Averaging Principle on Semi-axis for Semi-linear Differential Equations

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We establish an averaging principle on the real semi-axis for semi-linear equation

$$x' = \varepsilon(\mathcal{A}x + f(t) + F(t,x)) \tag{1}$$

with unbounded closed linear operator $\mathcal{A}$ and asymptotically Poisson stable (in particular, asymptotically stationary, asymptotically periodic, asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent) coefficients (see [1] for details). Under some conditions we prove that there exists at least one solution, which possesses the same asymptotically recurrence property as the coefficients, in a small neighborhood of the stationary solution to the averaged equation, and this solution converges to the stationary solution of averaged equation uniformly on the real semi-axis when the small parameter approaches to zero.

Consider the following differential equation

$$x'(t) = \mathcal{A}x(t) + f(t) + F(t, x(t)), \tag{2}$$

where $f \in C(\mathbb{R}_+, \mathfrak{B})$, $F \in C(\mathbb{R}_+ \times \mathfrak{B}, \mathfrak{B})$ and $\mathcal{A} : D(\mathcal{A}) \to \mathfrak{B}$ is a linear operator acting from $D(\mathcal{A}) \subseteq \mathfrak{B}$ to $\mathfrak{B}$.

We will consider a differential equation (2) when the linear operator $\mathcal{A}$ is an infinitesimal operator which generates a C_0-semigroup $\{U(t)\}_{t\geq 0}$.

Definition. *A semigroup of operators $\{U(t)\}_{t\geq 0}$ is said to be hyperbolic if there is a projection $\mathcal{P}$ and constants $\mathcal{N}, \nu > 0$ such that each $U(t)$ commutes with $\mathcal{P}$, $U(t) : Im\mathcal{Q} \mapsto Im\mathcal{Q}$ is invertible and for every $x \in E$*

$$|U(t)\mathcal{P}x| \leq \mathcal{N}e^{-\nu t}|x|, \quad \text{for } t \geq 0;$$

$$|U_{\mathcal{Q}}(t)x| \leq \mathcal{N}e^{\nu t}|x|, \quad \text{for } t < 0;$$

where $\mathcal{Q} := I - \mathcal{P}$ and, for $t < 0$, $U_{\mathcal{Q}}(t) := [U(-t)\mathcal{Q}]^{-1}$.

Denote by Ψ the family of all decreasing, positive bounded functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to+\infty} \psi(t) = 0$.

Below we will use the following conditions:

(G1): $F(t, 0) = 0$ for any $t \geq 0$;

(G2): there exists a positive constant L such that

$$|F(t, x_1) - F(t, x_2)| \leq L|x_1 - x_2|$$

for any $x_1, x_2 \in \mathfrak{B}$ and $t \in \mathbb{R}_+$;

(G3): there exists functions $\bar{F} \in C(\mathfrak{B}, \mathfrak{B})$ and $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (respectively, an element $\bar{f} \in \mathfrak{B}$ and function $\omega \in \Psi$) such that

$$\frac{1}{T}\Big|\int_t^{t+T} [F(s, x) - \bar{F}(x)]dt\Big| \leq \omega(T, r)$$

(respectively,

$$\frac{1}{T}\Big|\int_t^{t+T} [f(s) - \bar{f}]dt\Big| \leq \omega(T) \quad)$$

and $\omega(\cdot, r) \in \Psi$ for any $t \in \mathbb{R}_+$, $T > 0$, $r > 0$ and $x \in B[0, r]$.

The standard form of (2) is

$$x'(t) = \varepsilon(\mathcal{A}x(t) + f(t) + F(t, x(t))). \tag{3}$$

We will consider also the following equations

$$x'(t) = \mathcal{A}x(t) + f(\frac{t}{\varepsilon}) + F(\frac{t}{\varepsilon}, x(t))), \tag{4}$$

where $f_\varepsilon(t) := f(\frac{t}{\varepsilon})$ (respectively, $F_\varepsilon(t,x) := F(\frac{t}{\varepsilon}, x)$) for any $t \in \mathbb{R}_+$ (respectively, for any $(t,x) \in \mathbb{R}_+ \times \mathfrak{B}$), $\varepsilon \in (0, \varepsilon_0]$ and ε_0 is some fixed small positive number. Along with equations (3)-(4) we will consider the following averaged differential equation

$$x'(t) = \mathcal{A}x(t) + \bar{f} + \bar{F}(x(t)). \tag{5}$$

Theorem. *Suppose that the following conditions hold:*

1. *$-\mathcal{A}$ is a sectorial hyperbolic operator;*
2. *the function F satisfies conditions (**G1**)-(**G3**);*
3. *the functions f and F are Lagrange stable;*
4. $$L < \frac{\nu}{2\mathcal{N}} \ ,$$
   *where $\mathcal{N}$ and ν there are the numbers figuring in the **Definition**.*

Then there exists a positive number ε_0 such that for any $0 < \varepsilon \leq \varepsilon_0$

1. *equation (4) has a unique solution $\psi_\varepsilon \in C_b(\mathbb{R}_+, \mathfrak{B})$ with $\mathcal{P}\psi_\varepsilon(0) = 0$;*

2. *if the function* $f \in C_b(\mathbb{R}_+, \mathfrak{B})$ *is asymptotically stationary (respectively,* τ*-periodic, quasi-periodic, Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, positively Lagrange stable), then equation (1) has a unique solution* $\varphi_\varepsilon \in C_b(\mathbb{R}_+, \mathfrak{B})$ *with* $\mathcal{P}\varphi_\varepsilon(0) = 0$ *which is asymptotically stationary (respectively,* τ*-periodic, quasi-periodic, Bohr almost periodic, Bohr almost automorphic, Birkhoff recurrent, positively Lagrange stable);*

3. $$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}_+} |\psi_\varepsilon(t) - \bar{\psi}| = 0,$$

where $\bar{\psi}$ *is a unique stationary solution of equation (5).*

Bibliography

[1] D. N. Cheban, *Asymptotically Almost Periodic Solutions of Differential Equations.* Hindawi Publishing Corporation, New York, 2009, ix+186 pp.