Center Problem for Cubic Differential Systems with Two Affine Invariant Straight Lines of Total Transversal Multiplicity Tree

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Abstract

In this paper, using the first five Lyapunov quantities, we solve the center problem for cubic differential systems with two affine invariant straight lines one of which has transversal algebraic multiplicity two.

Keywords: cubic differential system, the problem of the center, multiple invariant straight line.

1 Introduction

We consider the real cubic differential systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \gcd(P,Q) = 1, \end{cases} \quad (1)$$

and the vector fields $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$.

The *straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_{\mathcal{L}} \in \mathbb{C}[x,y]$ such that the identity $\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_{\mathcal{L}}(x,y),\ (x,y) \in \mathbb{R}^2$, i.e., $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x,y)K_{\mathcal{L}}(x,y),\ (x,y) \in \mathbb{R}^2$, holds.

If $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$) is the greatest natural number such that $\mathcal{L}^{m_p(\mathcal{L})}$ (respectively, $\mathcal{L}^{m_a(\mathcal{L})}$) divides $\mathbb{X}(\mathcal{L})$ (respectively, $E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$), then we say that $\mathcal{L}$ has *parallel multiplicity*

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(*algebraic multiplicity*, or in brief, *multiplicity*) $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$). We have $1 \leq m_p(\mathcal{L}) \leq 3$, $1 \leq m_a(\mathcal{L}) \leq 7$ and $m_p(\mathcal{L}) \leq m_a(\mathcal{L})$. If $m_a(\mathcal{L}) > m_p(\mathcal{L})$, then we say that $\mathcal{L}$ has *transversally multiplicity* $m_t(\mathcal{L}) = m_a(\mathcal{L}) - m_p(\mathcal{L}) + 1$.

Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic systems with parallel multiple invariant straight lines (with the line at infinity and an affine real invariant straight line of total multiplicity four) were investigated in [3], [5] (in [6]).

Let $\mathcal{L}$ to be an invariant straight line for (1). A function $f = \exp\left(\frac{g}{\mathcal{L}^j}\right)$, $1 \leq j \leq m_a(\mathcal{L}) - 1$, $g \in \mathbb{C}[x, y]$, $deg(g) \leq j$, is called *exponential factor*) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K_f) \leq 2$ such that the identity $\mathbb{X}(f) \equiv f \cdot K_f(x, y)$ holds.

The critical point $(0, 0)$ of the system (1) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center.*

A critical point $(0, 0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral $F(x, y)$ (integrating factor $\mu(x, y)$) in a neighborhood of $(0, 0)$. Also, $(0, 0)$ is a center iff all Lyapunov quantities L_j vanish (see, [3]). If $F(x, y)$ $(\mu(x, y))$ has the form $f^{\alpha_1} \cdots f^{\alpha_\sigma}$, where f_j, $1 \leq j \leq \delta$ are invariant straight lines and f_j, $\delta \leq j \leq \sigma$ are exponential factors, then the system (1) is called Darboux integrable.

2 Condition of existence invariant straight lines

Denote $\kappa(x, y) = x(sx^3 + (q - a)x^2y + (n - c - 1)xy^2 + (l - f)y^3)$.

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be invariant straight lines for (1) and $m_a(\mathcal{L}_1) \geq 2$. The case $m_p(\mathcal{L}_1) \geq 2$ was investigated in [2].

Without lose of generality, we consider $\mathcal{L}_1 \equiv x - 1$. Suppose that $m_t(\mathcal{L}_1) \geq 2$. Then x^2 divides $\kappa(x, y)$. The direct calculus show that the straight lines $\mathcal{L}_1 \equiv x - 1 = 0$, $\mathcal{L}_2 \equiv \alpha x - y + \beta = 0, \beta \neq 0$ are invariant for (1) and $m_t(\mathcal{L}_1 \geq 2$ if and only if at least one of the following three

series of conditions hold

$$\begin{array}{l} b=-(\alpha+f\alpha\beta+f\beta^2)/\beta, d=((\alpha+\beta)(\alpha-2\beta-c\beta)-1)/\beta, \\ m=-c-1, g=(\alpha-\beta-a\alpha\beta-a\beta^2)/\beta, k=-a, l=f, \\ n=(\alpha+2\beta+c\beta)/\beta, p=-f, q=(1-\alpha^2+a\beta-\alpha\beta)/\beta, \\ r=0, s=-\alpha/\beta; \end{array} \tag{2}$$

$$\begin{array}{l} a=-(\alpha+\beta)(2+c+f\alpha+f\beta), b=-(\alpha+f\alpha\beta+f\beta^2)/\beta, \\ r=0, g=(2f\alpha+(c+2)(1+d\beta)+(\alpha+\beta)(df\beta-(c+2)\cdot \\ (\alpha-2\beta-c\beta))-f(\alpha+\beta)^2(\alpha-4\beta-2c\beta)+f^2\beta(\alpha+\beta)^3)/ \\ (f\beta), k=-a, l=f, n=-(1-\alpha^2+d\beta+c\alpha\beta)/\beta^2, p=-f, \\ q=-(df\beta^2+2\beta(d+\alpha+2\beta)+(c+2)(1-\alpha^2)+c\beta(d+3\alpha+ \\ 4\beta)+\beta(\alpha+\beta)(c^2+4f\beta+2cf\beta)+f^2\beta^2(\alpha+\beta)^2)/(f\beta^2), \\ s=\alpha(f\alpha+2d\beta+2\beta(\alpha+2\beta)+(c+2)(1-\alpha^2)+c\beta(d+3\alpha+ \\ 4\beta)-(\alpha+\beta)(f(\alpha-2\beta)(\alpha+\beta)-\beta(c^2+df))+ \\ cf\beta(\alpha+\beta)^2)/(f\beta^2), m=-c-1; \end{array} \tag{3}$$

$$\begin{array}{l} a=-(2+c)(\alpha+\beta), d=((\alpha+\beta)(\alpha-2\beta-c\beta)-1)/\beta, f=0, \\ b=-\alpha/\beta, g=(\alpha+\alpha\beta(c+1)(\alpha+\beta)-q\beta^2)/\beta, k=-a, l=0, \\ m=-c-1, r=0, n=(\alpha+2\beta+c\beta)/\beta, p=-f, \\ s=-(\alpha(q\beta+(\alpha+\beta)(\alpha+2\beta+c\beta)))/\beta. \end{array} \tag{4}$$

3 Problem of the center

In each of the conditions (2) and (3) we calculate the first five Lyapunov quantities $L_1, ..., L_5$. Solving the system $\{L_1 = 0, L_2 = 0, ..., L_5 = 0\}$, we obtain for (1) all the cases of the center in the sets of conditions (2) and (3). Under the conditions (4) for solving the center problem, it is necessary and sufficient as the first three Lyapunov quantities to vanish. In this way, the following Theorem is valid.

Theorem 1. *The cubic system* (1) *with two affine invariant straight lines of total transversal multiplicity three has at origin* $(0, 0)$ *center if and only if the first five Lyapunov quantities vanish* $L_1 = 0, L_2 = 0, ..., L_5 = 0$.

The system (1) in one of the cases of the existence of the center is reversible, and in the other cases it is Darboux integrable.

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