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Finiteness for Sets with Atoms

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Abstract

We present various definitions for finite sets in the framework of finitely supported sets. We compare these definitions and emphasize their independence.

Keywords: finitely supported sets; several independent definitions of finite sets with atoms.

1 Finitely Supported Sets

Finitely supported structures [1] are related to the permutation models of Zermelo-Fraenkel set theory with atoms (ZFA). More recently, finitely supported sets have been developed in Zermelo-Fraenkel (ZF) set theory by equipping classical ZF sets with actions of a group of finite permutations of some basic elements called atoms that can only be compared for equality. These atomic sets were used to study the renaming of the bound variables by fresh names in the theory of programming, as well as to describe automata, languages and Turing machines operating over infinite alphabets. The theory of finitely supported algebraic structures [1] allows a discrete representation of possibly infinite structures containing enough symmetries to be concisely handled. More exactly, this theory allows us to treat as equivalent the objects of a structure that have a certain degree of similarity; thus, we focus only on those objects that are 'really different' by using the notions of 'finite support'.

Let us consider a set A of infinitely many atoms. We denote by P_A the set of all bijections of A permuting only finitely many atoms. A

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P_A-set is a pair $(X, \cdot)$, where X is a ZF set and $\cdot$ is a group action of P_A on X. If $(X, \cdot)$ is a P_A-set, we say that a finite subset S of atoms supports x (namely, x is *finitely supported*) if for each $\pi \in P_A$ fixing pointwise any element of S, we have $\pi \cdot x = x$. If $(X, \cdot)$ is a P_A-set, we say that X is an *invariant set* whenever each $x \in X$ is finitely supported. If there is a finite set supporting x, then there exists a least finite set $supp(x)$ supporting x which is called *the support of* x. An empty supported element is *equivariant*.

Let $(X, \cdot)$ and $(Y, \triangleright)$ be P_A-sets. According to [1], the set A of atoms is an invariant set with the P_A-action $\diamond$ defined by $\pi \diamond a := \pi(a)$ for all $\pi \in P_A$ and $a \in A$. Non-atomic sets are trivially invariant, i.e. they are equipped with the action $\diamond$ defined as $(\pi, x) \mapsto x$. The Cartesian product $X \times Y$ is a P_A-set with the P_A-action $\otimes$ defined by $\pi \otimes (x, y) = (\pi \cdot x, \pi \triangleright y)$ for all $\pi \in P_A$ and all $x \in X$, $y \in Y$. The disjoint union $X + Y = \{(0, x) \,|\, x \in X\} \cup \{(1, y) \,|\, y \in Y\}$ is a P_A-set with the P_A-action $\star$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \triangleright y)$ if $z = (1, y)$. For $(X, \cdot)$ and $(Y, \triangleright)$ invariant sets, $(X \times Y, \otimes)$ and $(X + Y, \star)$ are also invariant sets. The powerset $\mathcal{P}(X)$ is a P_A-set with the P_A-action $\star$ defined by $\pi \star Z := \{\pi \cdot z \,|\, z \in Z\}$ for all $\pi \in P_A$ and $Z \subseteq X$. We denote by $\mathcal{P}_{fs}(X)$ the set formed from those subsets of X that are finitely supported as elements in $\mathcal{P}(X)$ with respect to the action $\star$. A subset Z of an invariant set $(X, \cdot)$ is called *finitely supported* if and only if $Z \in \mathcal{P}_{fs}(X)$. A subset Z of an invariant set $(X, \cdot)$ is *uniformly supported* if there is a finite set of atoms S such that $supp(x) \subseteq S$ for all $x \in X$. A function $f : Z \to T$ is *finitely supported* if $f \in \mathcal{P}_{fs}(X \times Y)$. The set of all finitely supported functions from Z to T is denoted by T^Z_{fs}. We have that Y^X is a P_A-set with the P_A-action $\widetilde{\star}$ defined by $(\pi \widetilde{\star} f)(x) = \pi \triangleright (f(\pi^{-1} \cdot x))$ for all $\pi \in P_A$, $f \in Y^X$ and $x \in X$. A function $f : Z \to T$ is supported by a finite set $S \subset A$ if and only if for all $x \in Z$ and all $\pi \in P_A$ fixing S pointwise we have $\pi \cdot x \in Z$, $\pi \triangleright f(x) \in T$ and $f(\pi \cdot x) = \pi \triangleright f(x)$. Two finitely supported sets X and Y are called *equipollent* if there exists a finitely supported bijection between them. The equipollence relation is an equivariant equivalence relation on the family of all finitely supported sets [2].

The *cardinality* of X is the equivalence class of all finitely supported sets that are equipollent to X, and is denoted by $|X|$. Two finitely supported sets X and Y have the same cardinality (i.e., $|X| = |Y|$) if and only if there exists a finitely supported bijection $f : X \to Y$.

2 Several Definitions of Finiteness

We present several non-equivalent definitions of finite in the framework of finitely supported sets, generalizing the classical approach in the ZF set theory [3] and provide various relationship results between them.

Definition 1. *Let X be a finitely supported set:*

1. *X is called* 1-finite *if $|X \times X| \neq |X|$.*
2. *X is called* 2-finite *if $|X + X| \neq |X|$.*
3. *X is called* 3-finite *if every finitely supported injective function $f : X \to X$ is also surjective (or, equivalently, there is no finitely supported injective function from $\mathbb{N}$ to X).*
4. *X is called* 4-finite *if every finitely supported chain of X is finite.*
5. *X is called* 5-finite *if every uniformly supported subset of X is finite.*
6. *X is called* 6-finite *if for every finitely supported countable chain of finitely supported sets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_i \subseteq \ldots \subseteq X$ is stationary.*
7. *X is called* 7-finite *if every finitely supported family of finitely supported subsets of X, totally ordered with respect to the inclusion relation, has a maximal element.*
8. *X is called* 8-finite *if any finitely supported subset of X is either finite or it has a finite complement.*
9. *X is called* 9-finite *if X is can be expressed as $\{x_1, \ldots, x_n\}$.*

We establish several implications between the different forms of finiteness in the framework of finitely supported sets.

Theorem 2. *Let X be a finitely supported set.*

- *X is 9-finite $\Rightarrow$ X is 7-finite $\Rightarrow$ X is 4-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*
- *X is 9-finite $\Rightarrow$ X is 5-finite $\Rightarrow$ X is 4-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*
- *X is 9-finite $\Rightarrow$ X is 8-finite $\Rightarrow$ X is 6-finite $\Leftrightarrow$ $\mathcal{P}_{fs}(X)$ is 3-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*
- *X is 9-finite $\Rightarrow$ X is 7-finite $\Rightarrow$ X is 6-finite;*
- *X is 5 -finite $\Rightarrow$ $\mathcal{P}_{fin}(X)$ is 5-finite, where $\mathcal{P}_{fin}(X)$ is the family of all finite subsets of X;*
- *X is 5 -finite $\Rightarrow$ $X_{fs}^{A^n}$, where A^n (the n-times Cartesian product of A) and X_{fs}^{nA} (where nA is the n-times disjoint union of A) are 5-finite;*
- *X is 5 -finite $\Rightarrow$ $X_{fs}^{\mathcal{P}_n(A)}$ is 5-finite, where $\mathcal{P}_n(A)$ is the set of all n-sized subsets of A.*

3 Independence of Several Definitions of Finite

We present examples of some finitely supported sets satisfying certain definitions of finite while they do not satisfy others, emphasizing in this way the independence of various definitions for finite sets. In this table, we assume that X is 5 -finite, Y has the property that $\mathcal{P}_{fs}(Y)$ is 5-finite (with $|X|, |Y| \geq 2$), and consider Z to be one of the sets kA or A^k for some $k \in \mathbb{N}$, $k > 0$.

Set	1 f	2 f	3 f	4 f	5 f	6 f	7 f	8 f	9 f
A;	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
nA, A^n, $n > 1$	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No
$\mathcal{P}_{fs}(Z)$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_n(A))$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(Z)^{\mathcal{P}_n(A)}_{fs}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(A)^{Z}_{fs}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
X^{Z}_{fs}	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fin}(X)$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$(\mathcal{P}_{fs}(Z)^{Y}_{fs}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$A \cup \mathbb{N}$	Yes	Yes	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(A \cup \mathbb{N})$	Yes	No	No	No	No	No	No	No	No
$A \times \mathbb{N}$	Yes	No	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_{fs}(A))$	?	No	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_{fs}(\mathcal{P}_{fs}(A)))$	No	No	No	No	No	No	No	No	No

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