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Algorithm to Minimize the Worst-Case Regret Function

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Abstract

This paper examines mathematical model of decision-making under conditions of uncertainty, when Savage's regret function is used as the objective function. Numerical algorithm for solving the corresponding problems are proposed, based on the implementation of parallel minimization of indicators for each state of nature and the indicator of greatest regret. The algorithms are developed based on the generalized gradient projection method.

Keywords: uncertainty, regret function, optimization, numerical algorithm.

Introduction

When making decisions under conditions of uncertainty, and sometimes under conditions of risk, it is necessary to take into account the discrepancy between the actual result obtained (win or loss) and the "allowed" one – the best option that corresponded to the case if the current state of nature was guessed. This disagreement is measured by a value called "regret". The concept of regret, namely in matrix form, was first introduced by Savage in [1].

We will follow work [2], which defines the Savage function of the form

$$R_s(u) = \max_{i \in I_m} (\bar{r}(u, q_i)), \tag{1}$$

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where $\bar{r}\,(u, q_i) = (r\,(u, q_i) - r_i^*)$, $r_i^* = r\left(u_{(i)}^*, q_i\right) = \underset{u \in D}{min}\; r\,(u, q_i)$.

Here $r\,(u, q_i)$ is the cost function for the case when the decision is made $u \in D$, and nature is determined by the state $q_i \in Q = \{q_1, q_2, \ldots, q_m\}$, $i \in I_m = \{i : i = 1, 2, \ldots, m\}$.

Let us assume, on a compact convex set U of n-dimensional Euclidean space $\mathrm{E^n}$, all functions $r(u, q_i)$, $i \in I_m$ and $F_j(u), j \in J_t = \{1, 2, ..., t\}$ are continuous and convex. Obviously, the functions also have the same properties $\bar{r}\,(u, q_i)$ and $R_s\,(u)$. Let subgradients $r'_u\,(u, q_i)$ of functions $r\,(u, q_i)$, $i \in I_m$, and subgradients $g_j(u)$ of functions $F_j(u)$, $j \in J_t$, exist for all boundary points of the set U, and they are uniformly bounded in U. The set $D = \{u \in U : F_j(u) \leq 0, j \in J_t\}$.

To minimize the function of worst-case regrets $R_s\,(u)$ on set D, parallel implementation of **m+1** iterative processes will be considered. First $\boldsymbol{m}$ of them is aimed at obtaining "sufficient" estimates for unknown values of r_i^*, $i \in I_m$; and process number **(m+1)** –- to determine approximations of the value $R_S^* = \underset{u \in D}{min} R_S\,(u)$. It is assumed that the set U has a relatively simple structure, in the sense that for an arbitrary point $u \in E^n$, one can precisely determine its projection onto U. We denote this projection by $\Pi_U\,(u)$.

If we define a function $F(u) = \underset{j \in J_t}{max} F_j(u)$, then the mathematical model in the new formulation is presented as follows:

$$\begin{cases} R_s(u) \to min \\ F(u) \leq 0 \\ u \in U \end{cases}.$$

Thus, the specified **m+1** processes are implemented according to the scheme:

$$\begin{cases} u_{(i)}^{k+1} = \Pi_U\left(u_{(i)}^k - h_{(i)k} \cdot \eta_{(i)}^k\right), i \in I_m \\ u^{k+1} = \Pi_U\left(u^k - h_k \cdot \eta^k\right). \end{cases} \tag{2}$$

$$\text{Here: } \eta_{(i)}^k = \begin{cases} \frac{r_u^{'}(u_{(i)}^k, q_i)}{\left\| r_u^{'}(u_{(i)}^k, q_i) \right\|} \text{ if } F\left(u_{(i)}^k\right) \le \bar{\varepsilon}_k \text{ and } r_u^{'}(u_{(i)}^k, q_i) \ne 0 \\ \frac{g(u_{(i)}^k)}{\left\| g(u_{(i)}^k) \right\|} \text{ if } F(u_{(i)}^k) > \bar{\varepsilon}_k \text{ and } g\left(u_{(i)}^k\right) \ne 0 \\ 0, \text{ in other cases, where } i \in I_m \end{cases} \tag{3}$$

$$\eta^k = \begin{cases} \frac{g_s^k(u^k)}{\|g_s^k(u^k)\|}, \text{ if } F(u^k) \le \bar{\varepsilon}_k \text{ and } g_s^k(u^k) \ne 0, \\ \frac{g(u^k)}{\|g(u^k)\|}, \text{ if } F\left(u^k\right) > \bar{\varepsilon}_k \text{ and } g\left(u^k\right) \ne 0, \\ 0, \text{ in other cases} \end{cases}, \tag{4}$$

where $g_s^k\left(u^k\right)$ – arbitrary subgradient function

$$R_S^k(u) = \max_{i \in I_m} \left[r(u, q_i) - r\left(u_{(i)}^k, q_i\right) \right] \tag{5}$$

and $g(u^k)$ – arbitrary subgradient function $F(u)$, calculated at point $u = u^k$.

Let $\mathbf{U}^*$ – set points minimum functions Savage in the $\mathbf{D}$ area. If it is additionally assumed that subgradients $g_s^k(u)$ and $g(u)$ of functions $R_s^k(u)$ and $F(u)$ are uniformly bounded on $\mathbf{U}$, then the following theorem holds.

Theorem. *Let for number sequences $h_{(i)k}, h_k$ and $\bar{\varepsilon}_k$, the following conditions are met:*

$$h_{(i)k}, h_k \ge 0;\ h_{(i)k}, h_k \to 0;\ \bar{\varepsilon}_k > 0;\ \bar{\varepsilon}_k \to 0;\ \frac{h_{(i)k}}{\bar{\varepsilon}_k};\ \frac{h_k}{\bar{\varepsilon}_k} \to 0,$$

$$\sum_{k=0}^{\infty} h_{(i)k}\bar{\varepsilon}_k = \infty, \sum_{k=0}^{\infty} h_k\bar{\varepsilon}_k = \infty. \tag{6}$$

When the above conditions (2) - (6) are met:

$$\lim_{k\to\infty} \min_{u^*\in U^*} \left\| u^k - u^* \right\| = 0, \lim_{k\to\infty} R_S^k \left(u^k \right) = R_S^*,$$

where R_S^* – minimum function value (1), and $R_S^k(u)$ is determined according to (5).

Conclusions. Suggested algorithm can apply in decision making under uncertain conditions when nature could be in a finite, comparatively small number states, and the decision maker chooses the minimum of worst-case regrets as an indicator.

References

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