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MINIMUM CONVEX COVER OF SPECIAL NONORIENTED GRAPHS

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A vertex set S of a graph G is *convex* if all vertices of every shortest path between two of its vertices are in S. We say that G has a *convex p-cover* if $X(G)$ can be covered by p convex sets. The *convex cover number* of G is the least $p \geq 2$ for which G has a convex p-cover. In particular, the *nontrivial convex cover number* of G is the least $p \geq 2$ for which G has a convex p-cover, where every set contains at least 3 elements. In this paper we determine convex cover number and nontrivial convex cover number of special graphs resulting from some graph operations. We examine graphs resulting from join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.

Keywords: *nonoriented graphs, convex covers, convex number, operations, join, cartesian product, lexicographic product, corona.*

ACOPERIREA CONVEXĂ MINIMĂ A GRAFURILOR SPECIALE NEORIENTATE

Mulţimea de vârfuri S a grafului G se numeşte *convexă* dacă pentru orice două vârfuri x şi y din S toate vârfurile ce aparţin tuturor lanţurilor de lungime minimă cu extremităţile x şi y se conţin în S. Se spune că G conţine o *p-acoperire convexă* dacă $X(G)$ poate fi acoperită cu p mulţimi convexe. *Numărul acoperirii convexe* al grafului G este cel mai mic număr $p \geq 2$ pentru care G conţine o p-acoperire convexă. În particular, *numărul acoperirii convexe netriviale* al lui G este cel mai mic număr $p \geq 2$ pentru care G conţine o p-acoperire convexă în care orice mulţime constă din cel puţin 3 vârfuri. În această lucrare noi determinăm numărul acoperirii convexe şi numărul acoperirii convexe netriviale a unor clase speciale de grafuri obţinute din următoarele operaţii pe grafuri: suma, produsul cartezian, produsul lexicografic, coroana.

Cuvinte-cheie: *grafuri neorientate, acoperiri convexe, numărul acoperirii convexe, operaţii, suma grafurilor, produs cartezian, produs lexicografic, coroană.*

Introduction

In this paper we consider only connected and nonoriented graphs. We denote by G a graph with vertex set $X(G)$ and edge set $U(G)$. An edge joining two vertices x and y in G is denoted by xy. The distance between vertices x and y in G is denoted by $d(x, y)$. The *diameter* of a graph is the length of the shortest path between the most distant nodes.

A set $S \subseteq X$ is a *clique* if any two vertices of S are adjacent in G. The *neighborhood* of a vertex x of $X(G)$ is the set of all vertices y of $X(G)$ such that x and y are adjacent, and it is denoted by $\Gamma(x)$. A vertex x is called *simplicial* if $\Gamma(x)$ is a clique. Also, a vertex x is called *universal* if $\Gamma(x) = X \backslash \{x\}$. Let S be a subset of $X(G)$. We say that $G[S]$ is the subgraph of G *induced* by S.

Now we remind some concepts from [1]. The *metric segment* $\langle x, y \rangle$ is the set of all vertices lying on a shortest path between vertices x and y in G. A set $S \subseteq X$ is called *convex* if $\langle x, y \rangle \subseteq S$ for any two vertices $x, y \in S$. The *convex hull* of $S \subseteq X$, denoted $d\text{-}conv(S)$, is the smallest convex set containing S.

A family of sets is called *convex cover* of G and is denoted by $\mathcal{P}(G)$ if the following conditions hold:

(i) Every set of $\mathcal{P}(G)$ is convex in G;

(ii) $X(G) = \bigcup_{S \in \mathcal{P}(G)} S$;

(iii) $S \not\subseteq \bigcup_{\substack{C \in \mathcal{P}(G) \\ C \neq S}} C$ for every $S \in \mathcal{P}(G)$.

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If $|\mathcal{P}(G)| = p$, then this family is called *convex p-cover* of G and is denoted by $\mathcal{P}_p(G)$ [2].

A convex cover $\mathcal{P}(G)$ of graph G is called *nontrivial convex cover* if every set $S \in \mathcal{P}(G)$ satisfies the inequalities: $3 \leq |S| \leq |X| - 1$. The minimum number of cliques that cover all the vertices of a graph is known as a *clique cover number* $\theta(G)$, introduced by Berge [3]. Also, *convex cover number* $\varphi_c(G)$ was defined as the least $p \geq 2$ for which G has a convex p-cover [2]. Similarly to $\varphi_c(G)$, we introduced *nontrivial convex cover number* $\varphi_{cn}(G)$ [4].

Let us remark that there are graphs for which there are no nontrivial convex covers. For instance, every convex simple graph has no nontrivial convex covers. A graph G is called *convex simple* if it does not contain nontrivial convex set [5]. Note that if G has a nontrivial convex cover, then $\varphi_c(G) \leq \varphi_{cn}(G)$.

The *minimum convex cover* $\mathcal{P}_{\varphi_c}(G)$ is the convex p-cover of graph G such that $p = \varphi_c(G)$. Similarly, the *minimum nontrivial convex cover* $\mathcal{P}_{\varphi_{cn}}(G)$ and the *minimum clique cover* $\mathcal{P}_\theta(G)$ of G can be defined.

By $P(G)$ we denote a family of convex sets, where $X(G) = \bigcup_{S \in P(G)} S$. We denote by $\mathcal{P}(P(G))$ a convex cover of G that consists of sets, which belong to $P(G)$. A nonempty subset S of $X(G)$ is *a nonconnecting* set in G if for every pair of vertices $x, y \in X(G) \backslash S$ with $d(x, y) = 2$ we have $\Gamma(x) \cap \Gamma(y) \cap S = \emptyset$.

A map $p_G: X(G * H) \to X(G)$, $p_G((g, h)) = g$, is a *projection* onto G and $p_H: X(G * H) \to X(H)$, $p_H((g, h)) = h$, is a *projection* onto H, where G and H are two graphs and symbol $*$ represents one of two operations: cartesian product or lexicographic product.

Convex covers of graphs were studied by many mathematicians. Any latest results on graph convex covers are given in papers [2, 4, 6-8]. Deciding whether a graph G has a convex p-cover or a nontrivial convex p-cover for a fixed $p \geq 2$ is known to be NP-complete [2, 4]. Besides, convexity was studied in some graph operations [9-11]. Further, there is particular interest in establishing of convex cover number and nontrivial convex cover number for special graphs resulting from graph operations, such as join of graphs, cartesian product of graphs, lexicographic product of graphs and corona of graphs.

Preliminary Results

First, note that for a given $P(G)$ that has no the set $X(G)$ we can easily obtain $\mathcal{P}(P(G))$ by removing from $P(G)$ all sets contained in the union of other sets of the family $P(G)$. It can easily be checked that Propositions 1, 2 and 3 are true.

Proposition 1. *Let G be a connected graph of order $n \geq 2$. Then for every vertex $x \in X(G)$ there is a convex set $S \subseteq X(G)$ such that $x \in S$ and $|S| = 2$.*

Proposition 2. *Let G be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_{\varphi_c}(G)$ such that for every set $S \in \mathcal{P}_{\varphi_c}(G)$ the condition $|S| \geq 2$ holds.*

Proposition 3. *Let G be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_\theta(G)$ such that for every set $S \in \mathcal{P}_\theta(G)$ the condition $|S| \geq 2$ holds.*

Theorem 1. *Let G be a connected graph of order $n \geq 3$ that contains a universal vertex. Then for every vertex $g \in X(G)$ the there is a convex set $S \subseteq X(G)$ such that $g \in S$ and $|S| = 3$.*

Proof. Let x be a universal vertex of G, i.e. $\Gamma(x) = X(G) \backslash \{x\}$. Suppose that $G[\Gamma(x)]$ is a disconnected graph. Further, for any two vertices $x_1 \in X(G_1)$ and $x_2 \in X(G_2)$, where G_1 and G_2 are two different connected components of $G[\Gamma(x)]$, we get a convex set $\{x, x_1, x_2\}$ that is nontrivial.

Now suppose that $G[\Gamma(x)]$ is a connected graph. In this case, every vertex y of $X(G)\backslash\{x\}$ has an adjacent vertex $z \in X(G)\backslash\{x\}$. Hence, set $\{x, y, z\}$ is convex and consists of three vertices. □

Consequence 1. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then G has a nontrivial convex cover.*

Consequence 2. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then $\varphi_c(G) = \varphi_{cn}(G)$.*

Join of Graphs

The *join* of graphs G and H, denoted $G + H$, is the graph with vertex set $X(G + H) = X(G) \cup X(H)$ and edge set $U(G + H) = U(G) \cup U(H) \cup \{xy | x \in X(G), y \in X(H)\}$.

Theorem 2 [9]. *Let G be a connected graph and K_m the complete graph of order m. Then a proper subset $C = S_1 \cup S_2$ of $X(G + K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$, is convex in $G + K_m$ if and only if either*

(i) *S_1 is a clique in G, or*

(ii) *$S_1 \subseteq X(G)\backslash S$ and $S_2 = X(K_m)$ for some nonconnecting set S of G.*

Theorem 3. *Let G be a noncomplete graph with diameter 2 and K_m the complete graph of order $m \geq 1$. Let $C = S_1 \cup S_2$ be a proper convex subset of $X(G + K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$. Then S_1 is convex in G.*

Proof. By Theorem 2, let us consider two cases. Firstly, if S_1 induces a complete subgraph of G, then evidently it is convex in G. It can be assumed that S_1 does not induce a complete subgraph of G. Thus, $S_1 \subseteq X(G)\backslash S$ and $S_2 = X(K_m)$ for some nonconnecting set S of G. Assume further that S_1 is not convex in G. Let x and y be two vertices of S_1 such that there exists a vertex $z \in \langle x, y\rangle_G$ that does not belong to S_1. Since the diameter of G is 2, we obtain $d_G(x, y) = d_{G+K_m}(x, y) = 2$ and $z \in \Gamma_G(x) \cap \Gamma_G(y)$. Hence, $z \in S_1$. From the definition of nonconnecting set, $\Gamma_G(x) \cap \Gamma_G(y) \cap S = \emptyset$ and consequently there is a contradiction. Furthermore, S_1 is convex in G. □

Theorem 4. *Let G be a connected graph of order $n \geq 1$ and K_m the complete graph of order $m \geq 1$. Then the following statements hold:*

1) *If G is complete, then $\varphi_c(G + K_m) = 2$;*

2) *If G is complete and $n + m \geq 4$, then $\varphi_{cn}(G + K_m) = 2$;*

3) *If G is noncomplete with diameter 2, then $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m) = \varphi_c(G)$;*

4) *If G is noncomplete with diameter at least 3, then $\varphi_c(G + K_m) = \varphi_{cn}(G + K_m) \leq \varphi_c(G)$.*

Proof.

1) Suppose $G = K_n$. Then, by the definition of the join of two graphs, it follows that $G + K_m$ also is complete. Here graphs K_n and K_m have at least one vertex. Further, we obtain $\varphi_c(K_n + K_m) = 2$.

2) Suppose $G = K_n$ and $n + m \geq 4$. As before, $G + K_m$ is complete. Since every nontrivial convex set has at least three elements, we have $\varphi_{cn}(K_n + K_m) = 2$.

3) Suppose G is a noncomplete graph with diameter 2. Let C be a proper convex subset of $X(G + K_m)$ that satisfies the conditions of Theorem 2. It follows from Theorem 3 that $X(G) \cap C$ is a convex set in G. Let $\mathcal{P}_{\varphi_c}(G + K_m)$ be a minimum convex cover of $G + K_m$. We obtain a family of sets $P(G) =$

$\cup_{S\in\mathcal{P}_{\varphi_c}(G+K_m)}\{X(G)\cap S\}$. It is clear that $P(G)$ has no the set $X(G)$. This yields that $|\mathcal{P}(P(G))|\leq\varphi_c(G+K_m)$. In fact, we get $\varphi_c(G)\leq\varphi_c(G+K_m)$.

By Proposition 2, a connected graph G on $n\geq 3$ vertices has a minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ such that for every set $S\in\mathcal{P}_{\varphi_c}(G)$ the condition $|S|\geq 2$ holds. Hence, we obtain a nontrivial convex cover $\mathcal{P}(G+K_m)$ by adding $X(K_m)$ to Y_i, where $Y_i\in\mathcal{P}_{\varphi_c}(G)$, for every i, $1\leq i\leq\varphi_c(G)$. Note that $|\mathcal{P}(G+K_m)|=\varphi_c(G)$ and $\varphi_c(G+K_m)\leq\varphi_{cn}(G+K_m)\leq\varphi_c(G)$. So, $\varphi_c(G+K_m)=\varphi_{cn}(G+K_m)=\varphi_c(G)$.

4) Now, assume that G is noncomplete and its diameter is at least 3. As above, it is easy to prove that every minimum convex cover of G, which satisfies Proposition 2, generates a nontrivial convex cover of $G+K_m$. Thence, $\varphi_{cn}(G+K_m)\leq\varphi_c(G)$. Note also that there are noncomplete graphs W, with diameter at least 3, for which the strict inequality $\varphi_{cn}(W+K_m)<\varphi_c(W)$ holds. For instance, graph represented in Figure 1 is the join of graphs W and K_1, where $X(K_1)=\{k\}$. This graph has a minimum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}}(W+K_1)=\{\{x_1,x_7,x_9,k\},\{x_2,x_8,x_{10},k\},\{x_3,x_5,k\},\{x_4,x_6,k\}\}$, but graph W has a minimum convex cover $\mathcal{P}_{\varphi_c}(W)=\{\{x_1,x_3\},\{x_5,x_7\},\{x_2,x_4\},\{x_6,x_8\},\{x_9\},\{x_{10}\}\}$ and further $\varphi_{cn}(W+K_1)=4$, but at the same time $\varphi_c(W)=6$. Hence, $\varphi_{cn}(W+K_1)<\varphi_c(W)$.

We stress that a nontrivial convex cover is a particular case of a convex cover. Since any vertex of $X(K_m)$ is universal in $G+K_m$, Consequence 2 implies $\varphi_c(G+K_m)=\varphi_{cn}(G+K_m)$. Thus, we get $\varphi_c(G+K_m)=\varphi_{cn}(G+K_m)\leq\varphi_c(G)$. □

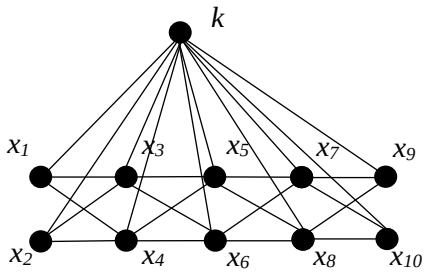


Fig.1.

Theorem 5 [9]. *Let G and H be connected noncomplete graphs. Then a proper subset $C=S_1\cup S_2$ of $X(G+H)$, where $S_1\subset X(G)$ and $S_2\subset X(H)$, is convex in $G+H$ if and only if S_1 and S_2 are cliques in G and H respectively.*

Theorem 6. *Let G and H be connected noncomplete graphs. Then the following equalities hold:*

$$\theta(G+H)=\varphi_c(G+H)=\varphi_{cn}(G+H)=max\{\theta(G),\theta(H)\}.$$

Proof. From Theorem 5, we know that every convex set of $G+H$ is a clique. Further, every convex cover of $G+H$ is a clique cover. Therefore, we have $\varphi_c(G+H)=\theta(G+H)$. Let $\mathcal{P}_{\varphi_c}(G+H)$ be a minimum convex cover of graph $G+H$. By Theorem 5, we obtain a family of sets $P(G)=\cup_{S\in\mathcal{P}_{\varphi_c}(G+H)}\{X(G)\cap S\}$. It is clear that $P(G)$ has no the set $X(G)$ and every set of $P(G)$ is a clique. This implies the inequality $|\mathcal{P}(P(G))|\leq\varphi_c(G+H)$. Thus, $\theta(G)\leq\varphi_c(G+H)$. Continuing in the same way, we see that $|\mathcal{P}(P(H))|\leq\varphi_c(G+H)$ for $P(H)=\cup_{S\in\varphi_c(G+H)}\{X(H)\cap S\}$, and further $\theta(H)\leq\varphi_c(G+H)$. Hence, we get $max\{\theta(G),\theta(H)\}\leq\varphi_c(G+H)$.

By Proposition 3, we consider minimum clique covers $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ of graphs G and H such that every set of $\mathcal{P}_\theta(G)$ and $\mathcal{P}_\theta(H)$ has at least two vertices. If $\theta(G) \geq \theta(H)$ then we construct a nontrivial clique cover $\mathcal{P}(G+H)$ that satisfies the equality $|\mathcal{P}(G+H)| = \theta(G)$. Since every convex set of $G+H$ is a clique, we unify sets X_i and Y_i, where $X_i \in \mathcal{P}_\theta(G)$ and $Y_i \in \mathcal{P}_\theta(H)$, for each $i, 1 \leq i \leq \theta(H)$, and after, unify X_i with Y_1 for each $i, \theta(H)+1 \leq i \leq \theta(G)$. By the analogy, if $\theta(G) < \theta(H)$, then it can be constructed a nontrivial clique cover $\mathcal{P}(G+H)$, where $|\mathcal{P}(G+H)| = \theta(H)$. We obtain $\varphi_c(G+H) \leq \varphi_{cn}(G+H) \leq max\{\theta(G), \theta(H)\}$. So, $\theta(G+H) = \varphi_c(G+H) = \varphi_{cn}(G+H) = max\{\theta(G), \theta(H)\}$. □

Cartesian Product of Graphs

The *cartesian product* of graphs G and H is the graph $G \times H$ on vertex set $X(G) \times X(H)$ in which vertices (g_1, h_1) and (g_2, h_2) are adjacent iff either $g_1 = g_2$ and $h_1h_2 \in U(H)$ or $h_1 = h_2$ *and* $g_1g_2 \in U(G)$.

Theorem 7 [9]. *Let G and H be two connected graphs. A subset C of $X(G \times H)$ is convex in $G \times H$ if and only if $p_G(C)$ is convex in G, $p_H(C)$ is convex in H, and $C = p_G(C) \times p_H(C)$.*

Theorem 8. *Let G be a connected graph of order n and K_m the complete graph of order m such that $n + m \geq 3$. Then the following statements hold:*

1) *If $m = 1$, then $\varphi_c(G \times K_m) = \varphi_c(G)$;*

2) *If $m = 1$ and $n \geq 4$, then $\varphi_{cn}(G \times K_m) = \varphi_{cn}(G)$;*

3) *If $m \geq 2$, then $\varphi_c(G \times K_m) = 2$;*

4) *If $m \geq 2$ and $n \geq 3$, or $m \geq 3$ and $n \geq 2$, or $m \geq 4$, then $\varphi_{cn}(G \times K_m) = 2$.*

Proof.

1) Suppose $m = 1$. Here we see that $G = G \times K_1$. Since $n + m \geq 3$, it is obvious that $\varphi_c(G \times K_m) = \varphi_c(G)$. Further assume that $n \geq 4$. In this case, graph $G \times K_1$ has a nontrivial convex cover if and only if G has a nontrivial convex cover. Consequently, we have $\varphi_{cn}(G \times K_m) = \varphi_{cn}(G)$. So, the statement 2) also holds.

3) Suppose $m \geq 2$. We choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = \{(g,k) | g \in X(G), k \in X(K_m) \backslash \{k_1\}\} \text{ and}$$

$$C_2 = \{(g,k) | g \in X(G), k \in X(K_m) \backslash \{k_2\}\}.$$

Since K_m is a complete graph, both sets C_1 and C_2 satisfy the conditions of Theorem 7. Furthermore, sets C_1 and C_2 form a convex 2-cover of graph $G \times K_m$ and $\varphi_c(G \times K_m) = 2$. If $n \geq 3$, then we see that C_1 and C_2 form a nontrivial convex 2-cover of $G \times K_m$ and further $\varphi_{cn}(G \times K_m) = 2$. Similarly, if $m \geq 3$ and $n \geq 2$, or $m \geq 4$, then we also get the equality $\varphi_{cn}(G \times K_m) = 2$. Thus, the statement 4) holds. □

Theorem 9. *Let G and H be two connected noncomplete graphs and $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \times H)} \{p_G(S)\}$. Then $|P(G)| = 1$ or $|P(G) \backslash \{X(G)\}| \geq 2$.*

Proof. Let $\mathcal{P}_{\varphi_c}(G \times H)$ be a minimum convex cover of $G \times H$. Let $|P(G)| = 1$ and $C \in P(G)$. It means that $C = X(G)$. Now, assume $|P(G) \backslash \{X(G)\}| = 1$. Further, for $S \in P(G) \backslash \{X(G)\}$ there is $S' \in \mathcal{P}_{\varphi_c}(G \times H)$ such that $p_G(S') = S$. If $\{X(G)\} \notin P(G)$, then we obtain a contradiction, because $X(G) \backslash S \neq \emptyset$, which means that $G \times H$ is not covered by convex sets. Suppose further that $\{X(G)\} \in P(G)$. From the definition of convex cover, we know that every set of $\mathcal{P}_{\varphi_c}(G \times H)$ has at least one vertex that belongs only to this set.

Hence, there is $h \in X(H)$ for which there is a vertex (g,h) of $G \times H$ that belongs to S' and does not belong to $S'' \in \mathcal{P}_{\varphi_c}(G \times H)$, where $p_G(S'') = X(G)$. By Theorem 7, for h that we have fixed before and for $g \in X(G)\backslash S$, vertices (g,h) remain uncovered in $G \times H$. It is a contradiction. □

Consequence 3. *Let G and H be two connected noncomplete graphs and $P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \times H)}\{p_H(S)\}$. Then $|P(H)| = 1$ or $|P(H)\backslash\{X(H)\}| \geq 2$.*

Theorem 10. *Let G and H be two connected noncomplete graphs. Then the following equalities hold:*

$$\varphi_c(G \times H) = \varphi_{cn}(G \times H) = min\{\varphi_c(G), \varphi_c(H)\}.$$

Proof. First, note that $|X(G)| \geq 3$ and $|X(H)| \geq 3$. By Proposition 2, there is a minimum convex cover $\mathcal{P}_{\varphi_c}(G)$ of G such that every set of $\mathcal{P}_{\varphi_c}(G)$ has at least two elements. Further, by Theorem 7, we obtain a nontrivial convex cover $\mathcal{P}(G \times H)$ that consists of sets $C_i = \{(g,h) | g \in S_i, h \in X(H)\}$, where $S_i \in \mathcal{P}_{\varphi_c}(G)$, $1 \leq i \leq \varphi_c(G)$. Note that $|\mathcal{P}(G \times H)| = \varphi_c(G)$. This implies $\varphi_{cn}(G \times H) \leq \varphi_c(G)$. For the same reason, if $\mathcal{P}_{\varphi_c}(H)$ is a minimum convex cover of H, then we obtain a nontrivial convex cover $\mathcal{P}(G \times H)$ of $G \times H$ such that $|\mathcal{P}(G \times H)| = \varphi_c(H)$ and further $\varphi_{cn}(G \times H) \leq \varphi_c(H)$. Then it stands to reason that $\varphi_c(G \times H) \leq \varphi_{cn}(G \times H) \leq min\{\varphi_c(G), \varphi_c(H)\}$.

Let $\mathcal{P}_{\varphi_c}(G \times H)$ be a minimum convex cover of graph $G \times H$. Using Theorem 7, we get two families of sets: $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \times H)}\{p_G(S)\}$ and $P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \times H)}\{p_H(S)\}$. Evidently, equalities $|P(G)| = 1$ and $|P(H)| = 1$ do not hold at the same time. By Theorem 9 and Consequence 3, we consider three cases:

Suppose $|P(G)| = 1$. In this case, the inequality $|P(H)\backslash\{X(H)\}| \geq 2$ holds. Consequently, for a convex cover $\mathcal{P}(P(H))$ we get $|\mathcal{P}(P(H))| \leq \varphi_c(G \times H)$ and $\varphi_c(H) \leq \varphi_c(G \times H)$. Now, suppose $|P(H)| = 1$. As above, we have $|\mathcal{P}(P(G))| \leq \varphi_c(G \times H)$ and $\varphi_c(G) \leq \varphi_c(G \times H)$. Similarly, if $|P(G)\backslash\{X(G)\}| \geq 2$ and $|P(H)\backslash\{X(H)\}| \geq 2$, we get $\varphi_c(G) \leq \varphi_c(G \times H)$ and $\varphi_c(H) \leq \varphi_c(G \times H)$. Combining these three cases, we obtain $min\{\varphi_c(G), \varphi_c(H)\} \leq \varphi_c(G \times H)$. Finally, $\varphi_c(G \times H) = \varphi_{cn}(G \times H) = min\{\varphi_c(G), \varphi_c(H)\}$.□

Lexicographic Product of Graphs

The *lexicographic product* of graphs G and H, denoted $G \circ H$, is the graph on vertex set $X(G \circ H) = X(G) \times X(H)$, where vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 g_2 \in U(G)$ or $g_1 = g_2$ and $h_1 h_2 \in U(H)$. The graph $G \circ H$ is called nontrivial if both graphs have at least two vertices.

Theorem 11 [11]. *Let C be a proper subset of a nontrivial connected lexicographic product $G \circ H$. If C induces a noncomplete subgraph of $G \circ H$, then C is convex if and only if the following conditions hold:*

(i) *$p_G(C)$ is convex in G;*

(ii) *$\{g\} \times X(H) \subseteq C$ for every nonsimplicial vertex $g \in p_G(C)$;*

(iii) *H is complete.*

Consequence 4. *Let C be a proper subset of a nontrivial connected lexicographic product $G \circ H$, where H is noncomplete. Then C is convex if and only if it induces a complete subgraph of $G \circ H$ and the following conditions hold:*

(i) *$p_G(C)$ induces a complete subgraph of G,*

(ii) *For every $g \in p_G(C)$, set $p_H(C^g)$ induces a complete subgraph of H, where*

$$C^g = \{(g,h) \in C | \ for \ any \ h \in H\}.$$

Theorem 12. *Let G be a connected graph of order n and K_m the complete graph of order m such that $n+m \geq 3$. Then the following statements hold:*

1) *If G is complete, then $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = 2$;*

2) *If G is complete and $n+m \geq 5$, or $n=2$ and $m=2$, then $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = 2$;*

3) *If G is noncomplete and $m=1$, then $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = \varphi_c(G)$;*

4) *If G is noncomplete, $n \geq 4$ and $m=1$, then $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_{cn}(G)$;*

5) *If G is noncomplete, has a simplicial vertex and $m \geq 2$, then $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$;*

6) *If G is noncomplete, has no simplicial vertices, $m \geq 2$, then $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$;*

7) *If G is noncomplete and $m \geq 2$, then $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$.*

Proof.

1) Suppose G is complete. Then, it is obvious that the obtained graph is complete and we get equalities $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = 2$. In addition, suppose $n+m \geq 5$, or $n=2$ and $m=2$. The obtained complete graph with at least 4 vertices has a nontrivial convex 2-cover. So, $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = 2$. The statement 2) also holds.

3) Suppose G is noncomplete. If $m=1$, then graphs $G \circ K_m$ and $K_m \circ G$ are equal to G and further we have $\varphi_c(G \circ K_m) = \varphi_c(K_m \circ G) = \varphi_c(G)$. In the same way, with condition $n \geq 4$, the statement 4) holds. In other words, $\varphi_{cn}(G \circ K_m) = \varphi_{cn}(K_m \circ G) = \varphi_{cn}(G)$. Assume that $m \geq 2$. If G has a simplicial vertex g', then we choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = \big((X(G)\backslash\{g'\}) \times X(K_m)\big) \cup \{(g', k) | k \in X(K_m)\backslash\{k_1\}\} \text{ and}$$

$$C_2 = \big((X(G)\backslash\{g'\}) \times X(K_m)\big) \cup \{(g', k) | k \in X(K_m)\backslash\{k_2\}\}.$$

Evidently, sets C_1 and C_2 satisfy the conditions of Theorem 11 and these sets form a nontrivial convex 2-cover of $G \circ K_m$. Further, we have $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = 2$. Statement 5) is satisfied.

Now assume that G has no simplicial vertices. We know from Theorem 11 that for every convex set C of $G \circ K_m$ the projection $p_G(C)$ is convex in G. Let $\mathcal{P}_{\varphi_c}(G \circ K_m)$ be a minimum convex cover of $G \circ K_m$. We get family $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c}(G \circ K_m)} \{p_G(S)\}$. Since noncomplete graph G has no simplicial vertices, it follows that $P(G)$ has no the set $X(G)$. Obviously, for convex cover $\mathcal{P}(P(G))$ of graph G we have $|\mathcal{P}(P(G))| \leq \varphi_c(G \circ H)$. Consequently, $\varphi_c(G) \leq \varphi_c(G \circ K_m)$.

Let $\mathcal{P}_{\varphi_c}(G)$ be a minimum convex cover of G. Then, sets $S_i = C_i \times X(K_m)$ form a convex cover of $G \circ K_m$, where $C_i \in \mathcal{P}_{\varphi_c}(G)$, $1 \leq i \leq \varphi_c(G)$, and further we get $\varphi_c(G \circ K_m) \leq \varphi_c(G)$. As a result, we have $\varphi_c(G \circ K_m) = \varphi_c(G)$. From Proposition 2 we obtain $\varphi_c(G \circ K_m) = \varphi_{cn}(G \circ K_m) = \varphi_c(G)$. So, the statement 6) also holds.

It follows from Consequence 4 that every proper convex subset of $K_m \circ G$ is a clique and further by Proposition 2 and Proposition 3 we have $\varphi_c(K_m \circ G) = \varphi_{cn}(K_m \circ G) = \theta(G)$. Furthermore, the statement 7) also holds. □

Theorem 13. *Let G and H be two connected noncomplete graphs. Then the following equalities hold:*

$$\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta(G \circ H) = \theta(G)\theta(H).$$

Proof. From Consequence 4 we know that every convex set of $G \circ H$ is a clique. Further, we have $\varphi_c(G \circ H) = \theta(G \circ H)$. Moreover, it can be easily checked that $\theta(G \circ H) = \theta(G)\theta(H)$. Taking into account

Proposition 2 and Proposition 3, we get $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H)$. Finally, $\varphi_c(G \circ H) = \varphi_{cn}(G \circ H) = \theta(G \circ H) = \theta(G)\theta(H)$. □

Corona of Graphs

The *corona* of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H, where $|X(G)| = n$, and then joining by an edge the ith vertex of G to every vertex in the ith copy of H.

We consider a general version of corona of graphs. Let G be a connected graph on n vertices. Let $\{g_1, g_2, \ldots, g_k\} \subseteq X(G)$ and $H_{g_1}, H_{g_2}, \ldots, H_{g_k}$, $1 \leq k \leq n$, be any connected graphs of order at least one. Then by $(G; \{g_1, g_2, \ldots, g_k\}) \odot (H_{g_1}, H_{g_2}, \ldots, H_{g_k})$ is denoted a graph obtained by taking one copy of G and after, by joining every vertex g_i to every vertex of H_{g_i}, $1 \leq i \leq k$. If $H_{g_1} = H_{g_2} = \cdots = H_{g_k} = H$, then we simply denote $(G; \{g_1, g_2, \ldots, g_k\}) \odot H$. If also $k = n$, then $(G; \{g_1, g_2, \ldots, g_k\}) \odot H$ is the corona $G \odot H$.

Theorem 14 [10]. *Let G be a connected graph and H be any graph, with $\{g_1, g_2, \ldots, g_k\} \subseteq X(G)$ and $H_{g_1}, H_{g_2}, \ldots, H_{g_k}$being the corresponding copies of H. A nonempty set $C \subseteq X(G \odot H)$ is convex in $G \odot H$ if and only if it satisfies one of the following conditions*:

(i) *C is a convex set in G;*

(ii) *C induces a complete subgraph of H_g for a vertex $g \in \{g_1, g_2, \ldots, g_k\}$;*

(iii) $(G \odot H)[C] = (G[S]; \{s_1, s_2, \ldots, s_l\}) \odot (H^*_{s_1}, H^*_{s_2}, \ldots, H^*_{s_l})$, *$S$ is convex in the graph G, $\{s_1, s_2, \ldots, s_l\} \subseteq S$, $\{s_1, s_2, \ldots, s_l\} \subseteq \{g_1, g_2, \ldots, g_k\}$ and $X(s_i + H^*_{s_i})$ is convex in $s_i + H_{s_i}$ for each i, $1 \leq i \leq l$, where $H^*_{s_i}$ is a subgraph of H_{s_i}.*

Theorem 15. *Let G and H be two connected graph of order n and m, with $\{g_1, g_2, \ldots, g_k\} \subseteq X(G)$,* where $1 \leq k \leq n$. *Then the following statements hold:*

1) *If $n = 1$, H is complete, then $\varphi_c(G \odot H) = 2$;*

2) *If $n = 1$, H is complete and $m \geq 3$, then $\varphi_{cn}(G \odot H) = 2$;*

3) *If $n = 1$, H is noncomplete with diameter 2, then $\varphi_c(G \odot H) = \varphi_{cn}(G \odot H) = \varphi_c(H)$;*

4) *If $n = 1$, H is noncomplete with diameter at least 3, then $\varphi_c(G \odot H) = \varphi_{cn}(G \odot H) \leq \varphi_c(H)$;*

5) *If $n \geq 2$, then $\varphi_c((G; \{g_1, g_2, \ldots, g_k\}) \odot H) = 2$;*

6) *If $n \geq 2$ and $k*m+n \geq 4$, then $\varphi_{cn}((G; \{g_1, g_2, \ldots, g_k\}) \odot H) = 2$.*

Proof. Suppose $n = 1$. In fact, $\varphi_c(K_1 \odot H) = \varphi_c(K_1 + H)$ and the statements 1), 2), 3), 4) hold.

5) Suppose $n \geq 2$. It can easily be checked that sets $X(H_{g_1})$ and $X(G) \cup \bigcup_{i=2}^{k} X(H_{g_i})$ satisfy the conditions of Theorem 14 and further form a convex 2-cover of graph $(G; \{g_1, g_2, \ldots, g_k\}) \odot H$. This implies $\varphi_c((G; \{g_1, g_2, \ldots, g_k\}) \odot H) = 2$.

6) Now suppose that $n \geq 2$ and $k * m + n \geq 4$. In other words, the cardinality of the set $X((G; \{g_1, g_2, \ldots, g_k\}) \odot H)$ is at least 4. Taking into account Theorem 14, we show the existence of nontrivial convex 2-covers of $(G; \{g_1, g_2, \ldots, g_k\}) \odot H$ in two cases:

a) If $m = 1$, then we choose a vertex $g' \in \Gamma(g) \backslash X(H_g)$ for a vertex $g \in \{g_1, g_2, \ldots, g_k\}$ and obtain a nontrivial convex 2-cover:

$$\mathcal{P}_2\big((G; \{g_1, g_2, \ldots, g_k\}) \odot H\big) = \{\{g, g'\} \cup X(H_g), X(G) \cup \bigcup_{g'' \in \{g_1, g_2, \ldots, g_k\}, g'' \neq g} X\big(H_{g''}\big)\}.$$

b) If $m \geq 2$, then we choose a vertex $h \in H_g$ for a vertex $g \in \{g_1, g_2, \ldots, g_k\}$ and obtain a nontrivial convex 2-cover:

$$\mathcal{P}_2\big((G;\{g_1,g_2,\dots,g_k\})\odot H\big)=\{\{g\}\cup X(H_g),\{h\}\cup X(G)\cup\bigcup_{g'\in\{g_1,g_2,\dots,g_k\},g'\neq g}X(H_{g'})\}.$$

The theorem is proved. □

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