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About Quadratic Functional Equations on Quasigroups

Fedir Sokhatsky, Halyna Krainichuk

Abstract

Functional equations over binary quasigroups are under consideration. An equation is called: *quadratic*, if each individual variable has either two appearances or none; *cancellable*, if a variable has two appearances and another none in a proper subterm; *reducible*, if it is equivalent to a system of equations such that every of which has less number of individual variables than the given one. Only the functional equations of unipotency, commutativity, associativity and mediality are noncancellable, and the irreducible ones are the same except the mediality. The criteria for the parastrophic equivalency of the equations up to the noncancellable equations were found.

Keywords: quasigroup, quadratic functional equation, reducible, cancelable, parastrophically equivalent equations.

1 Introduction

The concept of functional equation is taken from J.Aczél [1]. This article studies a class of functional equations over a quasigroup environment (briefly, qen), i.e. over a set $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2$ of functions defined on the same carrier Q, where Φ_0 is a set of nullary operations, Φ_1 a group of bijections, Φ_2 is a set of binary quasigroup operations and the set Φ is closed under compositions of binary and unary or nullary operations as well as unary and nullary operations from Φ. Moreover, we restrict our attention only to quadratic functional equations (each individual variable appears twice or never).

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The considered problem is to find conditions when a functional equation can be reduced to some shorter functional equations. This problem was under consideration in many works [2], [3], [4], [5] [6], [7]. We continue the research in this area, but we propose another approach for solving this problem.

2 Preliminaries

A functional equation is called: 1) *reducible*, if it is equivalent to a system of equations every of which has less number of individual variables than the given one; 2) *cancelable*, if it has a subterm having all appearances of an individual variable and none of another; 3) *parastrophically cancelable*, if it is parastrophically equivalent to a cancelable functional equation.

3 Equations in 1 and 2 individual variables

The equations $F(x,x) = a$ and $F(x,y) = F(y,x)$ are said to be a functional equation of *idempotency* and commutativity, respectively.

Theorem 1. *Quadratic equations in* 1 *individual variable are parastrophically noncancellable and parastrophically equivalent to the functional equation of idempotency.*

Theorem 2. *Let $\omega = \upsilon$ be a quadratic quasigroup functional equation in* 2 *individual variables, then the following propositions are equivalent:*
1) the equation $\omega = \upsilon$ is parastrophically noncancellable;
2) a subterm of the length 2 is repetirion-free;
3) the equation $\omega = \upsilon$ is parastrophically equivalent to commutativity.

4 Equations in 3 and 4 individual variables

The functional equations

$$F_1(F_2(x,y),z) = F_3(x,F_4(y,z)),$$

$$F_1(F_2(x,y),F_3(z,v)) = F_4(F_5(x,z),F_6(y,v)).$$

are said to be a equation of *associativity* and *mediality*, respectively.

Theorem 3. *Let $\omega = \upsilon$ be a quadratic quasigroup functional equation in* 3 *individual variables, then the following propositions are equivalent:*
1) the equation $\omega = \upsilon$ is parastrophically noncancellable;
2) every subterm (including outer one) of the length 2,3 is repetirion-free and two subterms of the length 2 have exactly 1 *variable in common;*
3) the equation $\omega = \upsilon$ is parastrophically equivalent to the functional equation of associativity.

Theorem 4. *Let $\omega = \upsilon$ be a quadratic quasigroup functional equation in* 4 *individual variables, then the following statements are equivalent:*
1) the equation $\omega = \upsilon$ is parastrophically noncancellable;
2) the equation satifies all of the following conditions:
2.1) its subterms of the length 2, 3 and 4 (including the outer ones) are repetition-free;
2.2) any 2 subterms of the length 2 has different sets of individual variables;
2.3) if $xy \cdot z$ is a subterm, then neither xz, nor yz is a subterm;
2.4) if $(xy \cdot z)u$ is a subterm, then zu is not a subterm;
2.5) if xy is a subterm, then there is no u such that xu and yu are subterms.
3) the equation $\omega = \upsilon$ is parastrophically equivalent to the functional equation of mediality.

5 Conclusion

Each noncancellable quadratic functional equation in n individual variables is parastrophically equivalent to unipotency if $n = 1$; commutativity if $n = 2$; assiciativity if $n = 3$; mediality if $n = 4$. It does not exist if $n > 4$. Any functional equation of mediality is reducible.

The atoms of the semilattice of quadratic varieties are varieties being defined by the parametrical unipotency, commutativity and associativity.

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Fedir Sokhatsky[1], Halyna Krainichuk[2]

[1]Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NANU, Lviv, Ukraine
E–mail: fmsokha@ukr.net
ORCID: https://orcid.org/0000-0003-4969-5651

[2]Vinnytsia National Technical University, Vinnytsia, Ukraine
E–mail: kraynichuk@ukr.net
ORCID: https://orcid.org/0000-0002-6822-4858