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POISSON STABLE MOTIONS OF MONOTONE AND STRONGLY SUB-LINEAR NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. This paper is dedicated to the study of the problem of existence of Poisson stable (Bohr/Levitan almost periodic, almost automorphic, almost recurrent, recurrent, pseudo periodic, pseudo recurrent and Poisson stable) motions of monotone sub-linear non-autonomous dynamical systems. The main results we establish in the framework of general non-autonomous (cocycle) dynamical systems.

We apply our general results to the study of the problem of existence of different classes Poisson stable solutions of some types of non-autonomous evolutionary equations (Ordinary Differential Equations, Functional-Differential Equations with finite delay and Difference Equations).

1. **Introduction.** The aim of this paper is to study the problem of existence of Poisson stable motions of two important classes of monotone non-autonomous dynamical systems. Namely

1. *strongly monotone* dynamical systems and
2. *sub-linear* (or *sub-homogeneous*) systems.

The classes of monotonic systems mentioned above have long attracted the attention of researchers. The research of these systems is devoted to the publications of many authors (see, for example, [6]-[20], [24], [26], [27, 28], [34]-[37], [47], [50], [53] and the references therein). Works of D. Cheban [6]-[10], D. Cheban and Z. Liu [11], J. Jiang and X.-Q. Zhao [26], W. Shen and Y. Yi [47], Y. Wang and X.-Q. Zhao [52], X.-Q. Zhao [53] and others are devoted to the problem of almost periodicity and almost automorphism of movements in such systems.

In this paper we study the problem of Poisson stability (in particular, periodicity, Bohr/Levitan almost periodicity, almost automorphism, almost recurrence, recurrence, pseudo periodicity, pseudo recurrence, Poisson stability) for monotone and strongly sub-linear non-autonomous dynamical systems.

The paper is organized as follows.

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In Section 2 we collected some notions and facts from the theory of dynamical systems (the both autonomous and non-autonomous) which we use in this paper: cocycles, skew-product dynamical systems, shift (Bebutov's) dynamical systems, Poisson stable (Bohr/Levitan almost periodic, almost automorphic and so on) motions and functions and their comparability by character of recurrence, monotone non-autonomous dynamical systems, global attractors of cocycle etc.

Section 3 is dedicated to the study the problem of existence and convergence to Poisson stable motions of monotone non-autonomous dynamical systems.

In Section 4 we study the uniformly stable monotone non-autonomous dynamical systems. The main result of Section 4 is contained in Theorem 4.4 which gives sufficient conditions for the existence of strongly comparable motions by character of recurrence with point on the base (driving) dynamical system.

Section 5 is dedicated to the study of the sub-linear non-autonomous dynamical systems. We establish in this Section some important properties of sub-linear systems.

In Section 6 we study the problem of existence of comparable by character of recurrence motions for sub-linear cocycles. The main result of Section 6 (Theorems 6.5 and 6.9) gives sufficient conditions on existence of comparable (respectively, strongly comparable) motions for sub-linear cocycles.

Section 7 is dedicated to the applications of our general results obtained in Sections 3-6 to the study of the problem of existence of different classes Poisson stable solutions of some types of non-autonomous evolutionary equations (Ordinary Differential Equations, Functional-Differential Equations with finite delay and Difference Equations).

2. **Preliminaries.**

In this section we collect some notions and facts for non-autonomous dynamical systems which we will use below; the reader may refer to [13], [16, Ch. IX], [39] for details.

Throughout the paper, we assume that X and Y are metric spaces and for simplicity we use the same notation ρ to denote the metrics on them, which we think would not lead to confusion. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{S}_+ := \{s \in \mathbb{S} |\ s \geq 0\}$, $\mathbb{S}_- := \{s \in \mathbb{S} |\ s \leq 0\}$ and $\mathbb{T} \subseteq \mathbb{S}$ be a subsemigroup of $\mathbb{S}$ such that $\mathbb{S}_+ \subseteq \mathbb{T}$. For given dynamical system $(X, \mathbb{T}, \pi)$ and given point $x \in X$, we denote by Σ_x (respectively, Σ_x^+) its *trajectory* (respectively, *semi-trajectory*), i.e. $\Sigma_x := \{\pi(t, x) : t \in \mathbb{T}\}$ (respectively, $\Sigma_x^+ := \{\pi(t, x) : t \in \mathbb{T}_+\}$), and call the mapping $\pi(\cdot, x) : \mathbb{T} \to X$ the *motion* through x at the moment $t = 0$. For given set $A \subseteq X$, we denote $\Sigma_A := \{\pi(t, x) : t \in \mathbb{T}, x \in A\}$; Σ_A^+ is defined similarly. We denote the *hull* (respectively, *semi-hull*) of a point x by $H(x) := \overline{\Sigma}_x$ (respectively, $H^+(x) := \overline{\Sigma}_x^+$), where by bar we mean closure. A point $x \in X$ is called *Lagrange stable*, "st. L" in short, (respectively, *positively Lagrange stable*, "st. L^+" in short) if $H(x)$ (respectively, $H^+(x)$) is compact.

2.1. **Cocycles and non-autonomous dynamical systems.**

Let $(Y, \mathbb{T}, \sigma)$ be a dynamical system on Y and E a metric space.

Definition 2.1. A triplet $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ (or briefly φ if no confusion) is said to be a *cocycle* on state space (or fibre) E with base $(Y, \mathbb{T}, \sigma)$ if the mapping $\varphi : \mathbb{T}_+ \times E \times Y \to E$ satisfies the following conditions:

1. $\varphi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
2. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}_+, u \in E$ and $y \in Y$;

3. the mapping φ is continuous.

Definition 2.2. Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle on E, $X := E \times Y$ and π be a mapping from $\mathbb{T}_+ \times X$ to X defined by $\pi := (\varphi, \sigma)$, i.e. $\pi(t, (u, y)) = (\varphi(t, u, y), \sigma(t, y))$ for all $t \in \mathbb{T}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{T}_+, \pi)$ is an autonomous dynamical system and called *skew-product dynamical system*.

Definition 2.3. Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of the group $\mathbb{S}$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e. $h(\pi(t, x)) = \sigma(t, h(x))$ for all $t \in \mathbb{T}_1$ and $x \in X$, and h is continuous and surjective), then the triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is called a *non-autonomous dynamical system* (NDS).

Example 1. *(*The non-autonomous dynamical system generated by cocycle φ*.)* An important class of NDS are generated from cocycles. Indeed, let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle, $(X, \mathbb{T}_+, \pi)$ be the associated skew-product dynamical system $(X = E \times Y, \pi = (\varphi, \sigma))$ and $h = pr_2 : X \to Y$ (the natural projection mapping), then the triplet $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is an NDS.

2.2. **Conditional compactness.** Lagrange stable (or called "compact") motions have been studied comprehensively, but it is not the case for non-Lagrange stable motions. The following concept of conditional compactness introduced in [13] is important for our study of noncompact motions.

Definition 2.4. Recall [25] that a *bundle* (or fiber space) is a triplet (X, h, Y), where X, Y are topological spaces and $h : X \to Y$ is a continuous surjective mapping. The space Y is called the *base space*, the space X is called the *total space*, and the map h is called the *projection* of bundle. For each $y \in Y$, the space $X_y := h^{-1}(y)$ is called the *fiber* of bundle over $y \in Y$.

Example 2. Let $X := W \times Y$. A triplet (X, h, Y), where $h := pr_2$ is the projection on the second factor, is a bundle which is called the product bundle over Y with fiber W.

Definition 2.5. Let (X, h, Y) be a fiber space. A set $M \subseteq X$ is said to be *conditionally pre-compact* if its intersection with the pre-image of any pre-compact subset $Y' \subseteq Y$, i.e., the set $h^{-1}(Y') \bigcap M$, is a pre-compact subset of X. A set M is called *conditionally compact* if it is closed and conditionally pre-compact.

Remark 2.6. 1. Let K be a compact space, Y is a noncompact metric space, $X := K \times Y$ and $h = pr_2 : X \to Y$. Then the triplet (X, h, Y) is a fiber space. The space X is conditionally compact, but it is not compact.

2. If Y is a compact set and $M \subseteq X$ is conditionally pre-compact, then M is a pre-compact set.

The following result provides a useful criterion for conditional compactness in applications.

Lemma 2.7 ([5])**.** *Let $\langle E, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ be a cocycle and $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be the NDS generated by the cocycle φ. Assume that $x_0 := (u_0, y_0) \in X = E \times Y$ and the set $Q^+_{(u_0,y_0)} := \overline{\{\varphi(t, u_0, y_0) : t \in \mathbb{T}_+\}}$ is compact. Then the semi-hull $H^+(x_0)$ is conditionally compact.*

Denote by $C(\mathbb{T})$ the family of all continuous functions $f : \mathbb{T} \to \mathbb{R}$ equipped with the compact-open topology. This topology can be generated by Bebutov distance

(see, e.g. [2], [48, ChIV])

$$d(f,g) := \sup_{l>0} \min\{\max_{|t|\leq l} |f(t)-g(t)|, 1/l\}.$$

Denote by $(C(\mathbb{T}), \mathbb{T}, \sigma)$ the shift dynamical system (or called Bebutov dynamical system), i.e., $\sigma(\tau, f) := f^\tau$, where $f^\tau(t) := f(t+\tau)$ for $t \in \mathbb{T}$. Note that the function $f \in C(\mathbb{T})$ is st. L^+ (respectively, st. L) if and only if the function f is bounded and uniformly continuous on $\mathbb{T}_+$ (respectively, on $\mathbb{T}$) (see, e.g. [48, ChIV]).

2.3. **Some general facts about non-autonomous dynamical systems.** In this subsection we recall some general facts about NDS, see [13] or [16, Ch. IX] for details.

Definition 2.8. A point $y \in Y$ is called *positively Poisson stable* if there exists a sequence $t_n \to +\infty$ such that $\sigma(t_n, y) \to y$ as $n \to \infty$.

Definition 2.9. Let $(Y, \mathbb{S}, \sigma)$ be a two-sided dynamical system. A point $y \in Y$ is said to be *negatively Poisson stable* if there exists a sequence $t_n \to -\infty$ such that $\sigma(t_n, y) \to y$ as $n \to \infty$. If y is Poisson stable in both directions, it is called *Poisson stable*.

Denote $\mathfrak{N}_y := \{\{t_n\} \subset \mathbb{T} : \sigma(t_n, y) \to y\}$, $\mathfrak{N}_y^{+\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to +\infty\}$. If $\mathbb{T} = \mathbb{S}$, then we put $\mathfrak{N}_y^{-\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to -\infty\}$ and $\mathfrak{N}_y^{\infty} := \{\{t_n\} \in \mathfrak{N}_y : t_n \to \infty\}$.

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be an NDS and $y \in Y$ be positively Poisson stable. Denote

$$\mathcal{E}_y^+ := \{\xi : \exists \{t_n\} \in \mathfrak{N}_y^{+\infty} \text{ such that } \pi(t_n, \cdot)|_{X_y} \to \xi\}, \tag{1}$$

where $X_y := h^{-1}(y) = \{x \in X : h(x) = y\}$ and $\to$ means the pointwise convergence. If the NDS is two-sided, y is negatively Poisson stable or Poisson stable and we replace $\mathfrak{N}_y^{+\infty}$ by $\mathfrak{N}_y^{-\infty}$ or $\mathfrak{N}_y$ in (1), then we get the definition of $\mathcal{E}_y^-$ or $\mathcal{E}_y$.

Let X^X denote the Cartesian product of X copies of the space X, equipped with product topology. The set X^X can be endowed with a semigroup structure with respect to composition of mappings from X^X (for more details, see e.g. [3, ChI] and [22]).

Definition 2.10. Let $(Y, \mathbb{T}, \sigma)$ be a semi-group dynamical system. A continuous mapping $\gamma : \mathbb{S} \to Y$ is called an *entire trajectory* of dynamical system $(Y, \mathbb{T}, \sigma)$ passing through the point $y \in Y$ at $t = 0$ if $\gamma(0) = y$ and $\sigma(t, \gamma(s)) = \gamma(t+s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

Denote by

- $C(\mathbb{T}, X)$ the space of all continuous functions $f : \mathbb{T} \to X$ equipped with the compact-open topology;
- Φ_y the family of all entire trajectories of $(Y, \mathbb{T}, \sigma)$ passing through the point $y \in Y$ at the initial moment $t = 0$ and $\Phi(Y) := \bigcup\{\Phi_y : y \in Y\}$.

Definition 2.11. Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{T}, \sigma)$ with the fiber E. A continuous mapping $\nu : \mathbb{S} \to E$ is called an entire trajectory of cocycle φ passing through the point $(u, y) \in E \times Y$ at $t = 0$ if $\varphi(t, \nu(s), \sigma(s, y))) = \nu(t+s)$ for any $t \in \mathbb{T}$ and $s \in \mathbb{S}$.

Remark 2.12. Let $\langle E, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle over $(Y, \mathbb{S}, \sigma)$ with the fiber E and $(X, \mathbb{S}_+, \pi)$ be the corresponding skew-product dynamical system ($X := E \times Y$ and $\pi := (\varphi, \sigma)$), then the following statements hold:

1. if $x = (u, y) \in X$ and ν_u is an entire trajectory of cocycle φ passing through the point $(u, y) \in X$ at the initial moment $t = 0$, then $\gamma_x := (\nu, Id_Y)$ is an entire trajectory of skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ passing through the point $x = (u, y) \in X$ at the initial moment $t = 0$, where Id_Y is the identity mapping on the space Y;
2. if $x = (u, y) \in X$ and $\alpha_x \in \Phi(X)$ is an entire trajectory of skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ passing through the point $x = (u, y) \in X$ at the initial moment $t = 0$, then $\nu(s) := pr_1\alpha_x(s)$ for any $s \in \mathbb{S}$ is an entire trajectory of the cocycle φ passing through the point (u, y) at the initial moment $t = 0$.

Remark 2.13. Note that:

1. if $\gamma \in \Phi_x$ then $\gamma^\tau \in \Phi_{\gamma(\tau)}$, where $\gamma^\tau(t) := \gamma(t+\tau)$ for $t \in \mathbb{S}$, and consequently Φ is a translation invariant subset of $C(\mathbb{S}, X)$;
2. if $\gamma_n \in \Phi_{x_n}$ and $\gamma_n \to \gamma$ in $C(\mathbb{S}, X)$ as $n \to \infty$, then $\gamma \in \Phi_x$ with $x := \lim\limits_{n\to\infty} x_n$ and consequently Φ is a closed subset of $C(\mathbb{S}, X)$.

Similar to the shift dynamical system $(C(\mathbb{T}), \mathbb{T}, \sigma)$ in Subsection 2.2, let $\big(C(\mathbb{S}, X), \mathbb{S}, \lambda\big)$ be the shift dynamical system (or Bebutov dynamical system, see e.g. [3, 16, 39, 45]) on the space $C(\mathbb{S}, X)$. By Remark 2.13 Φ is a closed and invariant (with respect to shifts) subset of $C(\mathbb{S}, X)$, and consequently on Φ is defined a shift dynamical system $(\Phi, \mathbb{S}, \lambda)$ induced from $\big(C(\mathbb{S}, X), \mathbb{S}, \lambda\big)$.

If $K \in 2^X$, then K^K denotes the collection of any maps from K to itself, provided with the product topology, or, what is the same thing, the topology of pointwise convergence. By Tychonoff's theorem, K^K is compact. K^K has a semigroup structure defined by the composition of maps.

Let $\mathcal{E} \subset X^X$ be a semigroup. *A right ideal* in $\mathcal{E}$ is a nonempty subset I such that $I\mathcal{E} \subseteq I$, where $I\mathcal{E} := \{\eta \circ \xi \ : \ \eta \in I, \xi \in \mathcal{E}\}$, and $\eta \circ \xi$ is the composition of η and ξ defined in the following way:

$$(\eta \circ \xi)(x) := \xi(\eta(x)), \quad \text{for} \quad x \in X.$$

Remark 2.14. It is worth noticing that we are using the original notation for the composition which was used in the works [1], [3] and [22]. Needless to say that this notation can be misunderstanding, and it would be possible to use the standard definition, but then we should change any terminology about right and left ideals, and the results already proved in the literature concerning these sets. For this reason, we prefer to keep the original notation (at least in the framework of this Subsection) and recommend the reader to be careful with this notation.

Definition 2.15. *A minimal right ideal* is one which does not properly contain a right ideal.

Lemma 2.16. [1, 3] *Every compact semigroup $\mathcal{E}$ with the continuous left multiplication contains at least one* idempotent element*, i.e., an element u with $u^2 = u$.*

Recall that the point $y \in Y$ is called (see, for example, [46] and [48]) positively (negatively) stable in the sense of Poisson if there exists a sequence $t_n \to +\infty$ ($t_n \to -\infty$ respectively) such that $\sigma^{t_n} y \to y$.

Lemma 2.17. [18, ChIII], [11] *Let $y \in Y$ be a positively Poisson stable point, $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a non-autonomous dynamical system and X be a conditionally compact space, then $\mathcal{E}_y^+$ is a nonempty compact sub-semigroup of the semigroup $X_y^{X_y}$ (w.r.t. composition of mappings).*

Lemma 2.18. [18, ChIII,p.159] *Suppose that the following conditions are fulfilled:*

1. *$y \in Y$ is a two-sided Poisson stable point;*
2. *$\langle (X, \mathbb{S}, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is a two-sided non-autonomous dynamical system;*
3. *X is a conditionally compact space;*
4. $$\inf_{t\leq 0} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$$
   for any $x_1, x_2 \in X_y \quad (x_1 \neq x_2)$.

Then the following statement hold:

1. *$\mathcal{E}_y^-$ and $\mathcal{E}_y^+$ are subgroups of the semigroup $\mathcal{E}_y$;*
2. *for any pair of points $x_1, x_2 \in X_y$ with $x_1 \neq x_2$ there are sequences $\{t_n^-\} \in \mathfrak{N}_y^{-\infty}$ and $\{t_n^+\} \in \mathfrak{N}_y^{+\infty}$ such that*
   $$\lim_{n\to\infty} \pi(t_n^{\pm}, x_i) = x_i \ (i = 1, 2);$$
3. $$\inf_{t\geq 0} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$$
   for any $x_1, x_2 \in X_y \quad (x_1 \neq x_2)$.

2.4. **Structure of the ω-limit set.** Let M be a subset of X. We denote the ω-limit set of M by

$$\omega(M) := \bigcap_{t\geq 0} \overline{\bigcup \{\pi(\tau, M) : \ \tau \geq t\}};$$

for a singleton set, for simplicity we also write $\omega(x)$ or ω_x for $\omega(\{x\})$. Note that $x \in \omega(M)$ if and only if there exists sequences $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{T}$ such that $t_n \to +\infty$ as $n \to \infty$ and $\lim_{n\to\infty} \pi(t_n, x_n) = x$.

Definition 2.19. Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be an NDS. A subset $A \subseteq X$ is said to be *(positively) uniformly stable* if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho(x, a) < \delta$ ($a \in A$, $x \in X$ and $h(a) = h(x)$) implies $\rho(\pi(t, x), \pi(t, a)) < \varepsilon$ for any $t \geq 0$. In particular, a point $x_0 \in X$ is called uniformly stable if the singleton set $\{x_0\}$ is so.

Remark 2.20. Let $A \subseteq X$ be uniformly stable and $B \subseteq A$, then B is also uniformly stable.

Lemma 2.21. [3, ChIV], [4] *If the set $A \subseteq X$ is uniformly stable and the mapping $h : X \to Y$ is open, then the closure $\overline{A}$ of A is uniformly stable.*

Corollary 2.22. *If $\Sigma_{x_0}^+$ is uniformly stable and h is open, then:*

1. *$H^+(x_0)$ is uniformly stable;*
2. *ω_{x_0} is uniformly stable, because $\omega_{x_0} \subseteq H^+(x_0)$.*

Remark 2.23. Note that if an NDS is generated by a skew-product dynamical system (or equivalently by a cocycle) in which case the homomorphism h is given by the natural projection mapping, then clearly h is open.

Theorem 2.24. [11] *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

1. *there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma_{x_0}^+$ is conditionally pre-compact;*
2. *$\omega_{x_0} \neq \emptyset$;*

3. the set ω_{x_0} is positively uniformly stable.

Then all motions on ω_{x_0} can be extended uniquely to the left and on ω_{x_0} is defined a two-sided dynamical system $(\omega_{x_0}, \mathbb{S}, \pi)$, i.e., the one-sided dynamical system $(X, \mathbb{S}_+, \pi)$ generates on ω_{x_0} a two-sided dynamical system $(\omega_{x_0}, \mathbb{S}, \pi)$.

Remark 2.25. Note that Theorem 2.24 is known if Y is a compact minimal set (see, e.g. [34] and references therein) or if each point of Y is Poisson stable (see [14]). In our case Y, generally speaking, can be non-compact and non-minimal, and there is no restriction on the element of Y.

2.5. Poisson stable motions and their comparability by character of recurrence.

2.5.1. *Classes of Poisson stable motions.* Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [39, 45, 46, 48] for details.

Definition 2.26. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t, x) = x$ (respectively, $\pi(t + \tau, x) = \pi(t, x)$) for all $t \in \mathbb{T}$.

Definition 2.27. A point $x \in X$ is called *quasi-periodic* with the base of frequency $\nu = (\nu_1, \nu_2, \dots, \nu_k)$ if the associated function $f(\cdot) := \pi(\cdot, x) : \mathbb{R} \to X$ satisfies the following conditions:

1. the numbers $\nu_1, \nu_2, \dots, \nu_k$ are rationally independent;
2. there exists a continuous function $\Phi : \mathbb{R}^k \to X$ such that
$$\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_k + 2\pi) = \Phi(t_1, t_2, \dots, t_k)$$
for all $(t_1, t_2, \dots, t_k) \in \mathbb{R}^k$;
3. $f(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_k t)$ for $t \in \mathbb{R}$.

Definition 2.28. For given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called a *ε-shift of x* (respectively, *ε-almost period of x*), if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

Definition 2.29. A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains a ε-shift (respectively, ε-almost period) of x.

Definition 2.30. If a point $x \in X$ is almost recurrent and its trajectory Σ_x is pre-compact, then x is called *(Birkhoff) recurrent*.

Definition 2.31. A point $x \in X$ is called *Levitan almost periodic* [32] (see also [3, 14, 31]), if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 2.32. A point $x \in X$ is called *almost automorphic* if it is st. L and Levitan almost periodic.

Definition 2.33. A point $x \in X$ is said to be *uniformly Poisson stable* or *pseudo periodic* in the positive (respectively, negative) direction if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists a ε-almost period $\tau > l$ (respectively, $\tau < -l$) of x. The point x is said to be uniformly Poisson stable or pseudo periodic if it is so in both directions.

Lemma 2.34. *If the point $x \in X$ is positively (respectively, negatively) pseudo periodic, then every point $p \in H(x)$ is positively (respectively, negatively) Poisson stable.*

Proof. Let $p \in H(x)$ be an arbitrary point, then there exists a sequence $\{\tau_k\} \subset \mathbb{T}$ such that

$$p = \lim_{k\to\infty} \pi(\tau_k, x). \tag{2}$$

Since the point $x \in X$ is positively (respectively, negatively) pseudo periodic, then there exists a sequence $t_n \to +\infty$ (respectively, $t_n \to -\infty$) as $n \to \infty$ such that for any positive number ε there exists a number $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\pi(t + t_n, x), \pi(t, x)) < \varepsilon$$

for all $t \in \mathbb{T}$ and $n \geq n_\varepsilon$. In particular, we have

$$\rho(\pi(t_n, \pi(\tau_k, x)), \pi(\tau_k, x)) = \rho(\pi(t_n + \tau_k, x), \pi(\tau_k, x)) < \varepsilon \tag{3}$$

for any $k \in \mathbb{N}$ and $n \geq n_\varepsilon$. Passing to the limit in (3) as $k \to \infty$ and taking into consideration (2) we obtain

$$\rho(\pi(t_n, p), p) \leq \varepsilon$$

for any $n \geq n_\varepsilon$, i.e., $p \in \omega_p$ (respectively, $p \in \alpha_p$). Lemma is proved. □

Definition 2.35 ([40, 41])**.** A point $x \in X$ is said to be *pseudo recurrent* if for any $\varepsilon > 0$, $p \in \Sigma_x$ and $t_0 \in \mathbb{T}$ there exists $L = L(\varepsilon, t_0) > 0$ such that

$$B(p, \varepsilon) \bigcap \pi([t_0, t_0 + L], p) \neq \emptyset,$$

where $B(p, \varepsilon) := \{x \in X : \rho(p, x) < \varepsilon\}$ and $\pi([t_0, t_0 + L], p) := \{\pi(t, p) : t \in [t_0, t_0 + L]\}$.

Definition 2.36. A point $x \in X$ is said to be [18, ChI] *strongly Poisson stable* (in the positive direction) if $p \in \omega_p$ for any $p \in H(x)$.

Remark 2.37. It is known that:

1. a strongly Poisson stable point is Poisson stable, but the converse is not true in general;
2. all the motions introduced above (Definitions 2.26–2.35) are strongly Poisson stable.

Definition 2.38 ([15, 44])**.** A point $x \in X$ is said to be asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically pseudo periodic, asymptotically pseudo recurrent, asymptotically Poisson stable) if there exists a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, pseudo periodic, pseudo recurrent, Poisson stable) point $p \in X$ such that

$$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, p)) = 0.$$

2.5.2. *Shcherbakov's comparability principle of motions by their character of recurrence.* In this subsection we present some notions and results stated and proved by B. A. Shcherbakov [45]–[46] (see also [18, Ch.I]).

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be two dynamical systems.

Definition 2.39. A point $x \in X$ is said to be *comparable with $y \in Y$ by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ-shift of y is a ε-shift for x, i.e., $\rho(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$.

Theorem 2.40. *Let $x \in X$ be comparable with $y \in Y$. If the point y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is the point x.*

Definition 2.41. A point $x \in X$ is called *uniformly comparable with $y \in Y$ by character of recurrence* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ-shift of $\sigma(t, y)$ is a ε-shift of $\pi(t, x)$ for all $t \in \mathbb{T}$, i.e. $\rho(\sigma(t+\tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t+\tau, x), x) < \varepsilon$ for all $t \in \mathbb{T}$ (or equivalently: $\rho(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{T}$).

Denote $\mathfrak{M}_x := \{\{t_n\} \subset \mathbb{T} : \{\pi(t_n, x)\} \text{ converges}\}$, $\mathfrak{M}_x^{+\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to +\infty \text{ as } n \to \infty\}$ and $\mathfrak{M}_x^{\infty} := \{\{t_n\} \in \mathfrak{M}_x : t_n \to \infty \text{ as } n \to \infty\}$.

Definition 2.42 ([12, 15])**.** A point $x \in X$ is said to be *strongly comparable with $y \in Y$ by character of recurrence* if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

Theorem 2.43. *Let X and Y be two complete metric spaces. Let the point $x \in X$ be uniformly comparable with $y \in Y$ by character of recurrence. If y is quasi-periodic (respectively, Bohr almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, pseudo periodic, pseudo recurrent), then so is x.*

3. **Existence and convergence to Poisson stable motions of monotone non-autonomous dynamical systems.** Assume that E is an ordered space. A subset U of E is called lower-bounded (respectively, upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (respectively, $a \geq U$). Such an a is said to be a lower bound (respectively, upper bound) for U. A lower bound α is said to be the *greatest lower bound* (g.l.b.) or *infimum*, if any other lower bound a satisfies $a \leq \alpha$. Similarly, we can define the *least upper bound* (l.u.b.) or *supremum*.

A bundle (X, h, Y) is said to be *ordered* if each fiber X_y is ordered. Note that only points on the same fiber may be order related: if $x_1 \leq x_2$ or $x_1 < x_2$, then it implies $h(x_1) = h(x_2)$. We assume that the order relation and the topology on X are compatible in the sense that $x \leq \tilde{x}$ if $x_n \leq \tilde{x}_n$ for all n and $x_n \to x$, $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$.

Definition 3.1. For given bundle (X, h, Y), an NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ defined on it is said to be *monotone* (respectively, *strictly monotone*) if $x_1 \leq x_2$ (respectively, $x_1 < x_2$) implies $\pi(t, x_1) \leq \pi(t, x_2)$ (respectively, $\pi(t, x_1) < \pi(t, x_2)$) for any $t > 0$.

For given NDS $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, let $\mathcal{S} \subseteq X$ be a nonempty closed ordered subset possessing the following properties:

1. $h(\mathcal{S}) = Y$;
2. $\mathcal{S}$ is positively invariant with respect to π, i.e., $\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is an NDS.

Below we will use the following assumptions:

(C1) For every conditionally compact subset K of $\mathcal{S}$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both infimum $\alpha_y(K)$ and supremum $\beta_y(K)$;

(C2) For every $x \in \mathcal{S}$, the semi-trajectory Σ_x^+ is conditionally pre-compact and its ω-limit set ω_x is nonempty and positively uniformly stable;

(C3) The NDS
$$\langle (\mathcal{S}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$$
is monotone.

We state a simple result for two points to be asymptotic which will be frequently used below.

Lemma 3.2. [11] *Suppose that the following conditions are fulfilled:*

1. *the points $x, x_0 \in \mathcal{S}$ with $h(x) = h(x_0)$ are proximal, i.e., there is a sequence $t_n \to +\infty$ as $n \to \infty$ such that*
$$\lim_{n\to\infty} \rho(\pi(t_n, x), \pi(t_n, x_0)) = 0;$$
2. *the set $\Sigma^+_{x_0} \subseteq \mathcal{S}$ is positively uniformly stable.*

Then the points x, x_0 are asymptotic, i.e., $\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, x_0)) = 0$.

Lemma 3.3. [11] *Assume that (C1)–(C3) hold. For given $x_0 \in \mathcal{S}$, let $K := \omega_{x_0}$, $y_0 := h(x_0)$ and $q \in \omega_q \subseteq \omega_{y_0}$. Then:*

1. *the set $K^1_q := \omega_{\alpha_q} \bigcap h^{-1}(q)$ (respectively, $\omega_{\beta_q} \bigcap h^{-1}(q)$) consists of a single point γ_q (respectively, δ_q), i.e., $K^1_q = \{\gamma_q\}$ (respectively, $\omega_{\beta_q} \bigcap h^{-1}(q) = \{\delta_q\}$), where $\alpha_q := \alpha_q(K)$, $\beta_q := \beta_q(K)$ and $K^1 := \omega_{\alpha_q}$;*
2. *let γ_q and δ_q be as in (i), then we have*
$$\gamma_q \le \alpha_q \le x \le \beta_q \le \delta_q$$
for any $x \in K_q$.

Lemma 3.4. *Under the conditions of Lemma 3.3 $\pi(t, \gamma_q) = \gamma_{\sigma(t,q)}$ (respectively, $\pi(t, \delta_q) = \delta_{\sigma(t,q)}$) for any $q \in Y$ and $t \in \mathbb{S}$.*

Proof. In virtue of Theorem 2.24, under conditions (**C1**)-(**C3**), on every ω_x ($x \in \mathcal{S} := V \times Y$) is defined a two-sided dynamical system $(\omega_x, \mathbb{S}, \pi)$. Since $\{\gamma_q\} = \omega_{\alpha_q} \bigcap X_q = K^1_q$, then $\pi(t, \gamma_q) \in K^1_{\sigma(t,q)}$ for any $t \in \mathbb{S}$. On the other hand, by Lemma 3.3 we have $K^1_{\sigma(t,q)} = \{\gamma_{\sigma(t,q)}\}$, and hence, $\pi(t, \gamma_q) = \gamma_{\sigma(t,q)}$ for any $q \in Y$ and $t \in \mathbb{S}$. □

Theorem 3.5. [18, Ch.V], [11] *Assume that (C1)–(C3) hold. For given $x_0 \in \mathcal{S}$, let $y_0 := h(x_0)$. If $y_0 \in \omega_{y_0}$, then the point γ_{y_0} (respectively, δ_{y_0}) is comparable with y_0 by character of recurrence and*
$$\lim_{t\to+\infty} \rho(\pi(t, \alpha_{y_0}), \pi(t, \gamma_{y_0})) = 0$$
$$(\textit{respectively, } \lim_{t\to+\infty} \rho(\pi(t, \beta_{y_0}), \pi(t, \delta_{y_0})) = 0).$$

Theorem 3.6. [18, Ch.V], [11] *Assume that (C1)–(C3) hold, $x_0 \in \mathcal{S}$ and $y_0 := h(x_0) \in Y$ is strongly Poisson stable. Then the point γ_{y_0} (respectively, δ_{y_0}) is strongly comparable with y_0 by character of recurrence and*
$$\lim_{t\to+\infty} \rho(\pi(t, \alpha_{y_0}), \pi(t, \gamma_{y_0})) = 0$$
$$(\textit{respectively, } \lim_{t\to+\infty} \rho(\pi(t, \beta_{y_0}), \pi(t, \delta_{y_0})) = 0).$$

Lemma 3.7. *Assume that (C1)–(C3) hold. For given $x_0 \in \mathcal{S}$, let $K := \omega_{x_0}$, $y_0 := h(x_0)$ and $q \in \omega_q \subseteq \omega_{y_0}$. Then*

1. $$\omega_x = \omega_{\alpha_q}$$
whenever $\gamma_q \le x < K^1_q$ and

2.

$$\omega_x = \omega_{\beta_q}$$

whenever $K_q^1 < x \leq \delta_q$.

Proof. Let $\{t_k\} \in \mathfrak{N}_q^{+\infty}$, then

$$\lim_{k\to\infty} \sigma(t_k, q) = q.$$

By condition (**C2**) the sequence $\{\pi(t_k, \alpha_q\}$ is pre-compact because $h(\pi(t_k, \alpha_q) = \sigma(t_k, \alpha_q)$. Since $K_q^1 = \omega_{\alpha_q} \bigcap X_q = \{\gamma_q\}$, then the sequence $\{\pi(t_k, \alpha_q\}$ converges and

$$\lim_{k\to\infty} \pi(t_k, \alpha_q) = \gamma_q. \tag{4}$$

Let $x \in S$ and $\gamma_q \leq x \leq \alpha_q$. Since the non-autonomous dynamical system $\langle (S, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is monotone, then

$$\pi(t_k, \gamma_q) \leq \pi(t_k, x) \leq \pi(t_k, \alpha_q) \tag{5}$$

for any $k \in \mathbb{N}$. Finally, note that by Theorem 3.5 we have $\{t_k\} \in \mathfrak{N}_{\gamma_q}^{+\infty}$, and hence,

$$\lim_{k\to\infty} \pi(t_k, \gamma_q) = \gamma_q. \tag{6}$$

Passing to the limit in (5) as $k \to \infty$ and taking into consideration (4) and (6) we obtain

$$\lim_{k\to\infty} \pi(t_k, x) = \lim_{k\to\infty} \pi(t_k, \gamma_q) = \gamma_q. \tag{7}$$

Note that

$$\rho(\pi(t_k, x), \pi(t_k, \gamma_q)) \leq \rho(\pi(t_k, x), \gamma_q)) + \rho(\gamma_q, \pi(t_k, \gamma_q)), \tag{8}$$

(respectively,

$$\rho(\pi(t_k, x), \pi(t_k, \alpha_q)) \leq \rho(\pi(t_k, x), \alpha_q)) + \rho(\pi(t_k, \alpha_q), \gamma_q) + \rho(\gamma_k, \pi(t_k, \gamma_q)) \quad) \tag{9}$$

passing to the limit in (8) (respectively, in (9)) as $k \to \infty$ and taking in consideration (6)-(7) (respectively, (4) and (6)-(7)) we obtain

$$\lim_{k\to\infty} \rho(\pi(t_k, x), \pi(t_k, \gamma_q)) = 0 \tag{10}$$

(respectively,

$$\lim_{k\to\infty} \rho(\pi(t_k, x), \pi(t_k, \alpha_q)) = 0). \tag{11}$$

From (10) (respectively, (11)), condition (**C2**) and Lemma 3.2 we have

$$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, \gamma_q)) = 0,$$

(respectively,

$$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, \alpha_q)) = 0)$$

and, hence $\omega_x = \omega_{\gamma_q}$ (respectively, $\omega_x = \omega_{\alpha_q}$).

The second statement of Lemma 3.7 can be proved using the same argument as above. Lemma is proved. □

Let X, Y be two complete metric spaces and

$$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h) \tag{12}$$

be a non-autonomous dynamical system.

Definition 3.8. A closed subset M of X is said to be a minimal set of non-autonomous dynamical system (NDS) (12) if it possesses the following properties:

a. $h(M) = Y$;
b. M is positively invariant, i.e., $\pi(t, M) \subseteq M$ for any $t \in \mathbb{T}_1$;
c. M is a minimal subset of X possessing properties a. and b..

Theorem 3.9. [18, Ch.IV] *Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h)$ be a non-autonomous dynamical system and $M \subset X$ be a nonempty, conditionally compact and positively invariant set. If the dynamical system $(Y, \mathbb{T}_2, \sigma)$ is minimal, then the subset M is a minimal subset of NDS (12) if and only if $H(x) = M$ for any $x \in M$.*

Theorem 3.10. [18, Ch.IV] *Suppose that $\langle (X,\mathbb{T}_1,\pi),(Y, \mathbb{T}_2,\sigma), h\rangle$ is a non-autonomous dynamical system, $(Y, \mathbb{T}_2, \sigma)$ is minimal and the space X is conditionally compact, then there exists a minimal subset M of NDS (12).*

Lemma 3.11. *Let $(X, \mathbb{S}_+, \pi)$ be a semigroup dynamical system and M be a nonempty conditionally compact minimal subset of $(X, \mathbb{S}_+, \pi)$. Then any point $x \in M$ is Poisson stable in the positive direction.*

Proof. Let $x \in M$ and $x_k := \pi(k, x)$ $(k \in \mathbb{N})$. Since the set M is minimal and $H(x_k)$ is a closed and positively invariant subset of M we have $H(x_k) = M$ and, consequently, there exists a number $\tau_k \in \mathbb{S}_+$ such that $\pi(\tau_k, x_k) \in B(x, 1/k)$. Thus we obtain

$$\rho(\pi(t_k, x), x) < 1/k$$

for any $k \in \mathbb{N}$, where $t_k := k + \tau_k \geq k$. Since $t_k \to +\infty$ as $k \to \infty$, then $x \in \omega_x$. Lemma is proved. □

Corollary 3.12. *Let M be a nonempty conditionally compact minimal subset of the semigroup dynamical system $(X, \mathbb{S}_+, \pi)$. Then every point $x \in M$ is strongly Poisson stable in the positive direction.*

Proof. Let $x \in M$ and $p \in H(x)$. Since M is minimal $H(x) = M$ and by Lemma 3.11 $p \in \omega_p$, i.e., x is strongly Poisson stable in the positive direction. □

Remark 3.13. Note that Lemma 3.11 (respectively, Corollary 3.12) for two-sided dynamical systems $(X, \mathbb{S}, \pi)$ generally speaking is not true.

Lemma 3.14. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally pre-compact;
b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$.

Then the following statements hold:

1. *there are a Poisson stable point $p \in \omega_{x_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that*

$$\lim_{k\to\infty} \rho(\pi(t_k, x_0), \pi(t_k, p)) = 0; \tag{13}$$

2. *if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle (\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, a point $p \in M \bigcap X_{y_0}$ and a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that (13) is fulfilled.*

Proof. Consider the non-autonomous dynamical system $\langle (X_0, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, where $X_0 := H^+(x_0) = \overline{\{\pi(t, x_0)|\ t \geq 0\}}$. Since X_0 is conditionally compact and $y_0 \in \omega_{y_0}$, then by Lemma 2.17 $\mathcal{E}^+_{y_0}$ is a nonempty compact semigroup. According to Lemma 2.16 there exists an idempotent $u \in \mathcal{E}^+_{y_0}$, and hence, $u(u(x_0)) = u(x_0)$.

Note that $p := u(x_0) \in \omega_{x_0}$, $u(p) = p$ and $u(x_0) = u(p)$. Let $\{t_k\} \subset \mathbb{T}_+$ such that $\pi(t_k, \cdot) \to u$ (point-wise on $X_0 \bigcap X_{y_0}$) as $k \to \infty$. In particular, we have

$$\lim_{k\to\infty} \pi(t_k, p) = p \quad \text{and} \quad \lim_{k\to\infty} \rho(\pi(t_k, x_0), \pi(t_k, p)) = 0.$$

To prove the second statement we note that by Theorem 3.10 there exists a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle (X_0, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$. Denote by $\tilde{\mathcal{E}}^+_{y_0} := \{\xi \in \mathcal{E}^+_{y_0} |\ \xi(x_0) \in M_{y_0} := M \bigcap X_{y_0}\}$. It is easy to show that $\tilde{\mathcal{E}}^+_{y_0} \neq \emptyset$. In fact, if $m \in M_{y_0} \neq \emptyset$ then there exists a sequence $t_n \to +\infty$ such that $\pi(t_n, x_0) \to m$, $h(\pi(t_n, x_0)) = \sigma(t_n, y_0) \to h(m) = y_0$ as $n \to \infty$. Thus $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$ and without loss of generality we may suppose that $\pi^{t_n} \to \xi \in \mathcal{E}^+_{y_0}$. On the other hand $\xi(x_0) = \lim\limits_{n\to\infty} \pi(t_n, x_0) = m \in M_{y_0}$ and, consequently, $\xi \in \tilde{\mathcal{E}}^+_{y_0}$. This means that under the conditions of Lemma $\tilde{\mathcal{E}}^+_{y_0}$ is a nonempty and compact sub-semigroup of the semigroup $\mathcal{E}^+_{y_0}$. Let $v \in \tilde{\mathcal{E}}^+_{y_0}$ be an idempotent of this semigroup. Since $v^2 = v$, then $v(v(x_0)) = v(x_0)$ and reasoning as above we obtain a sequence $\{t_k\} \subseteq \mathbb{S}_+$ such that

1. $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$;
2. $p := v(x_0) \in M$;
3. $$\lim_{k\to\infty} \rho(\pi(t_k, x_0), \pi(t_k, p)) = 0.$$

Lemma is completely proved. □

Corollary 3.15. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be an NDS with the following properties:*

a. there exists a point $x_0 \in X$ such that the positive semi-trajectory $\Sigma^+_{x_0}$ is conditionally pre-compact;
b. the point $y_0 := h(x_0)$ is Poisson stable, i.e., $y_0 \in \omega_{y_0}$;
c. the set ω_{x_0} is positively uniformly stable.

Then the following statements hold:

1. there is a Poisson stable point $p \in \omega_{x_0}$ such that

$$\lim_{t\to+\infty} \rho(\pi(t, x_0), \pi(t, p)) = 0; \tag{14}$$

2. if the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal, then there are a minimal subset $M \subseteq \omega_{x_0}$ of non-autonomous dynamical system $\langle (\omega_{x_0}, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ and a point $p \in M \bigcap X_{y_0}$ such that (14) is fulfilled, and hence, ω_{x_0} is a minimal subset of non-autonomous dynamical system $\langle (X_0, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$.

Proof. This statement directly follows from Lemmas 3.2 and 3.14. □

4. **Uniformly stable monotone non-autonomous dynamical systems.** In this section we suppose that conditions (**C1**)-(**C3**) are fulfilled. Assume that (E, P) is an ordered Banach space with $Int(P) \neq \emptyset$. For $u_1, u_2 \in E$, we write $u_1 \leq u_2$ if $u_2 - u_1 \in P$; $u_1 < u_2$ if $u_2 - u_1 \in P \setminus \{0\}$; $u_1 \ll u_2$ if $u_2 - u_1 \in Int(P)$.

A subset U of E is said [26] to be order convex if for any $a, b \in U$ with $a < b$, the segment $\{a + s(b - a) :\ s \in [0, 1]\}$ is contained in U. Let $V = [0, b]_E$ with $b \gg 0$ or $V = P$, or furthermore, V be an order convex subset of E.

Now fix $x_0 = (u_0, y_0) \in X = V \times Y$, then the set $\omega_{(u_0 y_0)}$ is a conditionally compact and positively invariant set. Assume that $(Y, \mathbb{S}, \sigma)$ is minimal and $y \in \omega_y$ for any $y \in Y$, then $h(\omega_{(x_0,y_0)}) = Y$. According to Corollary 3.15 the set $\omega_{(u_0,y_0)}$

is a minimal set of non-autonomous dynamical system $\langle (V\times Y,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$. We put $K:=\omega_{(x_0,y_0)}$.

Let (E_i,P_i), $1\leq i\leq n$, be ordered Banach spaces with $Int(P_i)\neq\emptyset$. For each $I=\{j_1,\ldots,j_m\}\subseteq N:=\{1,\ldots,n\}$, we define

$$E_I:=\prod_{k=1}^{m}E_{j_k},\quad P_I:=\prod_{k=1}^{m}P_{j_k}.$$

Then (E_I,P_I) is an ordered Banach space with

$$Int(P_I)=\prod_{k=1}^{m}Int(P_{j_k})\neq\emptyset.$$

Let $\leq_I$ (respectively, $<_I$ and $\ll_I$) be the orders induced by P_I in E_I. In the case where $I=N$, we use (E,P) to denote the ordered Banach space (E_N,P_N), and omit the order subscripts to get the orders $\leq$, $<$ and $\ll$ in E, respectively. For each $1\leq i\leq n$, let $Q_i:E\times Y:\to E_i$ be the projection mapping defined by $Q_i(u,y)=u_i$.

Condition (C4). For any two bounded full orbits γ_j $(j=1,2)$ with $\gamma_1(t)\leq\gamma_2(t)$ for any $t\in\mathbb{S}$, there exists $t_0>0$ such that whenever $Q_i\gamma_1(s)<Q_i\gamma_2(s)$ holds for some $i\in N$ and $s\in\mathbb{S}$, then $Q_i\gamma_1(t)\ll Q_i\gamma_2(t)$ for all $t\geq s+t_0$.

Definition 4.1. A skew-product semi-flow π on $V\times Y$ is said [26] to be componentwise strongly monotone if it is monotone and whenever $u_1\leq u_2$ with $u_{1i}<u_{2i}$, one has $Q_i\pi(t,(u_1,y))\ll Q_i\pi(t,(u_2,y))$ for all $t>0$.

In the case where $n=1$, the componentwise strong monotonicity is equivalent to the strong monotonicity. Clearly, the assumption (**C5**) is much weaker than the componentwise strong monotonicity.

For any $q\in Y$, by Lemma 3.7, we have $\gamma_q\leq K_q^1$. We define

$$M(K,\gamma_q,q):=\max\{\delta(z):\ (z,q)\in K_q^1\}\tag{15}$$

with

$$\delta(z):=Card\{i:\ (z,q)_i>(\gamma_q)_i\}$$

and

$$M(K):=\max\{M(K,\gamma_q,q):\ q\in Y\}.\tag{16}$$

Lemma 4.2. *Let (**C1**)–(**C3**) and (**C5**) hold. Assume that $(Y,\mathbb{S},\sigma)$ is minimal, $q\in\omega_q$ for any $q\in Y$ and there exist a point $(z,q)\in K_q^1$ and an index subset $I\subset N$ such that*

$$(pr_1\gamma_q)_i<z_i$$

for any $i\in I$, and

$$(pr_1\gamma_q)_j=z_j,$$

for any $j\in N\setminus I$.

Then the following two properties hold:

1. $$(pr_1\pi(t,\gamma_q))_i\ll(pr_1\pi(t,(z,q)))_i$$
   for any $i\in I$ and $t\in\mathbb{S}$;
2. $$(pr_1\pi(t,\gamma_q))_j=(pr_1\pi(t,(z,q)))_j$$
   for any $j\in N\setminus I$ and $t\in\mathbb{S}$.

Proof. This statement may be proved using the same ideas as in the proof of Lemma 4.1 from [26]. Below we will present the details of this proof. Since K is a minimal set (see Corollary 3.15, item (ii)) of non-autonomous dynamical system $\langle (V \times Y, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$, then by Lemma 3.11 there exists a sequence $\{\tau_k\} \subset \mathbb{S}_+$ such that $\tau_k \to +\infty$ and $\pi(\tau_k, (z, q)) \to (z, q)$ as $k \to \infty$. Let t be an arbitrary element of $\mathbb{S}$, then $\tau_k + t \geq 0$ for sufficiently large k and

$$\pi(\tau_k, \pi(t, (z, q))) \to \pi(t, (z, q))$$

as $k \to \infty$.

By the monotonicity of φ we have

$$\pi(\tau_k, \gamma_q(t)) = \gamma_q(t + \tau_k) \leq \pi(t + \tau_k, (z, q)) = \pi(\tau_k, \pi(t, (z, q))), \tag{17}$$

for any $k \geq 1$.

According to Theorem 3.5 and Lemma 3.4, passing to the limit in (17), we will have $\pi(t, \gamma_q) \leq \pi(t, (z, q))$, and hence,

$$\varphi(t, pr_1\gamma_q, q) \leq \varphi(t, z, q) \tag{18}$$

for any $t \in \mathbb{S}$.

Assume that the property (i) does not hold, then from (18) and condition (**C5**) it follows that there exist $i_0 \in I$ and $t_0 \in \mathbb{S}$ such that

$$\varphi_{i_0}(t, (pr_1\gamma_q, q)) = \varphi_{i_0}(t, (z, q)) \tag{19}$$

for any $t \leq t_0$.

Since $(pr_1\gamma_q)_{i_0} < z_{i_0}$, condition (**C4**) implies that there exists $\bar{t} > 0$ for which we will have

$$\varphi_{i_0}(\bar{t}, pr_1\gamma_q, q) \ll \varphi_{i_0}(\bar{t}, z, q).$$

By the minimality of ω_{γ_q} (see Lemma 3.14 and Corollary 3.15) and K, we have

$$\alpha_{\gamma_q} = \omega_{\gamma_q}, \quad \alpha_{(z,q)} = K, \tag{20}$$

and hence,

$$\pi(\bar{t}, \gamma_q) \in \alpha_{\gamma_q}, \quad \pi(\bar{t}, (z, q)) \in \alpha_{(z,q)}. \tag{21}$$

From (21) it follows that there exists a sequence $\tau_k \to -\infty$ such that $\tau_k \leq t_0$ and

$$\pi(\tau_k, (z, q)) \to \pi(\bar{t}, (z, q)). \tag{22}$$

From (22) it follows that $\{\tau_k - \bar{t}\} \in \mathfrak{N}_q$. By Theorem 3.5 $\mathfrak{N}_q \subseteq \mathfrak{N}_{\gamma_q}$, and hence, $\{\tau_k - \bar{t}\} \in \mathfrak{N}_{\gamma_q}$. From the las relation we have

$$\pi(\tau_k, \gamma_q) = \pi(\bar{t}, \pi(\tau_k - \bar{t}, \gamma_q)) \to \pi(\bar{t}, \gamma_q)$$

as $k \to \infty$.

By (19), we have

$$\varphi_{i_0}(\tau_k, pr_1\gamma_q, q) = \varphi_{i_0}(\tau_k, z, q) \tag{23}$$

for any $k \in \mathbb{N}$. Passing to the limit in (23) as $k \to \infty$ we obtain

$$\varphi_{i_0}(\bar{t}, pr_1\gamma_q, q) = \varphi_{i_0}(\bar{t}, z, q).$$

The last relation contradicts to (20). The obtained contradiction proves our statement.

It is easy to see that property (ii) follows from (16) and the arguments supporting the property (a). □

Remark 4.3. If the space Y is compact, then Lemma 4.2 coincides with Lemma 4.1 from [26].

Theorem 4.4. *Assume that the dynamical system $(Y, \mathbb{S}, \sigma)$ is minimal and $q \in \omega_q$ for any $q \in Y$. Under conditions (**C1**)-(**C4**) for any $(u_0, y_0) \in V \times Y$ the following statements hold:*

1. *for any $q \in Y$ the set*
$$\omega_{(u_0,y_0)} \bigcap X_q$$
consists of a single point $\{(u_q, q)\}$;
2. *the point (u_q, q) is strongly comparable by character of recurrence with the point $q \in Y$;*
3. $$\lim_{t \to +\infty} \rho(\varphi(t, u_0, y_0), \varphi(t, u_{y_0}, y_0)) = 0.$$

Proof. This statement may be proved using the same ideas as in the proof of Theorem 4.1 from [26]. Below we will present the details of this proof.

For any given $(u_0, y_0) \in V \times Y$, let $K = \omega_{(u_0,y_0)}$. According to Corollary 3.15 (item (ii)), under the conditions of Theorem 4.4, the set K is minimal. As defined in (15) and (16), we then get an integer $M(K)$. We will show that $M(K) = 0$. If we suppose that it is not true, then $M(K) = m$ and $1 \leq m \leq n$. By the definition, there exists a $y \in Y$ such that $M(K; \gamma_y) = m$, and hence there is a point $(z, y) \in K$ such that $z_i > (pr_1\gamma_y)_i$ for any $i \in I \subset N$ with $CardI = m$, and $z_j = (pr_1\gamma_y)_j$ for $j \in N \setminus I$. From Lemma 4.2 we have properties (i) and (ii). In particular, $(pr_1\gamma_y)_i \ll z_i$ for all $i \in I$. Without loss of generality, we can assume that $I = \{1, 2, \ldots, m\}$. By property (ii) of Lemma 4.2 and the minimality of K we obtain
$$z_j^* = (pr_1\gamma_y)_j$$
for any $(z^*, y) \in K$ and $j \in N \setminus I$.

This implies that $\delta(z) \leq \delta(z^*)$ for any $(z^*, y) \in K$. The same argument as above deduces that $\delta(z^*) \leq \delta(z)$. Thus, we have
$$\delta(z) = \delta(z^*)$$
for all $(z^*, y) \in K$. Applying Lemma 4.2 again, we get
$$\begin{aligned} (pr_1\gamma_y)_i &\ll z_i^*, \quad \text{for any } i \in I \text{ and } (z^*, y) \in K; \\ (pr_1\gamma_y)_j &= z_j^*, \quad \text{for any } j \in N \setminus I \text{ and } (z^*, y) \in K. \end{aligned} \tag{24}$$
For any index subset $I \subseteq N$, we define the projection mapping $Q_I : E \times Y \to E_I$ by $Q_I(u) = u_I$, where $(u_I)_i = u_i$, for any $i \in I$. Thus, (24) implies that
$$(pr_1\gamma_y)_I \ll_I Q_I(K \bigcap h^{-1}(y)). \tag{25}$$

We have specified $I = \{1, 2, \ldots, m\}$. By (25), it then follows that there exists an $\varepsilon_0 > 0$ 0 such that
$$Q_I(B_+^I(pr_1\gamma_y, \varepsilon_0)) \ll_I Q_I(K \bigcap h^{-1}(y)) \tag{26}$$
with
$$\begin{aligned} B_+^I(pr_1\gamma_y, \varepsilon_0) := \{u = (u_1, \ldots, u_m, (pr_1\gamma_y)_{m+1}, \ldots, (pr_1\gamma_y)_n) \in V : \\ u \geq pr_1\gamma_y \text{ and } \|u - pr_1\gamma_y\| \leq \varepsilon_0\}. \end{aligned} \tag{27}$$
Since the forward orbit $\{\pi(t, \gamma_y) : t \geq 0\}$ is uniformly stable, there exists $\delta_0 = \delta_0(\varepsilon_0) \leq \varepsilon_0$ such that
$$\|\varphi(t, u, y) - \varphi(t, pr_1\gamma_y, y)\| \leq \varepsilon_0$$

for any $t \geq 0$ whenever $\|u - pr_1\gamma_y\| \leq \delta_0$. Therefore, the subset $\omega_{(u,y)} \bigcap h^{-1}(y^*)$ satisfies

$$\sup_{(\tilde{u},y^*)\in\omega_{(u,y)}} \|u - pr_1\gamma_{y^*}\| \leq \varepsilon_0. \tag{28}$$

Combining (27) and (28) together with (ii) of Lemma 4.2, we have for any $u \in B_+^I(pr_1\gamma_y, \delta_0)$

$$\gamma_y \leq \omega_{(u,y)} \bigcap h^{-1}(y) \leq K \bigcap h^{-1}(y).$$

By the definition of α_y, it follows that

$$\gamma_y \leq \omega_{(u,y)} \bigcap h^{-1}(y) \leq \alpha_y.$$

Applying Lemma 3.7 (item (i)) and the minimality of $\omega_{(u,y)}$, we obtain that

$$\omega_{(u,y)} = \omega_{\alpha_y}$$

for any $u \in B_+^I(pr_1\gamma_y, \delta_0)$.

Let $L := \{t\gamma_y + (1-t)(z,y) |\ t \in [0,1]\} \subset h^{-1}(y)$ denote the segment with endpoints γ_y and $(z,y) \in K$, and let

$$A := \{\tau \in [0,1] \mid u_\tau := pr_1\gamma_y + \tau(z - pr_1\gamma_y),\ \omega_{(u_\tau,y)} = \omega_{\alpha_y}\}.$$

The above argument shows that there exists a $\tau_0 > 0$ such that $[0,\tau_0] \subset A$. Set $\tau_* := \sup\{\tau |\ \tau \in A\}$. By Theorem 3.6 and the monotonicity, we have $\tau \in A$ whenever $0 \leq \tau \leq \tau'$ with $\tau' \in A$. Therefore, A must be an interval, that is, $A = [0,\tau_*)$ or $[0,\tau_*]$. In either case, we shall show that the semi-trajectory $\{\pi(t,(x_{\tau_*},y)) |\ t \geq 0\}$ is not uniformly stable.

In the case where $A = [0,\tau_*)$, we have $\omega_{(u_{\tau_*},y)} \neq \omega_{\alpha_y}$. Choose $(\tilde{u},y) \in \omega_{(u_{\tau_*},y)}$. Then $\tilde{u} \neq pr_1\gamma_y$. Let $\varepsilon_0 := \|\tilde{u} - pr_1\gamma_y\| > 0$. By the stability of the forward orbit $\{\pi(t,(u_{\tau_*},y)) |\ t \geq 0\}$, there is a $\delta_0 > 0$ such that

$$\|\varphi(t,u,y) - \varphi(t,u_{\tau_*},y)\| < \frac{\varepsilon_0}{2} \tag{29}$$

whenever $t \geq 0$ and $\|u - u_{\tau_*}\| < \delta_0$. Let $0 < \tau_* - \tau \ll 1$ and choose $\tau_k \to +\infty$ such that $\pi(\tau_k,(u_{\tau_*},y)) \to (\tilde{u},y)$ as $k \to \infty$. Since $0 < \tau_* - \tau \ll 1$, it follows from (29) that

$$\|\varphi(\tau_k,u_\tau,y) - \varphi(\tau_k,u_{\tau_*},y)\| < \frac{\varepsilon_0}{2}. \tag{30}$$

Passing to the limit in (30) as $k \to \infty$, we obtain

$$\varepsilon_0 = \|\tilde{u} - pr_1\gamma_y\| \leq \frac{\varepsilon_0}{2},$$

which is a contradiction.

In the case where $A = [0,\tau_*]$, we have $\omega_{(u_{\tau_*},y)} = \omega_{\alpha_y}$ and $0 < \tau_* < 1$. For any $\tau \in (\tau_*,1)$, $\gamma_y \notin \omega_{(u_\tau,y)}$. Let ε_0 be the number defined in $B_+^I(pr_1\gamma_y,\varepsilon_0)$ of (26). Then the stability assumption implies that

$$\|\varphi(t,u_\tau,y) - \varphi(t,u_{\tau_*},y)\| < \varepsilon_0, \tag{31}$$

for any $t \geq 0$ as $0 < \tau - \tau_* \ll 1$. Let $\{\tau_k\}$ be such a sequence that $\pi(\tau_k,(u_{\tau_*},y)) \to \gamma_y$ as $k \to \infty$. Without loss of generality, we can suppose that $\pi(\tau_k,(u_\tau,y)) \to (\tilde{u},y)$ for $0 < \tau - \tau_* \ll 1$. Putting τ_k into (31) and letting $k \to \infty$, we obtain $\|\tilde{u} - pr_1\gamma_y\| \leq \varepsilon_0$. From the monotonicity and Lemma 4.2 (item (ii)), we have

$$\varphi_j(t,u_\tau,y) = \varphi_j(t,pr_1\gamma_y,y)$$

for any $j \in N \setminus I$ and $t > 0$. Thus, $\tilde{u} \in B_+^I(pr_1\gamma_y,\varepsilon_0)$. By (26), $\tilde{u}_I \ll_I Q_I(K \bigcap h^{-1}(y))$. Applying Lemma 3.7, we have $\omega_{(\tilde{u},y)} = \omega_{\alpha_y}$. Since $\omega_{(\tilde{u},y)} \subset$

$\omega_{(u_\tau,y)}$, the minimality implies that $\omega_{(u_\tau,y)} = \omega_{(\tilde{u},y)} = \omega_{\alpha_y}$, contradicting the definition of τ_*.

The above two contradictions show that $M(K) = 0$. Thus, $K \bigcap h^{-1}(y) = \{\gamma_y\}$ for any $y \in Y$. Now to finish the proof of Theorem it is sufficient to apply Theorem 3.6. □

5. **Sub-linear non-autonomous dynamical systems.** Let E be a real Banach space with a nonempty closed convex cone P such that $P \bigcap (-P) = \{0\}$ and $Int(P) \neq \emptyset$.

Definition 5.1. $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is said to be monotone (respectively, strictly or strongly monotone) if for any $u_1, u_2 \in P$ the inequality $u_1 \leq u_2$ (respectively, $u_1 < u_2$) implies $\varphi(t, u_1, y) \leq \varphi(t, u_2, y)$ (respectively, $\varphi(t, u_1, y) < \varphi(t, u_2, y)$ or $\varphi(t, u_1, y) \ll \varphi(t, u_2, y)$) for any $t > 0$ and $y \in Y$.

Definition 5.2. A monotone cocycle $\langle P, \varphi, (Y, \mathbb{T}, \sigma)\rangle$ is said to be:

1. sub-linear if
$$\varphi(t, \lambda u, y) \geq \lambda \varphi(t, u, y) \tag{32}$$
for any $\lambda \in (0, 1)$, $t > 0$ and $(u, y) \in P \times Y$;
2. strictly sub-linear (respectively, strongly sub-linear) if
$$\varphi(t, \lambda u, y) > \lambda \varphi(t, u, y) \tag{33}$$
$$(\text{respectively, } \varphi(t, \lambda u, y) \gg \lambda \varphi(t, u, y))$$
for any $\lambda \in (0, 1)$, $t > 0$ and $(u, y) \in Int(P) \times Y$.

Remark 5.3. Note that inequality (32) is equivalent to
$$\alpha \varphi(t, u, y) \geq \varphi(t, \alpha u, y)$$
$$(\text{respectively, } \alpha \varphi(t, u, y) > \varphi(t, \alpha u, y) \text{ or } \alpha \varphi(t, u, y) \gg \varphi(t, \alpha u, y))$$
for any $\alpha \in (1, +\infty)$, $t > 0$ and $(u, y) \in P \times Y$.

We define the part (Birkhoff) metric on $Int(P)$ by
$$p(u_1, u_2) := \inf\{\ln \alpha |\ \alpha \geq 1, \text{ and } \alpha^{-1} u_1 \leq u_2 \leq \alpha u_1\}.$$
Then $(Int(P), p)$ is a metric space (see, for example, [20, 29, 38] and the bibliography therein).

Lemma 5.4. [19],[20, ChIV],[26] *Let $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a monotone sub-linear non-autonomous (cocycle) dynamical system, then the following statements hold:*

a.
$$p(\varphi(t, u_1, y), \varphi(t, u_2, y)) \leq p(u_1, u_2)$$
for any $t \geq 0$ and $u_1, u_2 \in Int(P)$;

b. if $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is strongly sub-linear, then
$$p(\varphi(t, u_1, y), \varphi(t, u_2, y)) < p(u_1, u_2)$$
for any $t > 0$ and $u_1, u_2 \in Int(P)$, $u_1 \neq u_2$ and $\varphi(t, u_i, y) \in Int(P)$.

Condition (C). For given $y_0 \in Y$ there exists a number $t_0 = t_0(y_0) > 0$ such that if $u_1, u_2 \in Int(P)$ and $\alpha^{-1} u_1 \leq u_2 \leq \alpha u_1$ then $\alpha^{-1} \varphi(t_0, u_1, y_0) \ll \varphi(t_0, u_2, y_0) \ll \alpha \varphi(t_0, u_1, y_0)$.

Lemma 5.5. [53] *Under the condition (**C**) the following statement holds*

$$p(\pi(t_0, x_1), \pi(t_0, x_2)) < p(x_1, x_2),$$

where $x_i := (u_i, y_0)$ *and* $u_i \in Int(P)$ *(*$i = 1, 2$*).*

Lemma 5.6. *Let* $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ *be monotone, sub-linear and one of the following conditions hold:*

1. *there exist* $y_0 \in Y$ *and* $t_0 = t_0(y_0) > 0$ *such that*
$$\varphi(t_0, \lambda u, y_0) \ll \lambda\varphi(t_0, u, y_0) \tag{34}$$
for any $u \in Int(P)$ *and* $\lambda > 1$*;*
2. *there exist* $y_0 \in Y$ *and* $t_0 = t_0(y_0) > 0$ *such that*
$$\varphi(t_0, u_1, y_0) \ll \varphi(t_0, u_2, y_0)$$
for any $u_1, u_2 \in Int(P)$ *with* $u_1 < u_2$*;*.

Then there are $y_0 \in Y$ *and* $t_0 = t_0(y_0) > 0$ *such that the inequality* $\lambda^{-1}u_1 \leq u_2 \leq \lambda u_1$ *(*$\lambda > 1$*) implies*

$$\lambda^{-1}\varphi(t_0, u_1, y_0) \ll \varphi(t_0, u_2, y_0) \ll \lambda\varphi(t_0, u_1.y_0).$$

Proof. Assume that the cocycle φ is monotone, sub-linear and there are $y_0 \in Y$ and $t_0 = t_0(y_0) > 0$ such that the condition (34) is fulfilled. Let $\lambda > 1$ and $u_1, u_2 \in Int(P)$ such that $\lambda^{-1}u_1 \leq u_2 \leq \lambda u_2$, then we have

$$\lambda^{-1}\varphi(t_0, u_1, y_0) \ll \varphi(t_0, \lambda^{-1}u_1, y_0) \leq \varphi(t_0, u_2, y_0) \leq$$
$$\varphi(t_0, \lambda u_1, y_0) \ll \lambda\varphi(t_0, u_1.y_0)$$

and hence,

$$\lambda^{-1}\varphi(t_0, u_1, y_0) \ll \varphi(t_0, u_2, y_0) \ll \lambda\varphi(t_0, u_1.y_0).$$

Analogically we can prove the second statement of Lemma. □

Let $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ (shortly φ) be a cocycle over dynamical system with the fiber P and $(X, \mathbb{S}_+, \pi)$ be a skew-product dynamical system generated by φ, i.e., $X := P \times Y$ and $\pi := (\varphi, \sigma)$ $(\pi(t, (u, y)) := (\varphi(t, u, y), \sigma(t, y))$ for any $t \in \mathbb{S}_+$ and $(u, y) := x \in X)$.

Lemma 5.7. [30] *The following statements hold:*

1. *let* P *be a cone with nonempty interior in a Banach space* E*; if* $u, v \in Int(P)$ *and* $r > 0$ *is a number such that the norm closed ball of radius* r *and center* u *and* v *respectively, is contained in* P*, then*
$$p(u, v) \leq \ln(1 + \frac{||u - v||}{r}); \tag{35}$$
2. *if* P *is a normal cone in Banach space* E *and the norm* $||\cdot||$ *is monotone on* P*, then for any* $u, v \in Int(P)$*,*
$$||u - v|| \leq (2e^{p(u,v)} - e^{-p(u,v)} - 1)\min\{||u||, ||v||\}.$$

Let $u_0 \in Int(P)$ and $y_0 \in Y$ such that $Q := \overline{\varphi(\mathbb{S}_+, u_0, y_0)} \subset Int(P)$ is a compact subset, then there are two positive numbers r and R $(R > r)$ such that

$$B[Q, 2r] := \{u \in Int(P) |\ \rho(u, Q) \leq 2r\} \subset Int(P)$$

and

$$||u|| \leq R$$

for any $u \in B[Q, 2r]$, where $\rho(u, Q) := \inf\limits_{v \in Q} ||u - v||$.

Definition 5.8. The point $x_0 := (u_0, y_0)$ (respectively, the semi-trajectory $\Sigma^+_{x_0}$) is called positively uniformly stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $||\varphi(\tau, u, y_0) - \varphi(\tau, u_0, y_0)|| < \delta$ implies $||\varphi(t, u, y_0) - \varphi(t, u_0, y_0)|| < \varepsilon$ for any $t \geq \tau$.

Remark 5.9. Note that the point $x_0 = (u_0, y_0) \in E \times Y$is uniformly positively stable (in the sense of Definition 5.8) if and only if the set $\Sigma^+_{x_0}$ is uniformly positively stable (in the sense of Definition 2.19) with respect to non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ generated by cocycle φ.

This statement follows from the corresponding definitions and the fact that $X = E \times Y$ and $\pi = (\varphi, \sigma)$.

Lemma 5.10. *Let P be normal cone in Banach space E and $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ be a monotone and sub-linear cocycle. Suppose that $u_0 \in Int(P)$ such that $Q = \overline{\varphi(\mathbb{S}_+, u_0, y_0)} \subset Int(P)$, then the point $x_0 := (u_0, y_0) \in Int(P) \times Y$ is positively uniformly stable.*

Proof. Suppose that this statement is not true, then there are $r > \varepsilon_0 > 0$, $\delta_n \to 0$ ($\delta_n > 0$), $\{u_n\}$ with

$$||\varphi(\tau_n, u_n, y_0) - \varphi(\tau_n, u_0, y_0)|| \leq \delta_n,$$

$\tau_n \geq 0$ and $t_n \geq \tau_n$ such that

$$||\varphi(t_n, u_n, y_0) - \varphi(t_n, u_0, y_0)|| \geq \varepsilon_0$$

for any $n \in \mathbb{N}$.

Notice that

$$\gamma_n := V(\varphi(t_n, u_n, y_0), \varphi(t_n, u_0, y_0)) =$$
$$V(\varphi(t_n - \tau_n, \varphi(\tau_n, u_n, y_0), \sigma(\tau_n, y_0)), \varphi(t_n - \tau_n, \varphi(\tau_n, u_0, y_0), \sigma(\tau_n, y_0)) \leq$$
$$V(\varphi(\tau_n, u_n, y_0), \varphi(\tau_n, u_0, y_0)) \leq \ln(1 + \tfrac{||\varphi(\tau_n, u_n, y_0) - \varphi(\tau_n, u_0, y_0)||}{r}) \to 0$$

as $n \to \infty$, where $V(u, v) := p(u, v)$. On the other hand

$$\varepsilon_0 \leq ||\varphi(t_n, u_n, y_0) - \varphi(t_n, u_0, y_0)|| \leq (2e^{\gamma_n} - e^{-\gamma_n} - 1)R \to 0$$

as $n \to \infty$, i.e., $\varepsilon_0 \leq 0$. The obtained contradiction proves our statement. □

Corollary 5.11. *Under the conditions of Lemma 5.10 the semi-trajectory $\Sigma^+_{x_0}$ is positively uniformly stable and on the set $K := \omega_{x_0}$ is defined a two-sided dynamical system $(K, \mathbb{S}, \tilde{\pi})$ which is an extension of one-sided dynamical system $(K, \mathbb{S}_+, \pi)$, i.e., $\tilde{\pi}(t, x) = \pi(t, x)$ for any $(t, x) \in \mathbb{S}_+ \times K$.*

Proof. This statement follows from Lemma 5.10 and Theorem 2.24. □

Lemma 5.12. *Let $V : Int(P) \times Int(P) \to \mathbb{R}_+$ be a function defined by equality*

$$V(u, v) := p(u, v) \tag{36}$$

for any $(u, v) \in Int(P) \times Int(P)$.

Then the following statements hold:

1. *$V(u, v) = 0$ if and only $u = v$;*
2. $$|V(u, v) - V(u_0, v_0)| \leq V(u, u_0) + V(v, v_0) \tag{37}$$
   for any $u, v, u_0, v_0 \in Int(P)$;

3. *the mapping* $V : Int(P) \times Int(P) \to \mathbb{R}_+$ *is norm continuous, i.e.,*

$$\lim_{n\to\infty} V(u_n, v_n) = V(u_0, v_0)$$

for any $u_0, v_0 \in Int(P)$ *and* $\{u_n\}, \{v_n\} \subseteq Int(P)$ *such that*

$$\lim_{n\to\infty} ||u_n - u_0|| = \lim_{n\to\infty} ||v_n - v_0|| = 0.$$

Proof. The first statement is evident. The second statement follows from (36) and the triangle inequality for the metric p. The third statement follows from (35) and (37) (see also [53]). □

6. **Comparability by character of recurrence of motions for sub-linear cocycles.**

Lemma 6.1. *Let* $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ *be a monotone non-autonomous dynamical system. Assume that the following conditions are fulfilled:*

1. *the point* $y_0 \in Y$ *is positively Poisson stable;*
2. *there exists a nonempty closed positively invariant subset* $K \subseteq X$ *which is conditionally compact and* $h(K) = Y$*;*
3. $\mathcal{M}$ *is a nonempty* $\mathcal{E}^+_{y_0}$ *minimal subset of* K_{y_0}*;*
4. *the* $\mathcal{E}^+_{y_0}$ *minimal set* $\mathcal{M}$ *is positively distal.*

Then the $\mathcal{E}^+_{y_0}$ *minimal set* $\mathcal{M}$ *does not contain two different points* $x_1, x_2 \in \mathcal{M}$ *such that* $x_2 \geq x_1$*.*

Proof. Suppose that this statement is not true. Let $x_1, x_2 \in \mathcal{M}$, $x_1 \neq x_2$ and $x_2 \geq x_1$. Since the set K is conditionally compact, then K_{y_0} is a compact set and, consequently, $\mathcal{M} \subseteq K_{y_0}$ is so because it is closed. Taking in consideration that $\mathcal{M}$ is partially ordered and compact, then by Zorn's lemma it contains at least one maximal element $m \in \mathcal{M}$. Notice that $\mathcal{M}$ is a $\mathcal{E}^+_{y_0}$ minimal set and, consequently, there exists $\xi \in \mathcal{E}^+_{y_0}$ such that $\xi(x_1) = m$. For $\xi \in \mathcal{E}^+_{y_0}$ there exists a sequence $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that

$$\lim_{n\to\infty} \pi^{t_n}\big|K_{y_0} = \xi$$

and $\lim_{n\to\infty} \pi(t_n, x_1) = m$. By conditions of Lemma the set K is conditionally compact, then the sequence $\{\pi(t_n, x_2)\}$ is also conditionally compact because $\{h(\pi(t_n, x_2))\} = \{\sigma(t_n, y_0)\}$. Thus we can extract a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that the sequence $\{\pi(t_{n_k}, x_2)\}$ converges. Denote its limit by $p := \lim_{n\to\infty} \pi(t_{n_k}, x_2)$, then taking in consideration that

$$\pi(t_{n_k}, x_2) \geq \pi(t_{n_k}, x_1) \tag{38}$$

for any $k \in \mathbb{N}$ and passing to the limit in (38) as $k \to \infty$ we obtain $p \geq m$ and, consequently, $m = p$.

On the other hand $x_2 \neq x_1$, $x_1, x_2 \in \mathcal{M}$ and $\mathcal{M}$ is positively distal, i.e., there exists a positive number $\alpha = \alpha(x_1, x_2)$ such that

$$\rho(\pi(t_{n_k}, x_2), \pi(t_{n_k}, x_1)) \geq \alpha \tag{39}$$

for any $k \in \mathbb{N}$. Passing to the limit in (39) as $k \to \infty$ we obtain $\rho(p, m) \geq \alpha$, i.e., $p \neq m$. The obtained contradiction proves our statement. □

Lemma 6.2. *Let* $\langle (X, \mathbb{S}_-, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ *be a monotone non-autonomous dynamical system. Assume that the following conditions are fulfilled:*

1. *the point* $y_0 \in Y$ *is negatively Poisson stable;*

2. *there exists a nonempty closed negatively invariant subset $K \subseteq X$ which is conditionally compact and $h(K) = Y$;*
3. *$\mathcal{M}$ is a nonempty $\mathcal{E}^-_{y_0}$ minimal subset of K_{y_0};*
4. *the $\mathcal{E}^-_{y_0}$ minimal set $\mathcal{M}$ is negatively distal.*

Then the $\mathcal{E}^-_{y_0}$ minimal set $\mathcal{M}$ does not contain two different points $x_1, x_2 \in \mathcal{M}$ such that $x_2 \geq x_1$.

Proof. This statement follows from Lemma 6.1 replacing t by $-t$. □

Remark 6.3. Lemma 6.1 (respectively, Lemma 6.2) remains true, if we replace the positive (respectively, negative) distality of minimal set $\mathcal{M}$ by the following conditions:

$$\inf_{\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}} \rho(\pi(t_n, x_1), \pi(t_n, x_2)) > 0$$

(respectively,

$$\inf_{\{t_n\} \in \mathfrak{N}^{-\infty}_{y_0}} \rho(\pi(t_n, x_1), \pi(t_n, x_2)) > 0\ )$$

for any $x_1, x_2 \in \mathcal{M}$ with $x_1 \neq x_2$.

Lemma 6.4. *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

1. *the point $y_0 \in Y$ is Poisson stable in the both directions;*
2. *X is conditionally compact, $x_0 \in X$ and $h(x_0) = y_0$.*

Then the following statements are equivalent:

a. *$\mathfrak{N}^{+\infty}_{y_0} \subseteq \mathfrak{N}^{+\infty}_{x_0}$;*
b. *the point $x_0 \in X_{y_0}$ is a common fixed point of the semigroup $\mathcal{E}^+_{y_0}$, i.e., $\xi(x_0) = x_0$ for any $\xi \in \mathcal{E}^+_{y_0}$.*

Proof. We will show that a. implies b. In fact, for any $\xi \in \mathcal{E}^+_{y_0}$ there exists a sequence $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that $\xi = \lim\limits_{n\to\infty} \pi^{t_n}\big|X_{y_0}$ and, consequently, $\xi(x_0) = \lim\limits_{n\to\infty} \pi(t_n, x_0) = x_0$ for any $\xi \in \mathcal{E}^+_{y_0}$ because $\mathfrak{N}^{+\infty}_{y_0} \subseteq \mathfrak{N}^{+\infty}_{x_0}$.

Conversely. Let $x_0 \in X_{y_0}$ be a common fixed point of the semigroup $\mathcal{E}^+_{y_0}$, i.e., $\xi(x_0) = x_0$ for any $\xi \in \mathcal{E}^+_{y_0}$. We need to show that $\mathfrak{N}^{+\infty}_{y_0} \subseteq \mathfrak{N}^{+\infty}_{x_0}$. Let $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$. Consider the sequence $\{\pi(t_n, x_0)\}$. Since $\{\pi(t_n, x_0\} \subseteq X$ and X is conditionally compact, then $\{\pi(t_n, x_0)\}$ is relatively compact. Thus, for the convergence of $\{\pi(t_n, x_0)\}$ it is sufficient to show that it has at most one (exactly one) limiting point. Let x' be a limiting point of the sequence $\{\pi(t_n, x_0)\}$, then there exists a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that $\pi(t_{n_k}, x_0\} \to x'$ as $k \to \infty$. Since X is conditionally compact and $\{\sigma(t_n, y_0)\} \to y_0$ as $n \to \infty$, then the set

$$Q := \overline{\bigcup \{X_{\sigma(t_n, y_0)} |\ n \in \mathbb{N}\}}$$

is compact. Consider the sequence $\{\pi^{t_{n_k}}\big|X_{y_0}\} \subset Q^{X_{y_0}}$. By Tikhonov's theorem we can suppose that this sequence is convergent. Denote its limit by ξ, then $\xi \in \mathcal{E}^+_{y_0}$ and we have $x' = \lim\limits_{n\to\infty} \pi(t_{n_k}, x_0) = \xi(x_0) = x_0$. □

Theorem 6.5. *Let $\langle P, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ (shortly φ) be a cocycle over dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber P. Assume that the following conditions are fulfilled:*

1. *the point $y_0 \in Y$ is Poisson stable in the both directions;*
2. *the cocycle φ is monotone and strongly sub-linear;*

3. *for every* $x \in X = P \times Y$ *the positive semi-trajectory* Σ_x^+ *is conditionally compact;*
4. *there exists a point* $x_0 = (u_0, y_0) \in Int(P) \times Y$ *such that*
   (1) *the set* $Q := \overline{\varphi(\mathbb{S}_+, u_0, y_0)}$ *is compact;*
   (2) *the closure* $\overline{\Sigma_{x_0}^+}$ *of positive semi-trajectory* $\Sigma_{x_0}^+$ *belongs to* $Int(P) \times Y$.

Then the following statements hold:

1. (a) *the set* ω_{x_0} *is negatively distal, i.e., for any* $q \in \omega_{y_0}$ *and* $(v_1, q), (v_2, q) \in \omega_{x_0}$ *with* $v_1 \neq v_2$ *we have*

$$\inf_{t \leq 0} ||\varphi(t, v_1, q) - \varphi(t, v_2, q)|| > 0;$$

   (b) *the set* $\omega_{x_0} \bigcap X_{y_0}$ *consists of a single point* $p_0 = (p_{u_0}, y_0)$;
   (c) $\mathfrak{N}_{y_0}^{+\infty} \subseteq \mathfrak{N}_{p_0}^{+\infty}$;
   (d) $\lim_{t \to +\infty} \|\varphi(t, u_0, y_0) - \varphi(t, p_{u_0}, y_0)\| = 0$.
2. *for any* $u \in Int(P)$ *the set* $\omega_{(u,y_0)} \bigcap X_{y_0}$ *consists of a single point* $p = (p_u, y_0)$;
3. $\mathfrak{N}_{y_0}^{+\infty} \subseteq \mathfrak{N}_p^{+\infty}$;
4.

$$\lim_{t \to +\infty} \|\varphi(t, u, y_0) - \varphi(t, p_u, y_0)\| = 0;$$

5.

$$p_{u_1} = p_{u_2}$$

for any $u_i \in P$ $(i = 1, 2)$.

Proof. Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ be a non-autonomous dynamical system generated by cocycle φ, i.e., $(X, \mathbb{S}_+, \pi)$ is the skew-product dynamical system $(X = P \times Y, \pi = (\varphi, \sigma)$ and $h := pr_2)$. Denote by $\tilde{X} \dot{\times} \tilde{X} := \{(x_1, x_2) |\ x_1, x_2 \in \tilde{X},\ h(x_1) = h(x_2)\}$ and $\mathcal{V}(x_1, x_2) := V(u_1, u_2)$, where $(u_i, y) := x_i$ $(i = 1, 2)$ and $\tilde{X} := Int(P) \times Y$. By Lemma 5.4 we have

$$\mathcal{V}(\pi(t, x_1), \pi(t, x_2)) \leq \mathcal{V}(x_1, x_2)$$

for any $t \geq 0$ and $(x_1, x_2) \in \tilde{X} \dot{\times} \tilde{X}$ with $\pi(t, x_i) \in Int(P) \times Y$.

Denote by $K := \omega_{x_0}$ the omega limit set of x_0. Under the conditions of Theorem the set K is invariant, conditionally compact and $h(K) = \omega_{y_0}$. According to Lemma 5.10 the point x_0 and, consequently, also the set ω_{x_0} is positively uniformly stable. By Theorem 2.24 (see also Corollary 5.11) it is defined a two-sided non-autonomous dynamical system $\langle (K, \mathbb{S}, \pi), (\omega_{y_0}, \mathbb{S}, \sigma), h \rangle$. We will show that the constructed above non-autonomous dynamical system is negatively distal. In fact. If we suppose that it is not true, then there exists a point $q \in \omega_{y_0}$ and pair of points $x_1, x_2 \in K_q$ with $x_1 \neq x_2$ such that

$$\inf_{t \leq 0} \rho(\pi(t, x_1), \pi(t, x_2)) = 0.$$

This means that there exists a sequence $t_n \to -\infty$ such that $\rho(\pi(t_n, x_1), \pi(t_n, x_2)) \to 0$ as $n \to \infty$. On the other hand we have

$$\mathcal{V}(x_1, x_2) = \mathcal{V}(\pi^{-t_n} \pi^{t_n}(x_1), \pi^{-t_n} \pi^{t_n}(x_2)) \leq$$
$$\mathcal{V}(\pi^{t_n}(x_1), \pi^{t_n}(x_2)) = p(\varphi(t_n, u_1, q), \varphi(t_n, u_2, q)) \leq$$
$$\ln\left(1 + \tfrac{||\varphi(t_n, u_1, q) - \varphi(t_n, u_2, q)||}{r}\right) \to 0$$

as $n \to \infty$, where $x_i := (u_i, q)$ $(i = 1, 2)$ and, consequently, $x_1 = x_2$. The obtained contradiction proves our statement. By Lemma 2.18 there are two sequences $\{t_n^{\pm}\}$

such that $t_n^{\pm} \to \pm\infty$ and for any $x \in K_{y_0}$

$$\pi(t_n^{\pm}, x) \to x \tag{40}$$

as $n \to \infty$.

Let $x_i = (u_i, y_0)$ $(i = 1, 2)$. Consider the function $\beta : \mathbb{T} \to \mathbb{R}_+$ defined by $\beta(t) := \mathcal{V}(\pi(t, x_1), \pi(t, x_2))$. Note that the function β possesses the following properties:

1. $\beta \in C(\mathbb{T}, \mathbb{R}_+)$;
2. $$\lim_{n\to\infty} \beta(t + t_n^{\pm}) = \beta(t) \tag{41}$$
   for any $t \in \mathbb{S}$;
3. according to Lemma 5.4 the function α is monotone decreasing, i.e.,
   $$\text{if } t_2 \geq t_1 \ (t_1, t_2 \in \mathbb{S}) \text{ then } \beta(t_2) \leq \beta(t_1). \tag{42}$$

Now we will show that the function β figuring above is a constant. Let t_1, t_2 be two arbitrary numbers from $\mathbb{S}$. Without loss of generality we can suppose that $t_1 \leq t_2$, then by (42) we have $\beta(t_2) \leq \beta(t_1)$. On the other hand according to (41) there exists a natural number n_0 such that

$$t_2 + t_n^- \leq t_1 \tag{43}$$

for any $n \geq n_0$. Since the function β is monotone decreasing, then from (43) we obtain

$$\beta(t_1) \leq \beta(t_2 + t_n^-) \tag{44}$$

for any $n \geq n_0$. Passing to the limit in (44) as $n \to \infty$ and taking into consideration (41) we obtain

$$\beta(t_1) \leq \beta(t_2). \tag{45}$$

From (42) and (45) we have $\beta(t_1) = \beta(t_2)$ for any $t_1, t_2 \in \mathbb{S}$ and, consequently, $\beta(t) = \beta(0)$ for any $t \in \mathbb{S}$. From this it follows that

$$\mathcal{V}(\pi(t, x_1), \pi(t, x_2)) = \mathcal{V}(x_1, x_2)$$

and, consequently,

$$p(\varphi(t, u_1, y_0), \varphi(t, u_2, y_0)) = p(u_1, u_2) \tag{46}$$

for any $t \in \mathbb{S}$.

Denote by

$$\mathcal{E}_{y_0}^+ := \{\xi |\ \xi = \lim_{n\to\infty} \pi^{t_n} | K_{y_0},\ \{t_n\} \in \mathfrak{N}_{y_0}^{+\infty}\}.$$

Note that $K_{y_0} := \{x \in K |\ h(x) = y_0\}$ is a compact subset of K. By Lemma 2.17 $\mathcal{E}_{y_0}^+$ is a nonempty, compact sub-semigroup of semigroup $K_{y_0}^{K_{y_0}}$. According to Lemma 3.8 [6] (see also Lemma 4.3.10 [18, ChIV,p.212]) there exists a $\mathcal{E}_{y_0}^+$ minimal set $\mathcal{M}$.

Now we will establish that the set $\mathcal{M}$ consists of a single point. If we suppose that it is not so, then there are to different points $x_1, x_2 \in \mathcal{M}$ ($x_i := (u_i, y_0)$ and $u_i \in Q \subset Int(P)$). Let $\mathcal{V}(x_1, x_2) = p(u_1, u_2) = \ln \alpha_0 > 0$, then $\alpha_0 > 1$ because $x_1 \neq x_2$ (respectively, $u_1 \neq u_2$). By Lemma 5.4 (item b.) we have

$$p(\varphi(t, u_1.y_0), \varphi(t, u_2, y_0)) < p(u_1, u_2) \tag{47}$$

for any $t > 0$. Relations (47) and (46) are contradictory. The obtained contradiction proves that the set $\mathcal{M}$ contains a single point $\{p_0\}$. We will show that $K_{y_0} = \mathcal{M}$. In fact, since $\{p_0\} = \mathcal{M} \subseteq \omega_{x_0}$ then there exists a sequence $t_n \to +\infty$ as $n \to \infty$ such that $\pi(t_n, x_0) \to p_0$ as $n \to \infty$.

Note that $\{t_n\} \in \mathfrak{N}_{y_0}^{+\infty}$. On the other hand $\{p_0\} = \mathcal{M} = \mathcal{E}_{y_0}^+(p_0)$ and, consequently, $\xi(p_0) = p_0$ for any $\xi \in \mathcal{E}_{y_0}^+$. By Lemma 6.4 we have $\mathfrak{N}_{y_0}^{+\infty} \subseteq \mathfrak{N}_{p_0}^{+\infty}$.

From the above it follows that $\{t_n\} \in \mathfrak{N}_{y_0}^{+\infty} \subseteq \mathfrak{N}_{p_0}^{+\infty}$ and, consequently,

$$\rho(\pi(t_n, x_0), \pi(t_n, p_0)) \leq \rho(\pi(t_n, x_0), p_0) + \rho(p_0, \pi(t_n, p_0)) \to 0$$

as $n \to \infty$. By Lemma 3.2 we have

$$\lim_{t \to +\infty} \rho(\pi(t, x_0), \pi(t, p_0)) = 0.$$

This means that $\omega_{x_0} = \omega_{p_0}$ and hence, $\omega_{x_0} \bigcap X_{y_0} = K_{y_0} = \{p_0\}$.

If $x = (u, y_0) \in P \times Y$ with $u \gg 0$, we can choose a point $(\tilde{u}, y_0) \in K = \omega_{x_0}$ such that $u \gg \alpha\tilde{u}$ for some $\alpha \in (0, 1)$. Then we have

$$\varphi(t, u, y_0) \geq \varphi(t, \alpha\tilde{u}, y_0) \gg \alpha\varphi(t, \tilde{u}, y_0)$$

for any $t \geq 0$ and hence, $\omega_{(u,y_0)} \subset Int(P) \times Y$. Using the same arguments as above for the point $x_0 = (u_0, y_0)$ and its omega limit set ω_{x_0} we see that $\omega_x \bigcap X_{y_0}$ consists of a single point p and as above

$$\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, p)) = 0$$

or equivalently

$$\lim_{t \to +\infty} \rho(\varphi(t, u, y_0), \varphi(t, p_u, y_0)) = 0,$$

where $(p_u, y_0) = p$.

To prove the last statement it is sufficient to show that $\omega_x = \omega_{x_0}$ (or $\omega_x \bigcap X_q = \omega_{x_0} \bigcap X_q$ for any $q \in Y$) for any $x = (u, y_0) \in Int(P) \times Y$ with $u \gg 0$. □

Corollary 6.6. *Under the conditions of Theorem 6.5 if the point y_0 is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable), then the point p is so.*

Proof. This statement directly follows from Theorems 6.5 and 2.40. □

Theorem 6.7. [17] *Let $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ be a non-autonomous dynamical system and $x_0 \in X$ be a conditionally Lagrange stable point. Suppose that the following conditions are fulfilled:*

a. *Y is minimal;*
b. *$H(x_0) \bigcap X_{y_0}$ consists of a single point $\{x_0\}$, where $y_0 := h(x_0)$;*
c. *the set $H(x_0)$ is distal, i.e., $\inf\limits_{t \in \mathbb{S}_+} \rho(\pi(t, x_1), \pi(t, x_2)) > 0$ for any $x_1, x_2 \in H(x_0)$ with $x_1 \neq x_2$ and $h(x_1) = h(x_2)$.*

Then $\mathfrak{M}_{y_0} \subseteq \mathfrak{M}_{x_0}$.

Remark 6.8. 1. Note that Theorem 6.7 coincides with the result of B. A. Shcherbakov [45, Ch.III] (see also [46, Ch.III]) in the case, when the set Y is compact.

2. Under the conditions of Theorem 6.7 the set $H(x_0) \bigcap X_q$ consists of a single point for every $q \in \omega_{y_0} = Y$.

Theorem 6.9. *Let $\langle P, \varphi, (Y, \mathbb{S}, \sigma) \rangle$ (shortly φ) be a cocycle over dynamical system $(Y, \mathbb{S}, \sigma)$ with the fiber P. Assume that the following conditions are fulfilled:*

1. *the set Y is minimal;*
2. *the point $y_0 \in Y$ is strongly Poisson stable in the both directions, i.e., $q \in \omega_q \bigcap \alpha_q$ for any $q \in H(y_0)$;*
3. *for every $x \in X = P \times Y$ the positive semi-trajectory Σ_x^+ is conditionally compact;*
4. *the cocycle φ is monotone and strongly sub-homogeneous;*

5. *(1) the set* $Q := \overline{\varphi(\mathbb{S}_+, u_0, y_0)}$ *is compact;*
   (2) the positive semi-trajectory $\Sigma^+_{x_0} \subset Int(P) \times Y$ *with* $x_0 = (u_0, y_0)$*;*
   (3) $\omega_{x_0} \subset Int(P) \times Y$*.*

Then the following statements hold:

1. *for any* $u \in Int(P)$ *the set* $\omega_{(u,y_0)} \bigcap X_{y_0}$ *consists of a single point* $p = (p_u, y_0)$*;*
2. $\mathfrak{M}^{+\infty}_{y_0} \subseteq \mathfrak{M}^{+\infty}_{p}$*;*
3. $$\lim_{t \to +\infty} \rho(\varphi(t, u, y_0), \varphi(t, p_u, y_0)) = 0;$$
4. $p_u = p_{u_0}$ *for any* $u \in Int(P)$*.*

Proof. By Theorem 6.5 for any $u \in Int(P)$ the set $\omega_{(u,y_0)} \bigcap X_{y_0}$ consists of a single point $p := (p_u, y_0)$ and $\mathfrak{N}^{+\infty}_{y_0} \subseteq \mathfrak{N}^{+\infty}_{p}$. Denote by $H(p)$ the closure of the trajectory of p, then the set $H(p) = \omega_p$ is distal (see the proof of Theorem 6.5) in the negative direction, i.e.,

$$\inf_{t \leq 0} ||\varphi(t, v_1, q) - \varphi(t, v_2, q)|| > 0$$

for any $q \in \omega_{y_0}$ and $(v_1, q), (v_2, q) \in H(p)$ with $v_1 \neq v_2$. Now to finish the proof it is sufficient to apply Lemma 2.18 and Theorem 6.7. □

Corollary 6.10. *Under the conditions of Theorem 6.9*

- *if the point* y_0 *is stationary (respectively,* τ*-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff), then the point* p *is so;*
- *for any* $u \in Int(\mathbb{R}^n_+)$ *the point* $x := (u, y_0)$ *is asymptotically stationary (respectively, asymptotically* τ*-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff).*

Proof. This statement follows from Theorems 6.9 and 2.43. □

Remark 6.11. Theorems 6.5 and 6.9 (respectively, Corollaries 6.6 and 6.10) remain true if we replace the conditions of monotonicity and strongly sub-linearity of the cocycle φ (item (ii) of Theorem 6.5 or 6.9) by one of the following group of conditions:

1. (a) φ is monotone and sub-linear;
   (b) there exists a positive number t_0 such that $\varphi(t_0, \cdot, y_0)$ is strongly sub-linear, i.e.,
   $$\varphi(t_0, \lambda u, y_0) \ll \lambda \varphi(t_0, u, y_0)$$
   for any $\lambda > 1$ and $u \in Int(P)$.
2. (a) φ is monotone;
   (b) there exists a positive number t_0 such that $\varphi(t_0, \cdot, y_0)$ is strongly monotone, i.e.,
   $$\varphi(t_0, u_1, y_0) \ll \varphi(t_0, u_2, y_0)$$
   whenever $u_1 < u_2$ $(u_1, u_2 \in Int(P))$;
   (c) φ is sub-linear.

Proof. This statement can be proved using the same arguments as in the proof of Theorem 6.5 and using Lemma 5.6. □

7. Application.

7.1. **Ordinary differential equations.** Let $\mathbb{R}^n$ be an n-dimensional real Euclidean space with the norm $|\cdot|$. Let us consider a system of differential equations

$$u' = f(t,u), \tag{48}$$

where $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Along with equation (48) we consider its H-class, i.e., the family of equations

$$v' = g(t,v), \tag{49}$$

where $g \in H(f) := \overline{\{f^\tau : \tau\in\mathbb{R}\}}$, $f^\tau(t,u) := f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$ and by bar we mean the closure in $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, which is equipped with the compact-open topology and this topology can be generated by the following metric

$$d(f_1,f_2) := \sum_{k\geq 1}\frac{1}{2^k}\frac{d_k(f_1,f_2)}{1+d_k(f_1,f_2)},$$

where $d_k(f_1,f_2) := \sup\limits_{|t|\leq k,|x|\leq k}|f_1(t,x)-f_2(t,x)|$.

Condition (A1). The function f is said to be *regular*, that is, for every equation (49) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Denote by $\varphi(\cdot,v,g)$ the solution of equation (49), passing through the point $v\in\mathbb{R}^n$ at the initial moment $t=0$. Then the mapping $\varphi:\mathbb{R}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$ is well defined and satisfies the following conditions (see, e.g. [3, 39]):

1. $\varphi(0,v,g)=v$ for all $v\in\mathbb{R}^n$ and $g\in H(f)$;
2. $\varphi(t,\varphi(\tau,v,g),g^\tau)=\varphi(t+\tau,v,g)$ for all $v\in\mathbb{R}^n$, $g\in H(f)$ and $t,\tau\in\mathbb{R}_+$;
3. the mapping $\varphi:\mathbb{R}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y,\mathbb{R},\sigma)$ the shift dynamical system on Y induced from $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$, i.e., $\sigma(\tau,g)=g^\tau$ for $\tau\in\mathbb{R}$ and $g\in Y$. Then the equation (48) generates a cocycle $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{R},\sigma)\rangle$ and an NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X := \mathbb{R}^n\times Y$, $\pi := (\varphi,\sigma)$ and $h = pr_2 : X\to Y$.

Remark 7.1. Let F be a mapping from $H(f)\times\mathbb{R}^n\to\mathbb{R}^n$ defined by equality $F(g,x)=g(0,x)$, then F possesses the following properties:

1. F is continuous;
2. $F(g^\tau,x)=g(\tau,x)$ for any $(g,x,\tau)\in H(f)\times\mathbb{R}^n\times\mathbb{R}$;
3. equation (48) (and its H-class (49)) can be rewritten as follows
$$x' = F(\sigma(t,g),x)\quad (g\in H(f); \tag{50}$$
4. if φ is the cocycle generated by equation (50), then $\phi := \lambda\varphi$ $(\lambda\in(0,1))$ is the cocycle generated by equation
$$x' = \lambda F(\sigma(t,g),\lambda^{-1}x)\quad (g\in H(f)).$$

Let $\mathbb{R}^n_+ := \{x=(x_1,\dots,x_n)\in\mathbb{R}^n |\ x_i\geq 0 \text{ for } i=1,2,\dots,n\}$. Then it defines a partial order on $\mathbb{R}^n$: $u\leq v$ if and only if $v-u\in\mathbb{R}^n_+$.

Condition (A2). Equation (48) is monotone. This means that the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{R},\sigma)\rangle$ (or shortly φ) generated by (48) is monotone: if $u,v\in\mathbb{R}^n$ and $u\leq v$ then $\varphi(t,u,g)\leq\varphi(t,v,g)$ for all $t\geq 0$ and $g\in H(f)$.

Let K be a closed cone in $\mathbb{R}^n$ (for example, $K=\mathbb{R}^n_+$). The dual cone to K is the closed cone K^* in the dual space $\left(\mathbb{R}^n\right)^*$ of linear functions on $\mathbb{R}^n$, defined by

$$K^* := \{\lambda\in\left(\mathbb{R}^n\right)^* : \lambda(x)=\langle\lambda,x\rangle\geq 0 \text{ for any } x\in K\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

Recall ([49],[50, ChV]) that a function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be *quasi-monotone* if for any $(t,u),(t,v) \in \mathbb{R} \times \mathbb{R}^n$ and $\phi \in \left(\mathbb{R}^n\right)^*$ we have: $u \leq v$ and $\phi(u) = \phi(v)$ imply $\phi(f(t,u)) \leq \phi(f(t,v))$.

Lemma 7.2. [11] *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular and quasi-monotone function, then the following statements hold:*

1. *if $u \leq v$, then $\varphi(t,u,f) \leq \varphi(t,v,f)$ for any $t \geq 0$;*
2. *any function $g \in H(f)$ is quasi-monotone;*
3. *equation (48) is monotone.*

Remark 7.3. If the function $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n_+)$ is quasi-monotone, then F is also so, i.e., for any $(g,u),(g,v) \in H(f) \times \mathbb{R}^n_+$ and $\phi \in \left(\mathbb{R}^n_+\right)^*$ we have: $u \leq v$ and $\phi(u) = \phi(v)$ imply $\phi(g(0,u)) \leq \phi(g(0,v))$.

Theorem 7.4. [24] *Suppose that $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is quasi-monotone and $x_0, x_1 \in \mathbb{R}^n$. Let $\prec$ denote any one of the relations $=, <, \ll$. If $x_0 \prec x_1$ then $\varphi(t,,x_0,f) \prec \varphi(t,x_1,f)$, hence equation (48) is monotone. Conversely, if (48) is monotone then f is quasi-monotone.*

Lemma 7.5. *Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular and quasi-monotone function, then the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma) \rangle$, associated by equation (48), is monotone.*

Proof. This statement follows from Lemma 7.2 and Theorem 7.4. □

Condition (A3). Equation (48) is sub-linear. This means that the cocycle φ, generated by equation (48), is sub-linear, i.e.

$$\lambda \varphi(t,x,g) \leq \varphi(t, \lambda x, g)$$

for any $(\lambda, t, x, g) \in (0,1) \times \mathbb{R}_+ \times \mathbb{R}^n \times H(f)$.

Let E and F be two Banach spaces and $W \subseteq E$. Denote by $C(\mathbb{T} \times W, F)$ the space of all continuous functions $f : \mathbb{T} \times W \to F$ equipped with the compact-open topology, where $\mathbb{T} \subseteq \mathbb{S}$.

Lemma 7.6. *Assume that $f \in C(\mathbb{T} \times W, F)$, then the following statement hold:*

1. *if*
$$\lambda f(t,x) \leq f(t, \lambda x) \tag{51}$$
for any $(\lambda, t, x) \in (0,1) \times \mathbb{T} \times W$, then
$$\lambda g(t,x) \leq g(t, \lambda x) \tag{52}$$
for all $(\lambda, t, x, g) \in (0,1) \times \mathbb{T} \times W \times H(f)$;
2. *if*
$$\inf_{(t,x) \in \mathbb{T} \times W} |\lambda f(t,x) - f(t, \lambda x)| > 0, \tag{53}$$
for any $\lambda \in (0,1)$ then
$$\lambda g(t,x) < g(t, \lambda x) \tag{54}$$
for all $(\lambda, t, x, g) \in (0,1) \times \mathbb{T} \times W \times H(f)$.

Proof. Assume that condition (51) hold and $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{T}$ such that $f^{t_k} \to g$ as $k \to \infty$ in $C(\mathbb{T} \times W, F)$. From (51) we have

$$\lambda f(t + t_k, x) \leq f(t + t_k, \lambda x) \tag{55}$$

for any $(\lambda, t, x) \in (0,1) \times \mathbb{T} \times W$ and $k \in \mathbb{N}$. Passing to the limit in (55) as $k \to \infty$ we obtain (52).

To prove the second statement of Lemma we note that from (53) for any $\lambda \in (0,1)$ we have some positive number $\alpha = \alpha(\lambda)$ such that

$$|\lambda f(t,x) - f(t,\lambda x)| \geq \alpha \tag{56}$$

for any $(t,x) \in \mathbb{T} \times W$. If $g \in H(f)$ then there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that $f^{t_k} \to g$ as $k \to \infty$ in $C(\mathbb{T} \times W, F)$. From (56) we obtain

$$|\lambda f(t+t_k,x) - f(t+t_k,\lambda x)| \geq \alpha \tag{57}$$

for any $(t,x) \in \mathbb{T} \times W$ and $k \in \mathbb{N}$. Passing to the limit in (57) as $k \to \infty$ we will have

$$|\lambda g(t,x) - g(t,\lambda x)| \geq \alpha$$

for all $(t,x,g) \in \mathbb{T} \times W \times H(f)$ and, consequently,

$$\lambda g(t,x) \neq g(t,\lambda x) \tag{58}$$

for any $(t,x,g) \in \mathbb{T} \times W \times H(f)$. To finish the proof of the second statement of Lemma it is sufficient to note that from (52) and (58) it follows (54). □

Condition (A4). $f_i(t,x) \geq 0$ for all $x \in \Gamma_i$, $t \in \mathbb{R}$ and $i = 1, \ldots, n$, where $\Gamma_i := \{x \in \mathbb{R}^n_+ : \ x_i = 0\}$.

Remark 7.7. It is easy to see that if the function f satisfies Condition **(A4)**, then every function $g \in H(f)$ possesses the same property.

Theorem 7.8. [20, Ch.V] *Assume that $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is regular, quasi-monotone and $\langle \mathbb{R}^n_+, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ is the cocycle in $\mathbb{R}^n_+$ generated by equation (48) (respectively, by equation (50)).*

Then the condition

$$\mathcal{F}(g,x) \leq F(g,x) \ (\textit{respectively } \mathcal{F}(g,x) < F(g,x) \ \textit{or } \mathcal{F}(g,x) \ll F(g,x)) \tag{59}$$

for any $(g,x) \in H(f) \times \mathbb{R}^n_+$ (respectively, $(g,x) \in H(f) \times Int(\mathbb{R}^n_+)$) implies that $\phi(t,x,g) \leq \varphi(t,x,g)$ (respectively $\phi(t,x,g) < \varphi(t,x,g)$ or $\phi(t,x,g) \ll \varphi(t,x,g)$) for any $t \geq 0$, $g \in H(f)$ and $x \in \mathbb{R}^n_+$ (respectively, $(g,x) \in H(f) \times Int(\mathbb{R}^n_+)$), where $\mathcal{F} \in C(H(f) \times \mathbb{R}^n_+, \mathbb{R}^n)$ is some regular function and $\langle \mathbb{R}^n_+, \phi, (H(f), \mathbb{R}, \sigma)\rangle$ (shortly, ϕ) is the cocycle generated by equation

$$x' = \mathcal{F}(\sigma(t,g),x) \quad (g \in H(f)).$$

Lemma 7.9. *Assume that the function $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is regular, quasi-monotone and $f(t,0) \geq 0$ for any $t \in \mathbb{R}$. Then $\mathbb{R}^n_+$ is a positively invariant subset of the cocycle φ, generated by equation (48), i.e., $\varphi(t,x,g) \in \mathbb{R}^n_+$ for any $(t,x,g) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times H(f)$.*

Proof. Let $g \in H(f)$, then it is easy to check that under the condition of Lemma 7.9 the function g is also regular, quasi-monotone and $g(t,0) \geq 0$ for any $t \in \mathbb{R}$. Now to finish the proof of Lemma it is sufficient to apply Proposition 3.3 from [24, ChIII,p.49]. Lemma is proved. □

Lemma 7.10. [20, Ch.V] *Assume that $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is regular, quasi-monotone and the mapping $F : H(f) \times \mathbb{R}^n_+ \to \mathbb{R}^n$ ($F(g,x) = g(0,x)$ for any $(g,x) \in H(f) \times \mathbb{R}^n_+$) is sub-linear, i.e.,*

$$\lambda F(g,x) \leq F(g,\lambda x)$$

for any $(\lambda, g, x) \in (0,1) \times H(f) \times \mathbb{R}^n_+$. *Then the cocycle* φ, *generated by equation (48), is sub-linear. Moreover* φ *is strongly sub-linear if the following condition is satisfied:*

$$\lambda F_i(g,x) < F_i(g,\lambda x) \quad (F = (F_1, \dots, F_n)) \tag{60}$$

for all $i = 1, \dots, n$, $0 < \lambda < 1$, $x \in Int(\mathbb{R}^n_+)$ *and* $g \in H(f)$.

Lemma 7.11. *Assume that* $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ *is regular, quasi-monotone and sub-linear, i.e.,*

$$\lambda f(t,x) \le f(t, \lambda x)$$

for any $(\lambda, t, x) \in (0,1) \times \mathbb{R} \times \mathbb{R}^n_+$. *Then the cocycle* φ, *generated by equation (48), is sub-linear. Moreover* φ *is strongly sub-linear if the following condition is satisfied:*

$$\inf_{(t,x) \in \mathbb{R} \times Int(\mathbb{R}^n_+)} |\lambda f_i(t,x) - f_i(t, \lambda x)| > 0 \tag{61}$$

for any $\lambda \in (0,1)$ *and* $i = 1, \dots, n$.

Proof. Notice that by Lemma 7.6 the mapping $F \in C(H(f) \times \mathbb{R}^n_+, \mathbb{R}^n)$ $(F(g,x) = g(0,x)$ for any $(g,x) \in H(f) \times \mathbb{R}^n_+)$ is sub-linear. Let φ be the cocycle generated by equation (48) (respectively, by equation (50)), then it is easy to check that $\varphi_\lambda := \lambda\varphi$ is the cocycle generated by equation

$$x' = \mathcal{F}(\sigma(t,g), x) \quad (g \in H(f)),$$

where $\mathcal{F}(g,x) := \lambda F(g, \lambda^{-1}x)$ for any $(g,x) \in H(f) \times \mathbb{R}^n_+$. Since F is sub-linear we have

$$\mathcal{F}(g,x) \le F(g,x)$$

for any $(g,x) \in H(f) \times \mathbb{R}^n_+$. According to Lemma 7.10 the cocycle φ is sub-linear.

Now we will show that condition (61) implies (60)). In fact. If $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{R}$ such that $f^{t_k} \to g$ as $k \to \infty$ in the space $C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$. From (61) it follows that for any $\lambda \in (0,1)$ there exists a positive number $\alpha(\lambda)$ such that

$$|\lambda f_i(t + t_k, x) - f_i(t + t_k, \lambda x)| \ge \alpha(\lambda) \tag{62}$$

for any $(t,x) \in \mathbb{R} \times Int(\mathbb{R}^n_+)$, $k \in \mathbb{N}$ and $i = 1, \dots, n$. Passing to the limit in (62) as $k \to \infty$ we obtain

$$|\lambda F_i(g,x) - F_i(g, \lambda x)| \ge \alpha(\lambda) \tag{63}$$

for all $(g,x) \in H(f) \times Int(\mathbb{R}^n_+)$ and $i = 1, \dots, n$, where $F_i(g,x) = g_i(0,x)$. From (62) and (63) we obtain

$$\lambda F_i(g,x) < F_i(g, \lambda x)$$

for any $(\lambda, g, x) \in (0,1) \times H(f) \times Int(\mathbb{R}^n_+)$ and $i = 1, \dots, n$. To finish the proof of the second statement of Lemma it is sufficient to apply Lemma 7.10 (second statement). □

Lemma 7.12. *Let* $\psi \in C(\mathbb{T}, Y)$ *and* $h : \psi(\mathbb{T}) \to X$ *be a continuous function. Then the function* $\varphi \in C(\mathbb{T}, X)$ *defined by equality*

$$\varphi(t) := h(\psi(t)) \quad (t \in \mathbb{T}) \tag{64}$$

is comparable by character of recurrence with the function ψ, *i.e.,* $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$.

Proof. If we suppose that this statement is not true, then there exists a sequence $\{t_k\} \in \mathfrak{N}_\psi$ such that $\{t_k\} \notin \mathfrak{N}_\varphi$. This means that there are a positive number ε_0, compact subset $K \subset \mathbb{T}$ and a subsequence $\{t_{m_k}\} \subseteq \{t_k\}$ such that

$$\max_{t\in K} \rho(\varphi(t + t_{m_k}), \varphi(t)) \geq \varepsilon_0$$

for any $k \in \mathbb{N}$. From (65) it follows that there exists a sequence $\{\tau_k\} \subset K$ such that

$$\rho(\varphi(\tau_k + t_{m_k}), \varphi(\tau_k)) \geq \varepsilon_0 \tag{65}$$

for any $k \in \mathbb{N}$. Since K is a compact subset of $\mathbb{T}$, then without loss of generality we can suppose that the sequence $\{\tau_k\}$ converges. Denote by $\tau_0 := \lim_{k\to\infty} \tau_k$, $x_0 := \psi(\tau_0)$, $x_k := \psi(\tau_{m_k})$ and $y_k := \psi(\tau_{m_k} + t_{m_k})$, then (65) we can rewrite as follows

$$\rho(h(x_k), h(y_k)) \geq \varepsilon_0 \tag{66}$$

for any $k \in \mathbb{N}$.

On the other hand we have

$$\rho(h(x_k), h(y_k)) \leq \rho(h(x_k), h(x_0)) + \rho(h(x_0), h(y_k))$$

for any $k \in \mathbb{N}$. It is easy to see that

$$h(x_k) \to h(x_0)$$

as a$k \to \infty$. We will show that

$$h(y_k) \to h(x_0)$$

as $k \to \infty$. Indeed, taking into consideration that $\{t_k\} \in \mathfrak{N}_\psi$ we have

$$\sup_{t\in K} \rho(\psi(t + t_k), \psi(t)) \to 0$$

as $k \to \infty$ and, consequently,

$$\rho(\psi(\tau_k + t_{m_k}), \psi(\tau_k)) \to 0$$

as $k \to \infty$, i.e.,

$$\rho(y_k, x_k) \to 0 \tag{67}$$

as $k \to \infty$. Since $x_k \to x_0$ as $k \to \infty$, then from (67) we obtain $y_k \to x_0$ as $k \to \infty$. By continuity of function h we get

$$\lim_{k\to\infty} \rho(h(x_k), h(y_k)) = 0. \tag{68}$$

The relations (66) and (68) are contradictory. The obtained contradiction proves our statement. Lemma is proved. □

Corollary 7.13. *Under the conditions of Lemma 7.12 if the function $\psi \in C(\mathbb{T}, Y)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the function ψ defined by equality (64) is so.*

Remark 7.14. Note that in the case when $\mathbb{T} = \mathbb{R}$ Lemma 7.12 was established in the work [45, Ch.II].

Remark 7.15. 1. A sub-linear function $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ satisfies to condition (61) if and only if for any $\lambda \in (0, 1)$ there exists a positive number $\alpha(\lambda)$ such that

$$f(t, \lambda x) - \lambda f(t, x) \geq \alpha(\lambda)e$$

for any $(t, x) \in \mathbb{R} \times Int(\mathbb{R}^n_+)$, where $e := (1, \ldots, 1) \in Int(\mathbb{R}^n_+)$.

2. Consider the function

$$h(t)=2+\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\quad(t\in\mathbb{R})$$

which is bounded and Levitan almost periodic (see Corollary 7.13), but not Bohr almost periodic [31, Ch.V,p.212-213] (see also [32, ChIV,p.58-59]), because h is not uniformly continuous.

It is easy to check that the function $f\in C(\mathbb{R}\times\mathbb{R}_+,\mathbb{R})$, defined by equality

$$f(t,x)=-x+\frac{x}{x+1}+h(t)$$

possesses the following properties:

1. it is Levitan almost periodic in $t\in\mathbb{R}$ uniformly with respect to x on every compact subset from $\mathbb{R}_+$;
2. the function f is not Bohr almost periodic in $t\in\mathbb{R}$;
3. f satisfies to conditions (61). In fact, if $\lambda\in(0,1)$ we have

$$f(t,\lambda x)-\lambda f(t,x)\geq 1-\lambda=\alpha(\lambda)>0$$

for any $(t,x)\in\mathbb{R}\times\mathbb{R}_+$.

Definition 7.16. A solution $\varphi(t,u_0,f)$ of equation (48) is said to be compact on $\mathbb{R}_+$ if the set $Q^{(u_0,f)}:=\overline{\varphi(\mathbb{R}_+,u_0,f)}$ is a compact subset of $\mathbb{R}^n$, where $\varphi(\mathbb{R}_+,u_0,f):=\{\varphi(t,u_0,f):t\in\mathbb{R}_+\}$.

Definition 7.17 ([15, 45, 46])**.** A solution $\varphi(t,u_0,f)$ of equation (48) is called *compatible* (respectively, *strongly compatible*, *uniformly compatible*) if the motion $\pi(t,x_0)$ (here $x_0:=(u_0,f)$) is comparable (respectively, strongly comparable, uniformly comparable) with $\sigma(t,f)$ by character of recurrence.

Recall that a function $\varphi\in C(\mathbb{T},E)$ (respectively, $f\in C(\mathbb{T}\times W,E_2)$, where $W\subseteq E_1$) *possess the property* (A), if the motion $\sigma(\cdot,\varphi)$ generated by φ possesses this property in the shift dynamical system $(C(\mathbb{T},E),\mathbb{T},\sigma)$. A function $f\in C(\mathbb{T}\times W,E_2)$, where $W\subseteq E_1$, *possess the property* (A) *in* $t\in\mathbb{T}$ *uniformly w.r.t.* u *on every compact subset of* W if the motion $\sigma(\cdot,f)$ generated by f possesses this property in the shift dynamical system $(C(\mathbb{T}\times W,E),\mathbb{T},\sigma)$. In the quality of the property (A) there can stand Lagrange stability, periodicity, asymptotic periodicity, almost periodicity, asymptotic almost periodicity and so on.

In accordance with the above convention, the function $f\in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$ is called Poisson stable in $t\in\mathbb{R}$ uniformly w.r.t. u on every compact subset from $\mathbb{R}^n_+$ in the both directions if there exist two sequences $t_n^{\pm}\to\pm\infty$ as $n\to\infty$ such that $f^{t_n^{\pm}}$ converges to f in $C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$.

Theorem 7.18. *Let $f\in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$ be regular, Poisson stable in $t\in\mathbb{R}$ uniformly w.r.t. u on every compact subset from $\mathbb{R}^n_+$ in the both directions. Suppose that f is quasi-monotone and*

$$\inf_{(t,x)\in\mathbb{R}\times\mathbb{R}^n_+}\big(\lambda f_i(t,x)-f_i(t,\lambda x)\big)>0 \tag{69}$$

for any $\lambda\in(0,1)$ and $i=1,\dots,n$.

If for every $u\in\mathbb{R}^n_+$, the set $Q_+^{(u,f)}:=\overline{\varphi(\mathbb{R}_+,u,f)}$ is a compact subset of $\mathbb{R}^n_+$ and there exists $u_0\in Int(\mathbb{R}^n_+)$ such that $Q_+^{(u_0,f)}\subset Int(\mathbb{R}^n_+)$. Then for any $u\gg 0$ the following statements (1)-(5) hold:

1. *the set* $\omega_{(u_0,f)}\bigcap X_f$ *consists of a single point* $p_0=(p_{u_0},f)$, *where* $\omega_{(p_{u_0},f)}$ *is the* ω*-limit set of the motion* $\pi(t,x_0,f))$ $(x_0=(p_{u_0},f))$ *of the skew-product dynamical system* $(X,\mathbb{R}_+,\pi)$ $(X:=\mathbb{R}^n_+\times H(f),\ \pi:=(\varphi,\sigma))$ *and* $X_f:=\mathbb{R}^n_+\times\{f\}$;
2. *the solution* $\varphi(t,p_{u_0},f)$ *of equation (48) is defined on* $\mathbb{R}$, $\overline{\varphi(\mathbb{R},p_{u_0},f)}\subseteq Q_+^{(p_{u_0},f)}$ *and it is compatible;*
3. $$\lim_{t\to\infty}|\varphi(t,u_0,f)-\varphi(t,p_{u_0},f)|=0;$$
4. *for any* $u\in Int(\mathbb{R}^n_+)$ *the set* $\omega_{(u,f)}\bigcap X_f$ *consists of a single point* $p=(p_u,f)$ *and the solution* $\varphi(t,p_u,f)$ *of equation (48) is compatible and* $\lim\limits_{t\to\infty}|\varphi(t,u,f)-\varphi(t,p_u,f)|=0$;
5. $p_{u_1}=p_{u_2}$ *for any* $u_1,u_2\in Int(\mathbb{R}^n_+)$, *i.e., equation (48) has a unique bounded globally attracting in* $Int(\mathbb{R}^n_+)$ *solution* $p:\mathbb{R}\to Int(\mathbb{R}^n)$ *with* $\overline{p(\mathbb{R})}\subset Int(\mathbb{R}^n_+)$.

Proof. Note that sub-linearity and (61) imply (69). In fact, as induced in Remark 7.15 (item 1.), for any $\lambda\in(0,1)$ there exists a positive number $\alpha(\lambda)$ such that

$$f(t,\lambda x)-\lambda f(t,x)\gg\alpha(\lambda)e \tag{70}$$

for all $(t,x)\in\mathbb{R}\times Int(\mathbb{R}^n_+)$. This, together with the continuity of f, implies (69).

From (69) it follows that $f(t,0)\geq(1-\lambda)^{-1}e\gg 0$, which deduces that $g(t,0)\gg 0$ for all $t\in\mathbb{R}$.

By Theorem 7.8 we have $\phi(t,x,g)\ll\varphi(t,x,g)$ for any $t\geq 0$, $g\in H(f)$ and $x\in\mathbb{R}^n_+$, where φ is the cocycle generated by equation (48) (respectively, equation (50)) and ϕ is the cocycle defined by equation $x'=0$, i.e., $\phi(t,x,g)=x$ for any $x\in\mathbb{R}^n_+$, $t\geq 0$ and $g\in H(f)$. Thus we have $\varphi(t,u,g)\geq u$ for any $(t,u,g)\in\mathbb{R}_+\times\mathbb{R}^n_+\times H(f)$. This means that $\omega_{(u,f)}\subset Int(\mathbb{R}^n_+)\times H(f)$. The statements (1)-(5) follow immediately from Theorem 6.5. □

Here is an example illustrating the theorem proved above.

Example 3. Consider the differential equation

$$x'=-x+\frac{x}{x+1}+h(t),\ \ (x\in\mathbb{R}_+) \tag{71}$$

where $h(t)=2+\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}$ $(t\in\mathbb{R})$. Along this equation (71) we consider its H-class

$$x'=-x+\frac{x}{x+1}+g(t),\quad (g\in H(h)) \tag{72}$$

where $H(h):=\overline{\{h^\tau|\ \tau\in\mathbb{R}\}}$ and by bar is denoted the closure in the space $C(\mathbb{R},\mathbb{R})$. Let $f(t,x):=-x+\frac{x}{x+1}+h(t)$ $((t,x)\in\mathbb{R}\times\mathbb{R}_+)$, then

$$g(t,x)\leq -x+4:=\bar{h} \tag{73}$$

for any $(t,x,g)\in\mathbb{R}\times R_+\times H(h)$. Denote by $\langle\mathbb{R}_+,\varphi,H(h)\rangle$ (respectively, $(\mathbb{R}_+,\phi)$) the cocycle generated by family of equations (72) (respectively, semigroup dynamical system generated by equation $x'=-x+4$), then by Theorem 7.8 we have

$$\varphi(t,u,g)\leq\phi(t,u)=(u-4)e^{-t}+4$$

for any $(t,x,g)\in\mathbb{R}_+\times R_+\times H(h)$. Thus, all the conditions of the Theorem 7.18 are fulfilled (see Remark 7.14, item 2.) and, consequently, equation (71) has a unique bounded, strictly positive, Levitan almost periodic solution that is globally uniformly asymptotically stable.

Lemma 7.19. *Assume that $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is almost automorphic (respectively, recurrent) in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$ and $\varphi\in C(\mathbb{R},\mathbb{R}^n)$ is a bounded on $\mathbb{R}$ and compatible solution of equation (48). Then φ is also almost automorphic (respectively, recurrent).*

Proof. Since the function f is almost automorphic (respectively, recurrent), then it is Levitan almost periodic (respectively, almost recurrent) in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$. Taking into account that φ is a compatible solution of equation (48), then by Theorem 2.40 it is Levitan almost periodic (respectively, almost recurrent). Now to finish the proof of Lemma it is sufficient to show that the function φ is Lagrange stable. Note that φ is bounded on $\mathbb{R}$ and, consequently, $K := \overline{\varphi(\mathbb{R})}$ is a compact subset of $\mathbb{R}^n$. Since f is almost automorphic (respectively, recurrent), then it is Lagrange stable. This means, in particular, that f is bounded on $\mathbb{R}\times K$, i.e., there exists a $C>0$ such that $|f(t,\varphi(t))|\leq C$ for any $t\in\mathbb{R}$. Thus we have

$$|\varphi'(t)| = |f(t,\varphi(t))| \leq C$$

for all $t\in\mathbb{R}$ and, consequently, φ is uniformly continuous on $\mathbb{R}$. Thus the function φ is Lagrange stable and Levitan almost periodic (respectively, almost recurrent) and, consequently, it is almost automorphic (respectively, recurrent). Lemma is proved. □

Corollary 7.20. *Under the conditions of Theorem 7.18 if the function $f\in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) in time $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)}\bigcap X_f$ consists of a single point $p_0=(p_{u_0},f)$;*
2. *$\varphi(t,p_{v_0},f)$ is a stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) solution of equation (48);*
3. *$\lim\limits_{t\to+\infty}|\varphi(t,u_0,f)-\varphi(t,p_{u_0},f)|=0$, i.e., $\varphi(t,u_0,f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically almost automorphic, asymptotically almost recurrent in the sense of Bebutov, asymptotically recurrent, asymptotically Poisson stable);*
4. *for any $u\in Int(\mathbb{R}^n_+)$ there exists a unique point $p=(p_u,f)\in\omega_x$ $(x=(u,f))$ such that the solution $\varphi(t,p_u,f)$ of equation (48) defined on $\mathbb{R}$ with values from the compact $Q_+^{(u,f)}$;*
5. *equation (48) has a unique solution with the following properties:*
   (a) *it is bounded;*
   (b) *globally attracting in $Int(\mathbb{R}^n_+)$ with $p:\mathbb{R}\to Int(\mathbb{R}^n_+)$;*
   (c) *the solution p is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable).*

Proof. This statement follows from Theorem 2.40, 7.18 and Lemma 7.19. □

Definition 7.21. Let $f\in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$. The set $H(f)$ is said to be minimal, if it is a minimal set of shift dynamical system $(C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n),\mathbb{R},\sigma)$.

Definition 7.22. A function $f\in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$ is said to be strongly Poisson stable if every function $g\in H(f)$ is Poisson stable.

Theorem 7.23. *Under the conditions of Theorem 7.18 if additionally the set $H(f)$ is minimal and f is strongly Poisson stable, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *the solution $\varphi(t, p_{u_0}, f)$ of equation (48) is defined on $\mathbb{R}$, $\overline{\varphi(\mathbb{R}, p_{u_0}, f)} \subseteq Q_+^{(u_0,f)}$ and it is strongly compatible;*
3. $$\lim_{t\to\infty} |\varphi(t, u_0, f) - \varphi(t, p_{u_0}, f)| = 0;$$
4. *for any $u \in Int(\mathbb{R}^n_+)$ the set $\omega_{(u,f)} \bigcap X_f$ consists of a single point $p = (p_u, f)$, the solution $\varphi(t, p_u, f)$ is strongly compatible and $\lim\limits_{t\to\infty} |\varphi(t, u, f) - \varphi(t, p_u, f)| = 0$;*
5. *$p_{u_1} = p_{u_2}$ for any $u_1, u_2 \in Int(\mathbb{R}^n_+)$, i.e., equation (48) has a unique bounded globally attracting in $Int(\mathbb{R}^n_+)$ solution $p : \mathbb{R} \to Int(\mathbb{R}^n)$ with $\overline{p(\mathbb{R})} \subset Int(\mathbb{R}^n_+)$.*

Proof. This statement can be proved using the same arguments as in the proof of Theorem 7.18, but instead of Theorem 6.5 we need to apply Theorem 6.9. □

Corollary 7.24. *Under the conditions of Theorem 7.23 if the function $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{R}$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *$\varphi(t, p_{u_0}, f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (48);*
3. *$\lim\limits_{t\to+\infty} |\varphi(t, u_0, f) - \varphi(t, p_{u_0}, f)| = 0$, i.e., $\varphi(t, u_0, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable);*
4. *for any $u \in Int(\mathbb{R}^n_+)$ there exists a unique point $p = (p_u, f) \in \omega_x$ $(x = (u, f))$ such that the solution $\varphi(t, p_u, f)$ of equation (48) defined on $\mathbb{R}$ with values from the compact $Q_+^{(u,f)}$;*
5. *the solution $\varphi(t, p_u, f)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable);*
6. *$\lim\limits_{t\to+\infty} |\varphi(t, u, f) - \varphi(t, p_u, f)| = 0$.*

Proof. This statement follows from Theorems 7.23, 2.43 and Corollary 6.10. □

7.2. **Functional-differential equations with finite delay.** Let us first recall some notions and notations from [23]. Let $r > 0$, $C([a, b], \mathbb{R}^n)$ be the Banach space of all continuous functions $\varphi : [a, b] \to \mathbb{R}^n$ equipped with the sup–norm. If $[a, b] = [-r, 0]$, then we set $\mathcal{C} := C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. We will define $u_t \in \mathcal{C}$ for any $t \in [\sigma, \sigma + A]$ by the equality $u_t(\theta) := u(t + \theta)$, $-r \leq \theta \leq 0$. Consider a functional differential equation

$$u'(t) = f(t, u_t), \tag{74}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous.

Denote by $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ the space of all continuous mappings $f:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n$ equipped with the compact-open topology. On the space $C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ is defined (see, e.g. [16, ChI] and [45, ChI]) a shift dynamical system $(C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n),\mathbb{R},\sigma)$, where $\sigma(\tau,f):=f^\tau$ for any $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ and $\tau\in\mathbb{R}$ and f^τ is τ-translation of f, i.e. $f^\tau(t,u):=f(t+\tau,u)$ for any $(t,u)\in\mathbb{R}\times\mathcal{C}$. Let us set $H(f):=\overline{\{f^\tau:\tau\in\mathbb{R}\}}$.

Along with equation (74) let us consider the family of equations

$$v'(t)=g(t,v_t),\tag{75}$$

where $g\in H(f)$.

Condition (F1). The function f is said to be *regular*, that is, for every equation (75) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Remark 7.25. Denote by $\tilde{\varphi}(t,u,f)$ the solution of equation (74) defined on $[-r,+\infty)$ (respectively, on $\mathbb{R}$) with the initial condition $u\in\mathcal{C}$. By $\varphi(t,u,f)$ we will denote below the trajectory of equation (74), corresponding to the solution $\tilde{\varphi}(t,u,f)$, i.e., a mapping from $\mathbb{R}_+$ (respectively, $\mathbb{R}$) into $\mathcal{C}$, defined by $\varphi(t,u,f)(s):=\tilde{\varphi}(t+s,u,f)$ for any $t\in\mathbb{R}_+$ (respectively, $t\in\mathbb{R}$) and $s\in[-r,0]$. Below we will use the notions of "solution" and "trajectory" for equation (74) as synonymous concepts.

It is well-known [3, 39] that the mapping $\varphi:\mathbb{R}_+\times\mathcal{C}\times H(f)\to\mathcal{C}$ possesses the following properties:

1. $\varphi(0,v,g)=v$ for any $v\in\mathcal{C}$ and $g\in H(f)$;
2. $\varphi(t+\tau,v,g)=\varphi(t,\varphi(\tau,v,g),\sigma(\tau,g))$ for any $t,\tau\in\mathbb{R}_+$, $v\in\mathcal{C}$ and $g\in H(f)$;
3. the mapping φ is continuous.

Thus equation (74) generates a cocycle $\langle\mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ and an NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $Y:=H(f)$, $X:=\mathcal{C}\times Y$, $\pi:=(\varphi,\sigma)$ and $h:=pr_2:X\to Y$.

Remark 7.26. Denote by $\mathcal{B}:=\{f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n):f$ is continuous in t uniformly w.r.t u on any bounded subset of $\mathcal{C}$ and $f(B)$ is a bounded subset of $\mathbb{R}^n$ for any bounded subset $B\subset\mathbb{R}\times\mathcal{C}\}$, equipped with the topology of uniform convergence on every bounded subset of $\mathbb{R}\times\mathcal{C}$. This topology can be defined by the following metric

$$d(f_1,f_2):=\sum_{k\geq1}\frac{1}{2^k}\frac{d_k(f_1,f_2)}{1+d_k(f_1,f_2)},$$

where $d_k(f_1,f_2):=\sup\limits_{|t|\leq k,||\phi||_{\mathcal{C}}\leq k}|f_1(t,\phi)-f_2(t,\phi)|$. Note that

1. the metric space $(\mathcal{B},d)$ is complete;
2. the subset $\mathcal{B}\subset C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n))$ is translation invariant, i.e. $f^h\in\mathcal{B}$ for any $f\in\mathcal{B}$ and $h\in\mathbb{R}$;
3. the mapping $\sigma:\mathbb{R}\times\mathcal{B}\to\mathcal{B}$, defined by equality $\sigma(h,f):=f^h$ for any $(h,f)\in\mathbb{R}\times\mathcal{B}$, is continuous.

Thus on the space $(\mathcal{B},d)$ is defined a shift dynamical system $(\mathcal{B},\mathbb{R},\sigma)$.

Denote by $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ the shift dynamical system on Y induced from $(C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n),\mathbb{R},\sigma)$, i.e., $\sigma(\tau,g)=g^\tau$ for $\tau\in\mathbb{R}$ and $g\in Y$. Then the equation (74) generates a cocycle $\langle\mathcal{C},\varphi,(Y,\mathbb{R},\sigma)\rangle$ and an NDS $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$, where $X:=\mathcal{C}\times Y$, $\pi:=(\varphi,\sigma)$ and $h=pr_2:X\to Y$.

Remark 7.27. Let F be a mapping from $H(f)\times\mathcal{C}\to\mathbb{R}^n$ defined by equality $F(g,x)=g(0,x)$ for any $\mathcal{C}$, then F possesses the following properties:

1. F is continuous;

2. $F(g^\tau, x) = g(\tau, x)$ for any $(g, x, \tau) \in H(f) \times \mathcal{C} \times \mathbb{R}$;
3. equation (74) (and its H-class (75)) can be rewritten as follows

$$u'(t) = F(\sigma(t, g), u_t) \quad (g \in H(f)); \tag{76}$$

4. if φ is the cocycle generated by equation (76), then $\phi := \lambda\varphi$ ($\lambda \in (0, 1)$) is the cocycle generated by equation

$$u'(t) = \lambda F(\sigma(t, g), \lambda^{-1} u_t) \quad (g \in H(f)).$$

Let $\mathcal{C}_+ := \{u \in \mathcal{C} : \ u \geq 0$, i.e., $u(t) \geq 0$ for any $t \in [-r, 0]\}$ be the cone of nonnegative functions in $\mathcal{C}$. By $\mathcal{C}_+$ on the space $\mathcal{C}$ is defined a partial order: $u \leq v$ if and only if $v - u \in \mathcal{C}_+$.

Condition (F2). Equation (74) is monotone, that is, the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ generated by (74) possesses the following property: if $u \leq v$, then $\varphi(t, u, g) \leq \varphi(t, v, g)$ for any $t \geq 0$ and $g \in H(f)$.

Recall (see, e.g. [49], [50, ChV]) that a function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is said to be *quasi-monotone* if $(t, u), (t, v) \in \mathbb{R} \times \mathcal{C}$, $u \leq v$, and $u_i(0) = v_i(0)$ for some i, then $f_i(t, u) \leq f_i(t, v)$.

Lemma 7.28. [11] *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a quasi-monotone function, then the following statements hold:*

1. *if $u \leq v$, then $\varphi(t, u, f) \leq \varphi(t, v, f)$ for any $t \geq 0$;*
2. *any function $g \in H(f)$ is quasi-monotone;*
3. *$u \leq v$ implies $\varphi(t, u, g) \leq \varphi(t, v, g)$ for any $t \geq 0$ and $g \in H(f)$.*

Corollary 7.29. *Let $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ be a regular and quasi-monotone function, then the cocycle $\langle \mathcal{C}, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$, associated by equation (74), is monotone.*

Remark 7.30. If the function $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ is quasi-monotone, then F ($F(g, x) = g(0, x)$ for any $(g, x) \in H(f) \times \mathcal{C}$) is also so, i.e., for any $(g, u), (g, v) \in H(f) \times \mathcal{C}_+$, $u \leq v$, and $u_i(0) = v_i(0)$ for some i, then $F_i(g, u) \leq F_i(g, v)$.

Theorem 7.31. [50, Ch.IV] *Let $f, g \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ be regular and assume that either f or g is quasi-monotone. Assume also that $f(t, u) \leq g(t.u)$ for all $(t, u) \in \mathbb{R} \times \mathcal{C}_+$. If $u, v \in \mathcal{C}_+$ satisfy $u \leq v$, then $\varphi(t, u, f) \leq \varphi(t, v, g)$ for all $t \geq 0$.*

Corollary 7.32. *Assume that $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ is regular, quasi-monotone and $\langle \mathcal{C}_+, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ is the cocycle in $\mathcal{C}_+$ generated by equation (74) (respectively, by its H-class (75)). Then the condition*

$$\mathcal{F}(g, u) \leq F(g, u)$$

for any $(g, x) \in H(f) \times \mathcal{C}_+$ implies that

$$\phi(t, u, g) \leq \varphi(t, u, g)$$

for any $t \geq 0$, $g \in H(f)$ and $x \in \mathcal{C}_+$, where $\mathcal{F} \in C(H(f) \times \mathcal{C}_+, \mathbb{R}^n)$ is some regular function and $\langle \mathcal{C}_+, \phi, (H(f), \mathbb{R}, \sigma)\rangle$ (shortly, ϕ) is the cocycle generated by equation

$$u' = \mathcal{F}(\sigma(t, g), u_t) \quad (g \in H(f)).$$

Proof. This statement follows directly from Theorem 7.31. □

Condition (F3). The cone $\mathcal{C}_+$ is positively invariant with respect to cocycle φ generated by equation (74), i.e., $\varphi(t, \phi, g) \in \mathcal{C}_+$ for any $(t, \phi, g) \in \mathbb{R}_+ \times \mathcal{C}_+ \times H(f)$.

Lemma 7.33. *Assume that the function $f \in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ is regular, quasi-monotone and $f(t,0)\geq 0$ for any $t\in\mathbb{R}$. Then $\mathcal{C}_+$ is a positively invariant subset of the cocycle φ, generated by equation (74), i.e., $\varphi(t,x,g)\in\mathcal{C}_+$ for any $(t,x,g)\in\mathbb{R}_+\times\mathcal{C}_+\times H(f)$.*

Proof. Let $g\in H(f)$, then it is easy to check that under the condition of Lemma 7.33 the function g is also regular. According to Lemma 7.28 the function g is also quasi-monotone and $g(t,0)\geq 0$ for any $t\in\mathbb{R}$. Now to finish the proof of Lemma it is sufficient to apply Theorem 4.2 from [24, Ch.IV,p.72]. Lemma is proved. □

Condition (F4). For any bounded subset $A\subset\mathcal{C}$ the set $f(\mathbb{R}\times A)$ is bounded in $\mathbb{R}^n$.

Lemma 7.34. *Let $\varphi(t,u,f)$ be a bounded on $\mathbb{R}_+$ solution of equation (74), then under the condition (**F4**) the set $\varphi(\mathbb{R}_+,u,f)\subset\mathcal{C}$ is pre-compact.*

Proof. This statement follows from the Lemmas 2.2.3 and 3.6.1 in [23]. □

Definition 7.35. A solution $\varphi(t,u_0,f)$ of equation (74) is said to be:

- *(positively) uniformly stable*, if for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $||\varphi(t_0,u,f)-\varphi(t_0,u_0,f)||_{\mathcal{C}}<\delta$ ($t_0\in\mathbb{R}$, $u\in\mathcal{C}$) implies $||\varphi(t,x,f)-\varphi(t,x_0,f)||_{\mathcal{C}}<\varepsilon$ for any $t\geq t_0$;
- *compact on $\mathbb{R}_+$* if the set $Q:=\overline{\varphi(\mathbb{R}_+,u_0,f)}$ is a compact subset of $\mathcal{C}$, where by bar we mean the closure in $\mathcal{C}$ and $\varphi(\mathbb{R}_+,u_0,f):=\{\varphi(t,u_0,f):\ t\in\mathbb{R}_+\}$.

Condition (F5). Every solution $\varphi(t,v,g)$ of every equation (75) is bounded on $\mathbb{R}_+$ and uniformly stable.

Lemma 7.36. [11] *Let $\varphi(t,u,f)$ be a bounded on $\mathbb{R}_+$ solution of equation (74), then under the Condition (**F5**) the set $\varphi(\mathbb{R}_+,u,f)\subset\mathcal{C}$ is pre-compact.*

Condition (F6). Equation (74) is sub-linear. This means that the cocycle φ generated by equation (74) is strict sub-linear, i.e.,

$$\lambda\varphi(t,u,g)\leq\varphi(t,\lambda u,g)$$

for any $(\lambda,t,u,g)\in(0,1)\times\mathbb{R}_+\times Int(\mathcal{C}_+)\times H(f)$.

Lemma 7.37. *Assume that $f\in C(\mathbb{R}\times\mathcal{C},\mathbb{R}^n)$ is regular, quasi-monotone and sub-linear, i.e.,*

$$\lambda f(t,u)\leq f(t,\lambda u)$$

for any $(\lambda,t,u)\in(0,1)\times\mathbb{R}\times\mathcal{C}$. Then the cocycle φ, generated by equation (74), is sub-linear.

Proof. Notice that by Lemma 7.6 the mapping $F\in C(H(f)\times\mathcal{C},\mathbb{R}^n)$ ($F(g,u)=g(0,u)$ for any $(g,u)\in H(f)\times\mathcal{C}$) is sub-linear. Let φ be the cocycle generated by equation (74) (respectively, by its H-class (76)), then it easy to check that $\varphi_\lambda:=\lambda\varphi$ is the cocycle generated by equation

$$u'=\mathcal{F}(\sigma(t,g),u_t)\quad(g\in H(f)),$$

where $\mathcal{F}(g,u):=\lambda F(g,\lambda^{-1}u)$ for any $(g,u)\in H(f)\times\mathcal{C}$. Since F is sub-linear we have

$$\mathcal{F}(g,u)\leq F(g,u)$$

for any $(g,u)\in H(f)\times\mathcal{C}$. According to Theorem 7.31 (item (iii)) the cocycle φ is sub-linear. □

Let E_1, E_2 be two Banach spaces. Denote by $L(E_1, E_2)$ the space of all linear bounded operators $A : E_1 \to E_2$ equipped with the operator norm. Consider the linear equation

$$u'(t) = A(t)u_t, \tag{77}$$

where $A \in C(\mathbb{R}, L(\mathcal{C}, \mathbb{R}^n))$.

Condition (F7) [24, Ch.V,p.75]. The operator $A \in C(\mathbb{R}, L(\mathcal{C}, \mathbb{R}^n))$ (respectively, equation (77)) is said to be cooperative, if for all $t \in \mathbb{R}$ and $u \geq 0$ with $u(0) \in \partial(\mathbb{R}^n_+)$ one of the conditions (i)-(ii) are fulfilled:

1. $u_i(-r) > 0$ and $u_i(0) = 0$ for any $i \in \{1, \ldots, n\}$, or
2. $u_i(s) > 0$ for any $-r \leq s \leq 0$ and $i \in \{1, \ldots, n\}$, there exists $i_0 \in \{1, \ldots, n\}$ such that $u_{i_0}(0) = 0$ and $(A(t)u)_{i_0} > 0$.

We suppose that $U := \mathbb{R} \times D$, where D is an open subset of $\mathcal{C}$. Suppose that $\frac{\partial f}{\partial u}(t, v)$ exists and $\frac{\partial f}{\partial u} \in C(\mathbb{R} \times D, L(\mathcal{C}, \mathbb{R}^n))$. In this case, $u(t, t_0, u)$ is continuously differentiable (see, for example, [24, Ch.V] and the bibliography therein) in its last argument and $v(t, t_0, \kappa) := \frac{\partial f}{\partial u}(t, t_0, u)\kappa$ satisfies the variational equation

$$v'(t) = \frac{\partial f}{\partial u}(t, u_t(t_0, u))v_t.$$

A function f (or equation (74)) is said to be cooperative[24, Ch.V] if for all $(t, v) \in J \times D$ the function $\kappa \to \frac{\partial f}{\partial u}(t, v)\kappa$ satisfies Condition (**F7**).

Recall that a subset S of an ordered Banach space E is p-convex if it contains the line segment $\{tu + (1 - t)v |\ t \in [0, 1]\}$ by u, v whenever $u, v \in S$ and $u < v$.

Theorem 7.38. [24, Ch.V] *Let D be p-convex, $\frac{\partial f}{\partial u}(t, v)$ exists and $\frac{\partial f}{\partial u} \in C(\mathbb{R} \times D, L(\mathcal{C}, \mathbb{R}^n))$, and f be cooperative.*

Then

$$u_0, u_1 \in D,\ u_0 < u_1 \ \ \textit{implies} \ \ x(t, t_0, u_0) \ll x(t, t_0, u_1)$$

for all $t \geq t_0 + 2r$. In particular,

$$u_t(t_0, u_0) \ll u_t(t_0, u_1)$$

for all $t \geq t_0 + 3r$.

Theorem 7.39. *Suppose that the following assumptions are fulfilled:*

- *let $D = Int(\mathcal{C}_+)$, $\frac{\partial f}{\partial u}(t, v)$ exist, $\frac{\partial f}{\partial u} \in C(\mathbb{R} \times D, L(\mathcal{C}, \mathbb{R}^n))$ and f be cooperative;*
- *the function $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ is strongly Poisson stable in $t \in \mathbb{R}$ uniformly w.r.t. u on every compact subset from $\mathcal{C}_+$ in the both directions;*
- *the set $H(f)$ is minimal.*

*Then under conditions (**F1**)–(**F5**) the following statements hold:*

1. *for any $u \in \mathcal{C}_+$ the set $\omega_{(u,f)} \bigcap X_f$ consists of a single point $p_0 = (p_u, f)$, where $\omega_{(u,f)}$ is the ω-limit set of the motion $\pi(t, (u, f))$ of the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ ($X := \mathcal{C}_+ \times H(f)$, $\pi := (\varphi, \sigma)$) and $X_f := \mathcal{C}_+ \times \{f\}$;*
2. *the solution $\varphi(t, p_u, f)$ of equation (74) is defined on $\mathbb{R}$, $\overline{\varphi(\mathbb{R}, p_u, f)} \subseteq Q_+^{(u,f)}$ and it is strongly compatible;*
3. $$\lim_{t \to \infty} |\varphi(t, u, f) - \varphi(t, p_u, f)| = 0.$$

Proof. Let $f \in \mathcal{B}$ and $(\mathcal{B}, \mathbb{R}, \sigma)$ be the shift dynamical system on $\mathcal{B}$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced from $(\mathcal{B}, \mathbb{R}, \sigma)$. Consider the cocycle $\langle \mathcal{C}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ generated by equation (74) (see Condition (**F1**)). According to Theorem 7.38 we have: $u < v$ $(u, v \in D = Int(\mathcal{C}_+))$ implies

$\varphi(t,u,f) \ll \varphi(t,v,f)$ for any $t \geq 3r$. Now to finish the proof of Theorem it is sufficient to apply Lemma 2.7 and Theorems 4.4, 2.43. □

Corollary 7.40. *Under the conditions of Theorem 7.39 if the function $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{R}$, then the following statements hold:*

1. *the set $\omega_{(u,f)} \bigcap X_f$ consists of a single point $p = (p_u, f)$;*
2. *$\varphi(t, p_u, f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (74);*
3. *$\lim\limits_{t \to +\infty} |\varphi(t,u,f) - \varphi(t,p_u,f)| = 0$, i.e., $\varphi(t,u,f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable).*

Proof. These statements follow from Theorems 7.39, 2.43 and Corollary 6.10. □

Condition (F8). A function $f \in C(\mathbb{T} \times E_1, E_2)$ is completely continuous (compact) if for any bonded subset M from E_1 there exists a pre-compact subset $K = K(M)$ from E_2 such that $f(t, M) \subseteq K$ for any $t \in \mathbb{T}$.

Remark 7.41. If the function $f \in C(\mathbb{T} \times E_1, E_2)$ satisfies condition **(F8)** then for any bounded subset M from E_1 there exists a compact subset K from E_2 such that $g(t, M) \subseteq K$ for any $g \in H(f)$ and $t \in \mathbb{T}$.

Lemma 7.42. *Assume that $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ satisfies condition **(F8)** and $\alpha \in C(\mathbb{R}_+, \mathbb{R}^n)$ (respectively, $\alpha \in C(\mathbb{R}, \mathbb{R}^n)$) is a bounded on $\mathbb{R}_+$ (respectively, on $\mathbb{R}$) solution of equation (74). Then the set $\{\alpha_t |\ t \geq 0\}$ (respectively, $\{\alpha_t |\ t \in \mathbb{R}\}$) is pre-compact in $\mathcal{C}$.*

Proof. Note that the family of functions $\{\alpha_t |\ t \geq 0\}$ (respectively, $\{\alpha_t |\ t \in \mathbb{R}\}$) is bounded in $\mathcal{C}$ if the function $\alpha : \mathbb{R}_+ \to \mathbb{R}^n$ (respectively, $\alpha : \mathbb{R} \to \mathbb{R}^n$) is bounded. Since f is completely continuous, then there exists a $C > 0$ such that $|f(t, \alpha_t)| \leq C$ for any $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$). Thus we have

$$|\alpha'(t)| = |f(t, \alpha_t)| \leq C$$

for all $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{R}$) and, consequently, α is uniformly continuous on $\mathbb{R}_+$. Thus the family of functions $\{\alpha_t |\ t \geq 0\} \subseteq \mathcal{C}$ is bounded and equicontinuous. By Arzela-Ascoli theorem $\{\alpha_t |\ t \geq 0\}$ (respectively, $\{\alpha_t |\ t \in \mathbb{R}\}$) is pre-compact in $\mathcal{C}$. Lemma is proved. □

Theorem 7.43. *Suppose that the following assumptions are fulfilled:*

1. *Let $D = Int(\mathcal{C}_+)$, $\frac{\partial f}{\partial u}(t,v)$ exist, $\frac{\partial f}{\partial u} \in C(\mathbb{R} \times D, L(\mathcal{C}, \mathbb{R}^n))$ and f be cooperative;*
2. *the function $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ is Poisson stable in $t \in \mathbb{R}$ uniformly w.r.t. u on every compact subset from $\mathcal{C}_+$ in the both directions, i.e., there exist two sequences $t_n^{\pm} \to \pm\infty$ as $n \to \infty$ such that $f^{t_n^{\pm}}$ converges to f in $C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$;*
3. *for every $u \in \mathcal{C}_+$ the set $Q_+^{(u,f)} := \overline{\varphi(\mathbb{R}_+, u, f)}$ is a compact subset of $\mathcal{C}_+$;*
4. *there is a solution $\varphi(t, u_0, f)$ of equation (74) such that*

$$Q_+^{(u_0,f)} := \overline{\varphi(\mathbb{R}_+, u_0, f)}$$

is a compact subset of $Int(\mathcal{C}_+)$.

Then under the conditions ***(F1)-(F3)*** *and* ***(F6)-(F8)*** *the following statements hold:*

1. *the set* $\omega_{(u_0,f)} \bigcap X_f$ *consists of a single point* $p_0 = (p_{u_0}, f)$*, where* $\omega_{(u_0,f)}$ *is the* ω*-limit set of the motion* $\pi(t,(u_0,f))$ *of the skew-product dynamical system* $(X, \mathbb{R}_+, \pi)$ *(*$X := \mathcal{C}_+ \times H(f)$*,* $\pi := (\varphi, \sigma)$*) and* $X_f := \mathcal{C}_+ \times \{f\}$*;*
2. *the solution* $\varphi(t, p_{u_0}, f)$ *of equation (74) is defined on* $\mathbb{R}$*,*
$$\overline{\varphi(\mathbb{R}, p_{u_0}, f)} \subseteq Q_+^{(u_0,f)}$$
and it is compatible;
3. $$\lim_{t\to\infty} |\varphi(t, u_0, f) - \varphi(t, p_{u_0}, f)| = 0;$$
4. *for any* $u \in Int(\mathcal{C}_+)$ *the set* $\omega_{(u,f)} \bigcap X_f$ *consists of a single point* $p = (p_u, f)$*, the solution* $\varphi(t, p_u, f)$ *is compatible and*
$$\lim_{t\to\infty} |\varphi(t, u, f) - \varphi(t, v, f)| = 0;$$

1. *for any* $u_1, u_2 \in Int(\mathcal{C}_+)$ *we have* $p_{u_1} = p_{u_2}$*, i.e., equation (74) has a unique bounded globally attracting in* $Int(\mathcal{C}_+)$ *solution* $p : \mathbb{R} \to Int(\mathcal{C}_+)$ *with* $\overline{p(\mathbb{R})} \subset Int(\mathcal{C}_+)$*.*

Proof. Let $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ and $(C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n), \mathbb{R}, \sigma)$ be the shift dynamical system on $C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$. Denote by $Y := H(f)$ and $(Y, \mathbb{R}, \sigma)$ the shift dynamical system on $H(f)$ induced from $(C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n), \mathbb{R}, \sigma)$. Consider the cocycle $\langle \mathcal{C}_+, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ and non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, with $X = \mathcal{C}_+ \times Y$, $\pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$, generated by equation (74) (see Condition **(F1)**).

Note that the cocycle φ, generated by equation (74), under the conditions of Theorem 7.43 possesses the following conditions:

1. the set $\mathcal{C}_+$ is positively invariant with respect to cocycle φ, i.e., $\varphi(t, v, g) \in \mathcal{C}_+$ for any $(t, v, g) \in \mathbb{R}_+ \times \mathcal{C}_+ \times H(f)$;
2. φ is monotone and by Theorem 7.38 $\varphi(t, \cdot, f)$ strongly monotone for any $t \geq 3r$.

Let $x_0 := (u_0, f)$. Since $Q_+^{(u_0,f)} := \overline{\varphi(\mathbb{R}_+, u_0, f)}$ is a compact subset of $Int(\mathcal{C}_+)$, then $\overline{\Sigma_{x_0}^+} \subseteq Int(\mathcal{C}_+) \times H(f)$.

Now to finish the proof it is sufficient to apply Theorems 6.5 and Remark 6.11 (item (ii)). □

Lemma 7.44. *Assume that* $f \in C(\mathbb{R} \times \mathcal{C}, \mathbb{R}^n)$ *is completely continuous and almost automorphic (respectively, recurrent) in* $t \in \mathbb{R}$ *uniformly with respect to* u *on every compact subset from* $\mathcal{C}$ *and* $\alpha \in C(\mathbb{R}, \mathcal{C})$ *is a bounded on* $\mathbb{R}$ *and compatible solution of equation (74). Then* α *is also almost automorphic (respectively, recurrent).*

Proof. Since the function f is almost automorphic (respectively, recurrent), then it is Levitan almost periodic (respectively, almost recurrent) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathcal{C}$. Taking into account that φ is a compatible solution of equation (74), then by Theorem 2.40 it is Levitan almost periodic (respectively, almost recurrent). Now to finish the proof of Lemma it is sufficient to show that the function φ is Lagrange stable. Note that φ is bounded on $\mathbb{R}$ and by Lemma 7.42 $K := \overline{\{\alpha_t |\ t \in \mathbb{R}\}}$ is a compact subset of $\mathcal{C}$. Since φ is uniformly continuous on $\mathbb{R}$ (see the proof of Lemma 7.42), then it is easy to

check that the family of functions $\{\alpha_t|\ t\in\mathbb{R}\}$ is equicontinuous and, consequently, the mapping $t\to\alpha_t$ from $\mathbb{R}$ to $\mathcal{C}$ is uniformly continuous. Thus the function α is Lagrange stable and Levitan almost periodic (respectively, almost recurrent) and, consequently, it is almost automorphic (respectively, recurrent). Lemma is proved. □

Corollary 7.45. *Under the conditions of Theorem 7.43 if the function $f\in C(\mathbb{R}\times\mathcal{C}_+,\mathbb{R}^n)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) in time $t\in\mathbb{R}$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)}\bigcap X_f$ consists of a single point $p_0=(p_{u_0},f)$;*
2. *$\varphi(t,p_{u_0},f)$ is a stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) solution of equation (74);*
3. *$\lim\limits_{t\to+\infty}|\varphi(t,u_0,f)-\varphi(t,p_{u_0},f)|=0$, i.e., $\varphi(t,u_0,f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically almost automorphic, asymptotically almost recurrent in the sense of Bebutov, asymptotically recurrent, asymptotically Poisson stable);*
4. *for any $u\in Int(\mathcal{C}_+)$ such that $\overline{\varphi(\mathbb{R}_+,u,f)}\subset Int(\mathcal{C}_+)$ there exists a unique point $p=(p_u,f)\in\omega_x$ $(x=(u,f))$ such that the solution $\varphi(t,p_u,f)$ of equation (74) defined on $\mathbb{R}$ with values from the compact $Q_+^{(u,f)}$;*
5. *the solution $\varphi(t,p_u,f)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable);*
6. *$\lim\limits_{t\to+\infty}|\varphi(t,u,f)-\varphi(t,p_u,f)|=0$.*

Proof. These statements follow from Theorem 7.43, 2.40 and Lemma 7.42. □

Definition 7.46. Let $f\in C(\mathbb{R}\times\mathcal{C}_+,\mathbb{R}^n)$. The set $H(f)$ is said to be minimal, if it is a minimal set of shift dynamical system $(C(\mathbb{R}\times\mathcal{C}_+,\mathbb{R}^n),\mathbb{R},\sigma)$.

Definition 7.47. A function $f\in C(\mathbb{R}\times\mathcal{C}_+,\mathbb{R}^n)$ is said to be strongly Poisson stable if every function $g\in H(f)$ is Poisson stable.

Theorem 7.48. *Under the conditions of Theorem 7.43 if additionally the set $H(f)$ is minimal and f is strongly Poisson stable, then the following statements hold:*

1. *the set $\omega_{(u_0,f)}\bigcap X_f$ consists of a single point $p_0=(p_{u_0},f)$, where $\omega_{(u_0,f)}$ is the ω-limit set of the motion $\pi(t,(u_0,f))$ of the skew-product dynamical system $(X,\mathbb{R}_+,\pi)$ $(X:=\mathcal{C}_+\times H(f),\ \pi:=(\varphi,\sigma))$ and $X_f:=\mathcal{C}_+\times\{f\}$;*
2. *the solution $\varphi(t,p_{u_0},f)$ of equation (74) is defined on $\mathbb{R}$, $\overline{\varphi(\mathbb{R},p_{u_0},f)}\subseteq Q_+^{(u_0,f)}$ and it is uniformly compatible;*
3. $$\lim_{t\to\infty}|\varphi(t,u_0,f)-\varphi(t,p_{u_0},f)|=0;$$
4. *for any $u\in Int(\mathcal{C}_+)$ the set $\omega_{(u,f)}\bigcap X_f$ consists of a single point $p=(p_u,f)$, the solution $\varphi(t,p_u,f)$ is strongly compatible and $\lim\limits_{t\to\infty}|\varphi(t,u,f)-\varphi(t,p_u,f)|=0$;*
5. *for any $u_1,u_2\in Int(\mathcal{C}_+)$ we have $p_{u_1}=p_{u_2}$, i.e., equation (74) has a unique bounded globally attracting in $Int(\mathcal{C}_+)$ solution $p:\mathbb{R}\to Int(\mathcal{C}_+)$ with $\overline{p(\mathbb{R})}\subset Int(\mathcal{C}_+)$.*

Proof. These statements can be proved using the same arguments as in the proof of Theorem 7.43, but instead of Theorem 6.5 we need to apply Theorem 6.9. □

Corollary 7.49. *Under the conditions of Theorem 7.48 if the function $f \in C(\mathbb{R} \times \mathcal{C}_+, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{R}$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *$\varphi(t, p_{u_0}, f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (74);*
3. *$\lim\limits_{t\to+\infty} |\varphi(t, u_0, f) - \varphi(t, p_{u_0}, f)| = 0$, i.e., $\varphi(t, u_0, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable);*
4. *for any $u \in Int(\mathcal{C}_+)$ such that $\overline{\varphi(\mathbb{R}_+, u, f)} \in Int(\mathcal{C}_+)$ there exists a unique point $p = (p_u, f) \in \omega_x$ $(x = (u, f))$ such that the solution $\varphi(t, p_u, f)$ of equation (74) defined on $\mathbb{R}$ with values from the compact $Q_+^{(u,f)}$;*
5. *the solution $\varphi(t, p_u, f)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable);*
6. *$\lim\limits_{t\to+\infty} |\varphi(t, u, f) - \varphi(t, p_u, f)| = 0$.*

Proof. These statements follow from Theorems 7.48, 2.43 and Corollary 6.10. □

7.3. **Difference equations.** Let us consider a difference equation

$$u(t+1) = f(t, u(t)), \tag{78}$$

where $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$. Along with equation (78) we consider its H-class [3, 32, 39, 45, 46], i.e., the family of equations

$$v(t+1) = g(t, v(t)), \tag{79}$$

where $g \in H(f) = \overline{\{f^\tau : \tau \in \mathbb{Z}\}}$, $f^\tau(t, u) = f(t+\tau, u)$ for all $(t, u) \in \mathbb{Z} \times \mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$.

Remark 7.50. 1. Note that the compact-open topology on the space $C(\mathbb{Z}, \mathbb{R}^n)$

1. can be defined by distance

$$d(\varphi_1, \varphi_2) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(\varphi_1, \varphi_2)}{1 + d_i(\varphi_1, \varphi_2)},$$

where $d(\varphi_1, \varphi_2) := \max\{|\varphi_1(k) - \varphi_2(k)| : \ |k| \le i\}$;

2. coincides with the topology of pointwise convergence.

2. A function $\varphi \in C(\mathbb{Z}, \mathbb{R}^n)$ is Lagrange stable (i.e., the set $\Sigma_\varphi := \{\varphi^h : \ h \in \mathbb{Z}\}$ is relatively compact in the space $C(\mathbb{Z}, \mathbb{R}^n)$) if and only if the set $\varphi(\mathbb{Z})$ is bounded.

3. The topology on the space $C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$

1. can be defined by distance

$$d(f_1, f_2) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(f_1, f_2)}{1 + d_i(f_1, f_2)},$$

where $d(f_1,f_2):=\max\{|f_1(k,x)-f_2(k,x)|:\ |k|\leq i,|x|\leq i\}$;

2. coincides with the topology of pointwise convergence in $k\in\mathbb{Z}$ uniformly with respect to x on every compact subset from $\mathbb{R}^n$.

4. A function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is Lagrange stable (i.e., the set $\Sigma_f:=\{f^h:\ h\in\mathbb{Z}\}$ is relatively compact in the space $C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$) if and only if for any compact subset $K\subset\mathbb{R}^n$ the following two conditions are fulfilled:

1. the set $f(\mathbb{Z}\times K)$ is bounded;
2. the mapping $f:\mathbb{Z}\times K\to\mathbb{R}^n$ is uniformly continuous.

Denote by $\varphi(\cdot,v,g)$ the solution of equation (79), passing through the point $v\in\mathbb{R}^n$ at the initial moment $t=0$. Then it is correctly defined a mapping $\varphi:\mathbb{Z}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$, verifying the following conditions (see, for example, [3, 39]):

1) $\varphi(0,v,g)=v$ for all $v\in\mathbb{R}^n$ and $g\in H(f)$;
2) $\varphi(t,\varphi(\tau,v,g),g_\tau)=\varphi(t+\tau,v,g)$ for every $v\in\mathbb{R}^n$, $g\in H(f)$ and $t,\tau\in\mathbb{Z}_+$;
3) the mapping $\varphi:\mathbb{Z}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$ is continuous.

Denote by $Y:=H(f)$ and $(Y,\mathbb{Z},\sigma)$ a dynamical system of translations on Y, induced by dynamical system of translations $(C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{Z},\sigma)$. The triplet $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{Z},\sigma)\rangle$ is a cocycle on $(Y,\mathbb{Z},\sigma)$ with the fiber $\mathbb{R}^n$. Thus equation (78) generates a cocycle $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{Z},\sigma)\rangle$ and a non-autonomous dynamical system $\langle(X,\mathbb{Z}_+,\pi),(Y,\mathbb{Z},\sigma),h\rangle$, where $X:=\mathbb{R}^n\times Y$, $\pi:=(\varphi,\sigma)$ and $h:pr_2:X\to Y$.

Remark 7.51. Let F be a mapping from $H(f)\times\mathbb{R}^n\to\mathbb{R}^n$defined by equality $F(g,x):=g(0,x)$, then F possesses the following properties:

1. F is continuous;
2. $F(g^\tau,x)=g(\tau,x)$ for any $(g,x,\tau)\in H(f)\times\mathbb{R}^n\times\mathbb{R}$;
3. equation (78) (and its H-class (79) can be rewritten as follows
$$x(t+1)=F(\sigma(t,f),x(t))\ \ (g\in H(f)); \tag{80}$$
4. if φ is the cocycle generated by equation (80), then $\phi(t,v,g):=\lambda\varphi(t,\lambda^{-1}v,g)$ $(\lambda\in(0,1))$ is the cocycle generated by equation
$$x(t+1)=\mathcal{F}(\sigma(t,f),x(t))\ \ (g\in H(f)), \tag{81}$$
where $\mathcal{F}(g,x):=\lambda F(g,\lambda^{-1}x)$ for any $(g,x)\in H(f)\times\mathbb{R}^n$.

Condition (D1). Equation (78) is monotone. This means that the cocycle $\langle\mathbb{R}^n,\varphi,(H(f),\mathbb{Z},\sigma)\rangle$ (or shortly φ) generated by (78) is monotone, i.e., if $u,v\in\mathbb{R}^n$ and $u\leq v$ then $\varphi(t,u,g)\leq\varphi(t,v,g)$ for all $t\geq0$ and $g\in H(f)$.

Definition 7.52. A function $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ is said to be monotone (order-preserving) if for any $(t,u),(t,v)\in\mathbb{Z}\times\mathbb{R}^n$ $u\leq v$ implies $f(t,u)\leq f(t,v)$.

Lemma 7.53. *Let $f\in C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n)$ be a monotone function, then the following statements hold:*

1. *any function $g\in H(f)$ is monotone;*
2. *$u\leq v$ implies $\varphi(t,u,g)\leq\varphi(t,v,g)$ for any $t\geq0$ and $g\in H(f)$;*
3. *equation (78) is monotone.*

Proof. Let $g\in H(f)$, then there exists a sequence $\{h_k\}\subset\mathbb{Z}$ such that $g(t,u)=\lim\limits_{k\to\infty}f^{h_k}(t,u)$ for any $(t,u)\in\mathbb{Z}\times\mathbb{R}^n$. Let $u\leq v$ $(u,v\in\mathbb{R}^n)$. Since f is monotone, then we will have
$$f_i(t+h_k,u)\leq f_i(t+h_k,v)\ (i=1,\dots,n) \tag{82}$$

and passing to the limit in (82) as $k \to \infty$ we obtain that g is monotone too.

Let $u \leq v$ $(u, v \in \mathbb{R}^n)$ and $g \in H(f)$, then we have

$$\varphi(1, u, g) = g(0, u) \leq g(0, v) = \varphi(1, v, g). \tag{83}$$

Let k be an arbitrary number from $\mathbb{N}$. Suppose that

$$\varphi(t, u, g) \leq \varphi(t, v, g) \tag{84}$$

for any $2 \leq t \leq k$. If $t = k+1$, then taking into consideration the first statement of Lemma and (83)–(84) we obtain

$$\varphi(t, u, g) = \varphi(k+1, u, g) = \varphi(1, \varphi(k, u, g), g^k) \leq$$
$$\varphi(1, \varphi(k, v, g), g^k) = \varphi(k+1, v, g) = \varphi(t, v, g)$$

for any $t \geq 0$. Lemma is completely proved. □

Lemma 7.54. *Suppose that $f_i \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ $(i = 1, 2)$ and $x_1, x_2 \in \mathbb{R}^n$ and the function f_2 is monotone. Let $\prec$ denote any one of the relations $\leq, <, \ll$. If $x_1 \prec x_2$ and $f_1(t, x) \prec f_2(t, x)$ for any $(t, x) \in \mathbb{Z} \times \mathbb{R}^n$, then $\varphi(t, x_1, f_1) \prec \varphi(t, x_2, f_2)$, hence equation (78) is monotone.*

Proof. Let $x_1 \prec x_2$ $(x_1, x_2 \in \mathbb{R}^n)$ and $f_1(t, x) \prec f_2(t, x)$ for any $(t, x) \in \mathbb{Z} \times \mathbb{R}^n$ $(f_i \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ $(i = 1, 2))$, then we have

$$\varphi(1, x_1, f_1) = f_1(0, x_1) \prec f_2(0, x_2) = \varphi(1, x_2, f_2). \tag{85}$$

Let k be an arbitrary number from $\mathbb{N}$. Suppose that

$$\varphi(t, x_1, g) \prec \varphi(t, x_2, f_2) \tag{86}$$

for any $2 \leq t \leq k$. If $t = k+1$, then taking into consideration (85)–(86) we obtain

$$\varphi(t, x_1, f_1) = \varphi(k+1, x_1, f_1) = \varphi(1, \varphi(k, x_1, f_1), f_{1k}) \prec$$
$$\varphi(1, \varphi(k, x_2, f_2), f_{2k}) = \varphi(k+1, x_2, f_2) = \varphi(t, x_2, f_2)$$

for any $t \geq 0$. Lemma is proved. □

Lemma 7.55. *Assume that $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is monotone and $\langle \mathbb{R}^n_+, \varphi, (H(f), \mathbb{Z}, \sigma) \rangle$ is the cocycle in $\mathbb{R}^n_+$ generated by equation (78) (respectively, by equation (80)). Let $\prec$ denote one of the three relations $\leq, <, \ll$. Then the condition*

$$\mathcal{F}(g, x) \prec F(g, x)$$

for any $(g, x) \in H(f) \times \mathbb{R}^n_+$ implies that

$$\phi(t, x, g) \prec \varphi(t, x, g)$$

for any $t \geq 0$, $g \in H(f)$ and $x \in \mathbb{R}^n_+$, where $\mathcal{F} \in C(H(f) \times \mathbb{R}^n_+, \mathbb{R}^n)$ is some function and $\langle \mathbb{R}^n_+, \phi, (H(f), \mathbb{R}, \sigma) \rangle$ (shortly, ϕ) is the cocycle generated by equation

$$x(t+1) = \mathcal{F}(\sigma(t, g), x(t)) \quad (g \in H(f)).$$

Proof. This statement follows from Lemma 7.54. □

Lemma 7.56. *Assume that the function $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ monotone and $f(t, 0) \geq 0$ for any $t \in \mathbb{Z}$. Then the following statements hold:*

1. *$\mathbb{R}^n_+$ is a positively invariant subset of the cocycle φ, generated by equation (78), i.e., $\varphi(t, x, g) \in \mathbb{R}^n_+$ for any $(t, x, g) \in \mathbb{Z}_+ \times \mathbb{R}^n_+ \times H(f)$;*
2. *if $g(t, 0) \gg 0$ for any $g \in H(f)$ and $t \in \mathbb{Z}_+$, then $Int(\mathbb{R}^n_+)$ is positively invariant with respect to cocycle φ, i.e., $\varphi(t, x, g) \in Int(\mathbb{R}^n_+)$ for any $t \geq 0$, $x \in Int(\mathbb{R}^n_+)$ and $g \in H(f)$.*

Proof. Let $g \in H(f)$, then it is easy to check that under the condition of Lemma 7.56 the function g is also regular, monotone and $g(t,0) \geq 0$ for any $t \in \mathbb{R}$. Note that $F(g,x) = g(0,x) \geq 0$ for any $(x,g) \in \mathbb{R}^n_+ \times H(f)$. By Lemma 7.55 we have $\phi(t,x,g) \leq \varphi(t,x,g)$ for any $t \geq 0$, $g \in H(f)$ and $x \in \mathbb{R}^n_+$, where φ is the cocycle generated by equation (78) (respectively, equation (80)) and ϕ is the cocycle defined by equation $x(t+1) = x(t)$, i.e., $\phi(t,x,g) = x$ for any $x \in \mathbb{R}^n_+$, $t \geq 0$ and $g \in H(f)$. Thus we have $\varphi(t,x,g) \geq x$ for any $(t,x,g) \in \mathbb{Z}_+ \times \mathbb{R}^n_+ \times H(f)$. This means that $\varphi(t,x,g) \geq 0$, i.e., $\varphi(t,x,g) \in \mathbb{R}^n_+$ for any $(t,x,g) \in \mathbb{Z}_+ \times \mathbb{R}^n_+ \times H(f)$.

The second statement can be proved using the same argument as above. Lemma is completely proved. □

Lemma 7.57. *Assume that $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is monotone and the mapping $F : H(f) \times \mathbb{R}^n_+ \to \mathbb{R}^n$ ($F(g,x) = g(0,x)$ for any $(g,x) \in H(f) \times \mathbb{R}^n_+$) is sub-linear, i.e.,*

$$\lambda F(g,x) \leq F(g,\lambda x) \tag{87}$$

for any $(\lambda, g, x) \in (0,1) \times H(f) \times \mathbb{R}^n_+$. Then the cocycle φ, generated by equation (78), is sub-linear. Moreover φ is strongly sub-linear if the following condition is satisfied:

$$\lambda F_i(g,x) < F_i(g,\lambda x) \quad (F = (F_1, \dots, F_n)) \tag{88}$$

for all $i = 1, \dots, n$, $0 < \lambda < 1$, $x \in Int(\mathbb{R}^n_+)$ and $g \in H(f)$.

Proof. Let $\lambda \in (0,1)$. The function $x_\lambda(t) = \lambda\varphi(t, \lambda^{-1}v, g)$ is the solution of equation (81) passing through the point $v \in \mathbb{R}^n_+$ at the initial moment $t = 0$, where $\mathcal{F}(g,y) := \lambda F(g, \lambda^{-1}y)$ for any $(g,y) \in H(f) \times \mathbb{R}^n_+$. From (87) we obtain $\mathcal{F}(g,y) \leq F(g,y)$ for any $(g,y) \in H(f) \times \mathbb{R}^n_+$. By Lemma 7.55 (item (i)) we have

$$\lambda\varphi(t,v,g) \leq \varphi(t,\lambda v,g),$$

i.e., the cocycle φ is sub-linear.

The strong sub-linearity of φ under condition (88) follows from Lemma 7.55. □

Lemma 7.58. *Assume that $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is monotone and*

$$\inf_{(t,x) \in \mathbb{Z} \times Int(\mathbb{R}^n_+)} |\lambda f_i(t,x) - f_i(t,\lambda x)| > 0 \tag{89}$$

for any $\lambda \in (0,1)$ and $i = 1, \dots, n$. Then the cocycle φ, generated by equation (80), is strongly sub-linear.

Proof. Now we will show that condition (89) implies (88). In fact. If $g \in H(f)$, then there exists a sequence $\{t_k\} \subset \mathbb{Z}$ such that $f^{t_k} \to g$ as $k \to \infty$ in the space $C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$. From (89) it follows that for any $\lambda \in (0,1)$ there exists a positive number $\alpha(\lambda)$ such that

$$|\lambda f_i(t + t_k, x) - f_i(t + t_k, \lambda x)| \geq \alpha(\lambda) \tag{90}$$

for any $(t,x) \in \mathbb{Z} \times Int(\mathbb{R}^n_+)$, $k \in \mathbb{N}$ and $i = 1, \dots, n$. Passing to the limit in (90) as $k \to \infty$ we obtain

$$|\lambda F_i(g,x) - F_i(g,\lambda x)| \geq \alpha(\lambda) \tag{91}$$

for all $(g,x) \in H(f) \times Int(\mathbb{R}^n_+)$ and $i = 1, \dots, n$, where $F_i(g,x) = g_i(0,x)$. From (91) we obtain

$$\lambda F_i(g,x) < F_i(g,\lambda x)$$

for any $(\lambda, g, x) \in (0,1) \times H(f) \times Int(\mathbb{R}^n_+)$ and $i = 1, \dots, n$. To finish the proof of the second statement of Lemma it is sufficient to apply Lemma 7.57. □

Let $f \in C(\mathbb{R}, \mathfrak{B})$ be a Bohr almost periodic function and $\tilde{f} : \mathbb{Z} \to \mathfrak{B}$ be a function defined by equality $\tilde{f}(k) := f(k)$ for any $k \in \mathbb{Z}$. By Theorem 1.27 [21, ChI] the function $\tilde{f}$ is also Bohr almost periodic.

Example 4. Since the function $\varphi(t) := 2+\cos t+\cos\sqrt{2}t$ $(t \in \mathbb{R})$ is Bohr almost periodic, then by Theorem 1.27 [21, Ch.I] the function $\varphi(k) := 2+\cos k+\cos\sqrt{2}k$ $(k \in \mathbb{Z})$ is almost periodic. Consider the function $f \in C(\mathbb{Z}, \mathbb{R})$ defined by equality

$$f(k) = \frac{1}{2 + \cos k + \cos\sqrt{2}k} \quad (k \in \mathbb{Z}).$$

According to Corollary 7.13 this function is Levitan almost periodic. Since the number $\sqrt{2}$ is irrational, then there exists a sequence $\{k_m\} \subset \mathbb{Z}$ such that $k_m \to +\infty$, $\cos k_m \to -1$ and $\cos\sqrt{2}k_m \to -1$ as $m \to \infty$ (see, for example, [51, ChIII,p.157]) and, consequently, $f(k_m) \to +\infty$ as $m \to \infty$. Thus the function $f(k)$ is not Bohr almost periodic.

Now we consider a function $h \in C(\mathbb{Z}, \mathbb{R})$ defined as follows

$$h(k) = \sin\frac{1}{2 + \cos k + \cos\sqrt{2}k} \quad (k \in \mathbb{Z}). \tag{92}$$

The function h is bounded and by Corollary 7.13 it is Levitan almost periodic.

Remark 7.59. 1. The function $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ satisfies to condition (89) if and only if for any $\lambda \in (0, 1)$ there exists a positive number $\alpha(\lambda)$ such that

$$f(k, \lambda x) - \lambda f(k, x) \geq \alpha(\lambda)e \tag{93}$$

for any $(k, x) \in \mathbb{Z} \times Int(\mathbb{R}^n_+)$, where $e := (1, \ldots, 1) \in Int(\mathbb{R}^n_+)$.

2. It is easy to check that the function $f \in C(\mathbb{Z} \times \mathbb{R}_+, \mathbb{R})$, defined by equality

$$f(k, x) = \frac{x}{2} + \frac{x}{x+1} + 2 + \sin\frac{1}{2 + \cos k + \cos\sqrt{2}k},$$

satisfies condition (89). In fact, if $\lambda \in (0, 1)$ we have

$$f(k, \lambda x) - \lambda f(k, x) \geq 1 - \lambda = \alpha(\lambda) > 0$$

for any $(k, x) \in \mathbb{Z} \times \mathbb{R}_+$.

Definition 7.60. A solution $\varphi(k, u_0, f)$ of equation (78) is said to be pre-compact on $\mathbb{Z}_+$ if the set $Q := \overline{\varphi(\mathbb{Z}_+, u_0, f)}$ is a compact subset of $\mathbb{R}^n$, where by bar is denoted the closure in $\mathbb{R}^n$ and $\varphi(\mathbb{Z}_+, u_0, f) := \{\varphi(k, u_0, f) : \ t \in \mathbb{Z}_+\}$.

Definition 7.61. A solution $\varphi(k, u_0, f)$ of equation (78) is called [15],[45],[46] compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(k, x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t, f)$.

Recall that the function $\varphi \in C(\mathbb{Z}, \mathbb{R}^n)$ (respectively, $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$) possesses the property (**A**), if the motion $\sigma(\cdot, \varphi)$ (respectively, $\sigma(\cdot, f)$) generated by the function φ (respectively, f) possesses this property in the dynamical system $(C(\mathbb{Z}, \mathbb{R}^n), \mathbb{Z}, \sigma)$ (respectively, $(C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n), \mathbb{Z}, \sigma)$).

In the quality of the property (**A**) there can stand stability in the sense of Lagrange (st. L), uniform stability (un. st. $\mathcal{L}^+$) in the sense of Lyapunov, periodicity, almost periodicity, asymptotical almost periodicity and so on.

For example, a function $f \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R}^n)$ is called *almost periodic (respectively, recurrent, asymptotically almost periodic etc) in* $t \in \mathbb{R}$ *uniformly with respect to*

(w.r.t.) w on every compact subset from $\mathbb{R}^n$, if the motion $\sigma(\cdot, f)$ is almost periodic (respectively, recurrent, asymptotically almost periodic, etc.) in the dynamical system $(C(\mathbb{Z}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{Z},\sigma)$.

Theorem 7.62. *Let $f\in C(\mathbb{Z}\times\mathbb{R}^n_+,\mathbb{R}^n)$ be regular, Poisson stable in $t\in\mathbb{Z}$ uniformly w.r.t. u on every compact subset from $\mathbb{R}^n_+$ in the both directions. Suppose that f is monotone and*

$$\inf_{(k,x)\in\mathbb{Z}\times\mathbb{R}^n_+}\left(\lambda f_i(k,x)-f_i(k,\lambda x)\right)>0 \tag{94}$$

for any $\lambda\in(0,1)$ and $i=1,\dots,n$.

If for every $(u,g)\in\mathbb{R}^n_+\times H(f)$, the set $Q_+^{(u,g)}:=\overline{\varphi(\mathbb{Z}_+,u,g)}$ is a compact subset of $\mathbb{R}^n_+$ and there exists $u_0\in Int(\mathbb{R}^n_+)$ such that $Q_+^{(u_0,f)}\subset Int(\mathbb{R}^n_+)$. Then for any $u\gg 0$ the following statements (1)-(5) hold:

1. *the set $\omega_{(u_0,f)}\bigcap X_f$ consists of a single point $p_0=(p_{u_0},f)$, where $\omega_{(p_{u_0},f)}$ is the ω-limit set of the motion $\pi(t,x_0,f))$ $(x_0=(p_{u_0},f))$ of the skew-product dynamical system $(X,\mathbb{Z}_+,\pi)$ $(X:=\mathbb{R}^n_+\times H(f),\ \pi:=(\varphi,\sigma))$ and $X_f:=\mathbb{R}^n_+\times\{f\}$;*
2. *the solution $\varphi(t,p_{u_0},f)$ of equation (78) is defined on $\mathbb{Z}$, $\overline{\varphi(\mathbb{Z},p_{u_0},f)}\subseteq Q_+^{(p_{u_0},f)}$ and it is compatible;*
3. $$\lim_{k\to\infty}|\varphi(k,u_0,f)-\varphi(k,p_{u_0},f)|=0;$$
4. *for any $u\in Int(\mathbb{R}^n_+)$ with $Q_+^{(u,f)}\subset Int(\mathbb{R}^n_+)$ the set $\omega_{(u,f)}\bigcap X_f$ consists of a single point $p=(p_u,f)$ and the solution $\varphi(t,p_u,f)$ of equation (78) is compatible and $\lim_{k\to\infty}|\varphi(k,u,f)-\varphi(k,p_u,f)|=0$;*
5. *$p_{u_1}=p_{u_2}$ for any $u_1,u_2\in Int(\mathbb{R}^n_+)$, i.e., equation (78) has a unique bounded globally attracting in $Int(\mathbb{R}^n_+)$ solution $p:\mathbb{Z}\to Int(\mathbb{R}^n)$ with $\overline{p(\mathbb{Z})}\subset Int(\mathbb{R}^n_+)$.*

Proof. Note that sub-linearity and (93) imply (94). In fact, as induced in Remark 7.59 (item 1.), for any $\lambda\in(0,1)$ there exists a positive number $\alpha(\lambda)$ such that

$$f(k,\lambda x)-\lambda f(k,x)\gg\alpha(\lambda)e \tag{95}$$

for all $(k,x)\in\mathbb{Z}\times Int(\mathbb{R}^n_+)$. This, together with the continuity of f, implies (94).

From (94) it follows that $f(k,0)\geq(1-\lambda)^{-1}e\gg 0$, which deduces that $g(k,0)\gg 0$ for all $k\in\mathbb{Z}$.

By Lemma 7.55 we have $\phi(k,u,g)\ll\varphi(k,u,g)$ for any $k\geq 0$, $g\in H(f)$ and $u\in\mathbb{R}^n_+$, where φ is the cocycle generated by equation (78) (respectively, equation (80)) and ϕ is the cocycle defined by equation $x(k+1)=x(k)$, i.e., $\phi(t,u,g)=u$ for any $x\in\mathbb{R}^n_+$, $k\geq 0$ and $g\in H(f)$. Thus we have $\varphi(k,u,g)\geq u$ for any $(k,u,g)\in\mathbb{Z}_+\times\mathbb{R}^n_+\times H(f)$. This means that $\omega_{(u,f)}\subset Int(\mathbb{R}^n_+)\times H(f)$. The statements (1)-(5) follow immediately from Theorem 6.5. □

The example below illustrates Theorem 7.62.

Example 5. Consider the difference equation

$$x(k+1)=\frac{x}{2}+\frac{x}{x+1}+h(k),\ (x\in\mathbb{R}_+) \tag{96}$$

where $h(k)=2+\sin\frac{1}{2+\cos k+\cos\sqrt{2}k}$ $(k\in\mathbb{Z})$. Along this equation (71) we consider its H-class

$$x(k+1)=\frac{x}{2}+\frac{x}{x+1}+g(k),\quad (g\in H(h)) \tag{97}$$

where $H(h) := \overline{\{h^\tau |\ \tau \in \mathbb{Z}\}}$ and by bar is denoted the closure in the space $C(\mathbb{Z}, \mathbb{R})$. Let $f(k,x) := \frac{x}{2} + \frac{x}{x+1} + h(k)$ $((k,x) \in \mathbb{Z} \times \mathbb{R}_+)$, then

$$g(k,x) \leq \frac{x}{2} + 4 := \bar{h}$$

for any $(k,x,g) \in \mathbb{Z} \times R_+ \times H(h)$. Denote by $\langle \mathbb{R}_+, \varphi, H(h) \rangle$ (respectively, $(\mathbb{R}_+, \phi)$) the cocycle generated by family of equations (97) (respectively, semigroup dynamical system generated by equation $x(k+1) = \frac{x(k)}{2} + 4$), then by Lemma 7.55 we have

$$\varphi(k,u,g) \leq \phi(k,u) = (u-8)(\frac{1}{2})^k + 8 \tag{98}$$

for any $(k,x,g) \in \mathbb{Z}_+ \times R_+ \times H(h)$. Thus, all the conditions of the Theorem 7.62 are fulfilled (see Remark 7.59, item 2.) and, consequently, equation (96) has a unique bounded on $\mathbb{Z}$, strictly positive, Levitan almost periodic solution that is globally uniformly asymptotically stable.

Lemma 7.63. *Suppose that the function $\varphi \in C(\mathbb{Z}, \mathbb{R}^n)$ is almost recurrent (respectively, Levitan almost periodic) and bounded, then it is recurrent (respectively, almost automorphic).*

Proof. Note that an almost recurrent (respectively, Levitan almost periodic) is recurrent (respectively, almost automorphic) if and only if it is Lagrange stable. Thus, to prove this statement it is sufficient to establish that under the conditions of Lemma the function φ is Lagrange stable. Since the function φ is bounded, then by Remark 7.50 (item 2.) it is Lagrange stable. □

Corollary 7.64. *Under the conditions of Theorem 7.62 if the function $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) in time $k \in \mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *$\varphi(t, p_{u_0}, f)$ is a stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable) solution of equation (78);*
3. *$\lim\limits_{k \to +\infty} |\varphi(k, u_0, f) - \varphi(k, v_0, f)| = 0$, i.e., $\varphi(k, u_0, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically almost automorphic, asymptotically almost recurrent in the sense of Bebutov, asymptotically recurrent, asymptotically Poisson stable);*
4. *for any $u \in Int(\mathbb{R}^n_+)$ with $Q_+^{(u,f)} \subset Int(\mathbb{R}^n_+)$ there exists a unique point $p = (p_u, f) \in \omega_x$ $(x = (u,f))$ such that the solution $\varphi(k, p_u, f)$ of equation (78) defined on $\mathbb{Z}$ with values from the compact $Q_+^{(u,f)}$;*
5. *equation (78) has a unique solution with the following properties:*
   (a) *it is bounded;*
   (b) *globally attracting in $Int(\mathbb{R}^n_+)$ with $p : \mathbb{Z} \to Int(\mathbb{R}^n_+)$;*
   (c) *the solution p is stationary (respectively, τ-periodic, Levitan almost periodic, almost automorphic, almost recurrent in the sense of Bebutov, recurrent, Poisson stable).*

Proof. This statement follows from Theorems 2.40 and 7.62. □

Definition 7.65. Let $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$. The set $H(f)$ is said to be minimal, if it is a minimal set of shift dynamical system $(C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n), \mathbb{Z}, \sigma)$.

Definition 7.66. A function $f \in C(\mathbb{Z} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is said to be strongly Poisson stable if every function $g \in H(f)$ is Poisson stable.

Definition 7.67. Let $f \in C(\mathbb{Z} \times \mathcal{C}_+, \mathbb{R}^n)$. The set $H(f)$ is said to be minimal, if it is a minimal set of shift dynamical system $(C(\mathbb{Z} \times \mathcal{C}_+, \mathbb{R}^n), \mathbb{Z}, \sigma)$.

Theorem 7.68. *Under the conditions of Theorem 7.62 if additionally the set $H(f)$ is minimal and f is strongly Poisson stable, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *the solution $\varphi(t, p_{u_0}, f)$ of equation (78) is defined on $\mathbb{Z}$, $\overline{\varphi(\mathbb{Z}, p_{u_0}, f)} \subseteq Q_+^{(u_0,f)}$ and it is uniformly compatible;*
3. $$\lim_{k \to \infty} |\varphi(k, u_0, f) - \varphi(k, p_{u_0}, f)| = 0;$$
4. *for any $u \in Int(\mathcal{C}_+)$ with $Q_+^{(u,f)} \subset Int(\mathbb{R}^n_+)$ the set $\omega_{(u,f)} \bigcap X_f$ consists of a single point $p = (p_u, f)$, the solution $\varphi(t, p_u, f)$ is strongly compatible and $\lim\limits_{k \to \infty} |\varphi(k, u, f) - \varphi(k, p_u, f)| = 0$.*

Proof. These statements can be proved using the same arguments as in the proof of Theorem 7.62, but instead of Theorem 6.5 we need to apply Theorem 6.9. □

Corollary 7.69. *Under the conditions of Theorem 7.68 if the function $f \in C(\mathbb{Z} \times \mathcal{C}_+, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) in time $t \in \mathbb{Z}$, then the following statements hold:*

1. *the set $\omega_{(u_0,f)} \bigcap X_f$ consists of a single point $p_0 = (p_{u_0}, f)$;*
2. *$\varphi(k, p_{u_0}, f)$ is a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable) solution of equation (78);*
3. *$\lim\limits_{k \to +\infty} |\varphi(k, u_0, f) - \varphi(k, p_{u_0}, f)| = 0$, i.e., $\varphi(k, u_0, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent in the sense of Birkhoff, asymptotically strongly Poisson stable);*
4. *for any $u \in Int(\mathcal{R}^n_+)$ with $\overline{\varphi(\mathbb{Z}_+, u, f)} \subset Int(\mathbb{R}^n_+)$ there exists a unique point $p = (p_u, f) \in \omega_x$ $(x = (u, f))$ such that the solution $\varphi(k, p_u, f)$ of equation (78) defined on $\mathbb{Z}$ with values from the compact $Q_+^{(u,f)}$;*
5. *the solution $\varphi(k, p_u, f)$ is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, strongly Poisson stable);*
6. *$\lim\limits_{k \to +\infty} |\varphi(k, u, f) - \varphi(k, p_u, f)| = 0$.*

Proof. These statements follow from Theorems 7.68, 2.43 and Corollary 6.10. □

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