}$, then

$$\begin{cases} {}^{\beta_1}B({}^{\alpha_1}A(x_1^n),\ ...,\ {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})) &=& {}^{\beta_1}B({}^{\alpha_1}A(y_1^n),\ ...,\ {}^{\alpha_1}A(y_{n(m-1)+1}^{mn})), \\ \dots & \dots & \dots \\ {}^{\beta_n}B({}^{\alpha_1}A(x_1^n),\ ...,\ {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})) &=& {}^{\beta_n}B({}^{\alpha_1}A(y_1^n),\ ...,\ {}^{\alpha_1}A(y_{n(m-1)+1}^{mn})), \\ \dots & \dots & \dots \\ {}^{\beta_1}B({}^{\alpha_n}A(x_1^n),\ ...,\ {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})) &=& {}^{\beta_1}B({}^{\alpha_n}A(y_1^n),\ ...,\ {}^{\alpha_n}A(y_{n(m-1)+1}^{mn})), \\ \dots & \dots & \dots \\ {}^{\beta_n}B({}^{\alpha_n}A(x_1^n),\ ...,\ {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})) &=& {}^{\beta_n}B({}^{\alpha_n}A(y_1^n),\ ...,\ {}^{\alpha_n}A(y_{n(m-1)+1}^{mn})). \end{cases}$$

As the system of principal parastrophes $\{{}^{\beta_1}B,\ ...,\ {}^{\beta_m}B\}$, is orthogonal, we get that

$$\begin{cases} {}^{\alpha_1}A(x_1^n) &=& {}^{\alpha_1}A(y_1^n), \\ \dots & \dots & \dots \\ {}^{\alpha_n}A(x_1^n) &=& {}^{\alpha_n}A(y_1^n), \\ \dots & \dots & \dots \\ {}^{\alpha_1}A(x_{n(m-1)+1}^{mn}) &=& {}^{\alpha_1}A(y_{n(m-1)+1}^{mn}), \\ \dots & \dots & \dots \\ {}^{\alpha_n}A(x_{n(m-1)+1}^{mn}) &=& {}^{\alpha_n}A(y_{n(m-1)+1}^{mn}). \end{cases}$$

Now, using the fact that the system $\{{}^{\alpha_1}A,\ ...,\ {}^{\alpha_n}A\}$, is orthogonal, we obtain $x_i = y_i$, $i = \overline{1,\ mn}$, i.e. the system $\{C_1,\ C_2,\ ...,\ C_{mn}\}$ is orthogonal.

As $\{{}^{\alpha_1}A,\ ...,\ {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B,\ ...,\ {}^{\beta_m}B\}$ are principal parastrophes of A, and respectively B, we have that each of the operations $\{C_1,\ C_2,\ ...,\ C_{mn}\}$ is a principal parastrophe of D, where $D(x_1^{mn}) = B\left(A(x_1^n),\ ...,\ A(x_{n(m-1)+1}^{mn})\right)$, so each of them is a principal parastrophe C_1, i.e. $(Q,\ C_1)$ is a self-orthogonal mn-ary quasigroup. □

Corollary 1. *If there exist n-ary self-orthogonal quasigroups of order q then there exist n^k-ary self-orthogonal quasigroups of order q, for every $k \geq 2$.*

Corollary 2. *There exist 2^k-ary self-orthogonal quasigroups of any order $q \neq 1, 2,\ 3,\ 6\}$, for every $k \geq 1$.*

Corollary 2 follows from Proposition 2 and the fact that there exist self-orthogonal latin squares of any order $q \neq 1, 2, 3, 6$. [2]

Example 2. *The set of all principal parastrofes of the n-ary quasigroup $(\mathbb{Z}_q,\ A)$, where $A(x_1^n) = 2x_1 + x_2 + ... + x_n$,* is $A, {}^{(12)}A, ..., {}^{(1n)}A$. Moreover,

this system of principal parastrophes is orthogonal if and only if $n+1 \not\equiv 0 \pmod{q}$. Hence, there exist self-orthogonal n-ary quasigroups of order $q \geq 3$ for any $n \geq 2$ such that $n+1 \not\equiv 0 \pmod{q}$.

Corollary 3. *There exist self-orthogonal p^k-ary quasigroups of order p for every odd prime p and every $k \geq 1$.*

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