MOLDOVA STATE UNIVERSITY
Vladimir Andrunachievici Institute
of Mathematics and Computer Science

ALEXANDRU LAZARI

STOCHASTIC PROCESSES WITH FINAL SEQUENCES OF STATES

LINEAR RECURRENCE METHOD FOR DECISION-MAKING PROBLEMS

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Reviewers

Dmitrii LOZOVANU, Dr. Habil., Professor
Anatol GODONOAGA, PhD, Associate Professor

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Introduction

The mathematical study of processes evolving in time has developed at the intersection of several major areas: algebra, probability theory, numerical analysis, optimization, theoretical computer science, and game theory. A common feature of these areas is the attempt to describe complex temporal behavior through a finite set of rules, states, and parameters. Recurrence relations provide an algebraic description of evolution, while stochastic processes introduce uncertainty into this evolution. Markov models, in turn, offer a particularly effective framework for representing systems whose future behavior depends on their present state.

This book consists of 10 chapters. It builds on and extends the author's previous results presented in [102] and [91]. Its main contribution is the development of a unified theory for connecting homogeneous linear recurrences with the analysis of stochastic processes evolving toward prescribed final sequences of states. Beginning with the fundamental properties and structural theory of homogeneous linear recurrences, the book progressively develops their applications to stochastic processes, compositions, optimization problems, stochastic games, hidden Markov models and hierarchical stochastic processes.

The importance of the proposed approach lies in its ability to provide a common mathematical language for a broad class of temporal and probabilistic models. The treatment is both theoretical and constructive: fundamental concepts and structural results are complemented by explicit formulas, matrix representations, algorithms and computational procedures. In this way, the book contributes not only to the theoretical study of recurrences and stochastic processes, but also to the development of practical methods for analyzing complex systems with sequential, hidden and hierarchical dynamics.

Chapter 1 introduces homogeneous linear recurrences and establishes the algebraic foundation. It presents the main definitions and properties, direct solution

formulas, methods for increasing the order of a recurrence, and linear combinations of recurrence sequences. The chapter also considers extensions over the set of integers, recurrence functions, reversibility and continuity, algebraic operations, polynomials, inhomogeneous recurrences, recurrences over matrix sets, and degenerate cases. These topics provide the basic language needed in the subsequent chapters.

Chapter 2 is devoted to order minimization methods and fundamental decision criteria. It examines the dimension of spaces of homogeneous linear recurrences and derives corresponding inequalities. Two minimization procedures are developed: the elimination of characteristic zeros and the rank-based method. The chapter also introduces minimal generating vectors and criteria for recognizing whether a sequence satisfies a homogeneous linear recurrence. Further sections address rational fractions, convergence, periodicity, and inclusion criteria. These results are essential for obtaining the simplest recurrence representation of a sequence or stochastic quantity.

Chapter 3 develops several advanced aspects of homogeneous linear recurrences. It studies subsequences, higher-dimensional recurrences, and recurrences defined over numerical rings. Special attention is given to recurrences associated with sign-based ring subsets, as well as to the Littlewood, Newman, and Borwein classes. The final part of the chapter discusses stability and perturbations, emphasizing how changes in coefficients or initial conditions influence the behavior of recurrences.

Chapter 4 applies recurrence methods to probability distributions and computational problems. It begins with techniques for improving the efficiency of matrix computations and continues with probabilistic characteristics of random variables. Numerical differentiation of rational fractions and composed functions is used in the analysis of recurrence distributions. The chapter then examines homogeneous linear recurrence distributions, including their parallelization, and concludes with a combinatorial result concerning the cardinality of sets of numbers with prescribed digits and a prescribed sum of digits.

Chapter 5 establishes the connection between homogeneous linear recurrences and the asymptotic behavior of Markov processes. It presents an algorithm for determining limiting probabilities and investigates the computation of differential compo-

nents through recurrence relations. An alternative approach based on the separation of eigenvalues is also considered. The chapter concludes with enhanced methods for determining both the limiting probability matrix and the differential components of a finite-state Markov process. These methods demonstrate how algebraic and spectral techniques can complement direct probabilistic analysis.

Chapter 6 introduces stochastic systems with final sequences of states. After presenting the fundamental definitions, the chapter considers systems with independent states, unit transition times, a final critical state, random transition times, interdependent states, rational transition times, and nonnegative real transition costs. The general formulation allows the probability, expected duration, and other characteristics of reaching a prescribed final sequence to be described by systems of homogeneous linear recurrences.

Chapter 7 studies compositions and optimization problems for stochastic systems. Sequential and parallel compositions are introduced for systems with final sequences of states, both in the independent and interdependent cases. The chapter also discusses geometric and signomial programming and develops optimization methods for stochastic systems subject to final-sequence conditions. These results provide a framework for selecting system parameters or transition rules in order to maximize or minimize a specified objective.

Chapter 8 extends the theory to stochastic systems with multiple final sequences of states. It begins with zero-order Markov processes and proceeds to general stochastic systems. Several methods for handling multiple sequences are developed, including state-merging strategies, the Knuth-Morris-Pratt approach, and the Aho-Corasick approach. The latter algorithms were originally developed for efficient pattern recognition and string matching; the Knuth-Morris-Pratt method and related state-automaton ideas provide natural tools for recognizing several possible final sequences simultaneously. The chapter concludes with an enhanced method and the determination of total alert time for Markov processes with multiple final sequences.

Chapter 9 applies the preceding theory to games defined on stochastic systems with final sequences of states. It first considers homogeneous linear recurrence games

and then develops games on zero-order Markov processes and general stochastic systems. Winning probabilities, optimal cooperative strategies, and real transition costs are analyzed. The chapter also adapts state-merging methods, the Knuth-Morris-Pratt approach, and extended network methods to stochastic games with one or several final sequences. This creates a link between stochastic-process analysis, finite-state pattern recognition, and the theory of optimal decision-making.

Chapter 10 completes the development by considering hidden and hierarchical Markov models with final sequences of states. It examines hidden models with one or multiple sequences of observable states, models with hierarchical emissions, and nested Markov models in which the inner processes and the outer process are subject to final-sequence conditions. The final section extends these ideas to nested stochastic systems. These constructions make it possible to represent systems with several levels of temporal organization, partial observability, and interacting state structures.

Taken as a whole, the book presents final sequences of states as a flexible organizing principle for the analysis of stochastic processes. The recurrence-based approach makes it possible to obtain explicit formulas and efficient algorithms, while the Markovian approach provides a natural interpretation in terms of states, transitions, probabilities, and costs. The resulting framework is intended for researchers, graduate students, and practitioners interested in recurrence relations, stochastic processes, Markov and hidden Markov models, optimization, stochastic games, and algorithmic methods for the analysis of systems evolving over time.

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1 HOMOGENEOUS LINEAR RECURRENCES

This chapter develops the fundamental theory of homogeneous linear recurrence sequences and functions. It introduces their definitions, main properties, and direct representation formulas, then examines changes in recurrence order and the behavior of linear combinations. Extensions over the set of integers, reversibility, continuity, and algebraic operations are also considered. The chapter further discusses inhomogeneous linear recurrences, linear recurrences defined over matrix sets, and degenerate cases, thereby establishing the theoretical framework required for subsequent analysis and applications.

1.1 Definitions and main properties

Let K be a subset of $\mathbb{C}$. In what follows, the definitions and the main properties of the homogeneous linear recurrence sequences over K are introduced.

Definition 1.1 *Consider* $a=(a_n)_{n=0}^{\infty}\subseteq\mathbb{C}$ *and* $q=(q_k)_{k=0}^{m-1}\in K^m$ *such that*

$$a_n=\sum_{k=0}^{m-1}q_k a_{n-1-k}\ ,\ \forall n\geq m\,.$$

The sequence a *is called a homogeneous linear* $m-$*recurrence over* K*. If* $q_{m-1}\neq 0$ *then the sequence* a *is called non-degenerate, otherwise it is called degenerate. The set* K *is called the generating set, the vector* q *is called the generating vector and the tuple* $I_m^{[a]}=(a_n)_{n=0}^{m-1}$ *is called the initial state of* a*.*

Definition 1.2 *A sequence* $a=(a_n)_{n=0}^{\infty}$ *is called a homogeneous linear recurrence over* K *if there exists* $m\in\mathbb{N}^*$ *such that the sequence is a homogeneous linear* $m-$*recurrence over the set* K*.*

In what follows, we consider several sets defined for non-degenerate homogeneous linear recurrences over the set K. Denote:

- $Rol[K][m]$ – the set of non-degenerate homogeneous linear $m-$recurrences over the set K;
- $Rol[K]$ – the set of non-degenerate homogeneous linear recurrence sequences over the set K;
- $G[K][m](a)$ – the set of the generating vectors of length m of the sequence $a \in Rol[K][m]$;
- $G[K](a)$ – the set of all generating vectors of the sequence $a \in Rol[K]$.

Also, we define several functions associated to the sequence $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$:

Definition 1.3 *The function*

$$G^{[a]} : \mathbb{C} \to \mathbb{C},\ G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n,$$

is called the generating function of the sequence a.

Definition 1.4 *The function*

$$G_t^{[a]} : \mathbb{C} \to \mathbb{C},\ G_t^{[a]}(z) = \sum_{n=0}^{t-1} a_n z^n,$$

is called the partial-generating function of order t *of the sequence* a.

Definition 1.5 *The function*

$$R_t^{[a]} :\ \mathbb{C} \to \mathbb{C},\ R_t^{[a]}(z) = G^{[a]}(z) - G_t^{[a]}(z),$$

is called the remainder of order t *of the sequence* a.

For describing the properties of the linear recurrence $a \in Rol[K][m]$ with generating vector $q \in G[K][m](a)$, we define its characteristic polynomial.

Definition 1.6 *The polynomial*

$$H_m^{[q]} : \mathbb{C} \to \mathbb{C},\ H_m^{[q]}(z) = 1 - zG_m^{[q]}(z),$$

is called the unitary characteristic polynomial and every polynomial of the form

$$H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z),\ \alpha \in \mathbb{C}^*,$$

is called the characteristic polynomial of the sequence a.

Definition 1.7 *The equation* $H_m^{[q]}(z) = 0$ *is called the characteristic equation of* a.

Denote:

- $H[K][m](a)$ − the set of all characteristic polynomials of degree m of a;
- $H[K](a)$ − the set of all characteristic polynomials of the sequence a.

Appropriately, we mark with "*" the sets that refer to both non-degenerate and degenerate recurrences. In this way, we define the corresponding sets

$$Rol^*[K][m],\ Rol^*[K],\ G^*[K][m](a),$$

$$G^*[K](a),\ H^*[K][m](a),\ H^*[K](a)$$

with analogous meanings.

Property 1.1 *The following equality holds:*

$$Rol[K] = \bigcup_{m=1}^{\infty} Rol[K][m].$$

Proof. The equality is obvious and it directly results from Definition 1.2. □

Property 1.2 *Let* $a \in Rol[K][m]$ *and* $q \in G[K][m](a)$. *Then* $H_m^{[q]}(0) = 1$.

Proof. We have $H_m^{[q]}(0) = 1 - 0 \cdot G_m^{[q]}(0) = 1$. □

Property 1.3 *The sequence* $a = (a_k)_{k=0}^{\infty} \subseteq \mathbb{C}$ *satisfies the formula*

$$a_n = \frac{1}{n!} G^{[a](n)}(0),\ \forall n \in \mathbb{N}.$$

Proof. For the sequence $a = (a_k)_{k=0}^{\infty} \subseteq \mathbb{C}$ and $\forall n \in \mathbb{N}$, we consider its generating function $G^{[a]}(z) = \sum_{k=0}^{\infty} a_k z^k$. By computing the first n derivatives, we obtain

$$G^{[a](n)}(z) = \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)\, a_k z^{k-n} =$$

$$= \sum_{k=0}^{\infty} (k+n)(k+n-1)\ldots(k+1)\, a_{k+n} z^k = n! a_n + \sum_{k=1}^{\infty} \frac{(k+n)!}{k!} a_{k+n} z^k,$$

which implies $G^{[a](n)}(0) = n! a_n$, hence

$$a_n = \frac{1}{n!} G^{[a](n)}(0).$$ □

Example 1.1 *For the Fibonacci sequence, determine:*

- *the order of the recurrence and the generating set;*
- *the initial state and the generating vector;*
- *The partial-generating function of order 5;*
- *The unitary characteristic polynomial;*
- *The solutions of the characteristic equation.*

Solution. Let $F=(F_n)_{n=0}^{\infty}$ be the Fibonacci sequence. It is known that the relations

$$F_0=0,\ F_1=1,\ F_n=F_{n-1}+F_{n-2},\ \forall n\geq 2,$$

hold. Therefore,

$$F\in Rol[\mathbb{N}][2],\ I_2^{[F]}=(0,1),\ q=(1,1)\in G[\mathbb{N}][2](F),$$

i.e. it is a homogeneous linear 2-recurrence over the set $\mathbb{N}$ with initial state $(0,1)$ and the generating vector $(1,1)$. As a result, we have

$$F_0=0,\ F_1=1,\ F_2=F_1+F_0=1,\ F_3=F_2+F_1=2,\ F_4=F_3+F_2=3,\ \ldots.$$

Next, we obtain the partial-generating function of order 5:

$$G_5^{[F]}(z)=\sum_{t=0}^{4}F_t z^t=z+z^2+2z^3+3z^4.$$

The unitary characteristic polynomial is

$$H_2^{[q]}(z)=1-zG_2^{[q]}(z)=1-z(1+z)=1-z-z^2\in H[\mathbb{N}][2](F).$$

The characteristic equation is $1-z-z^2=0$ and its solutions are $z_{0,1}=\dfrac{-1\mp\sqrt{5}}{2}$. □

Example 1.2 *A space probe continuously moves through interstellar space with constant speed v (m/s). Prove that the dynamics of the motion is a homogeneous linear recurrence and determine its initial state and its generating vector.*

Solution. Let d_n be the distance traveled by the space probe in the first n seconds, for each nonnegative integer n. The sequence $(d_n)_{n=0}^{\infty}$ represents the dynamics of the motion. Obviously, we have the relations

$$d_0=0,\ d_1=v,\ d_n=d_{n-1}+v,\ \forall n\geq 1.$$

We can easily observe that the sequence $d=(d_n)_{n=0}^{\infty}$ is not a homogeneous linear 1−recurrence for $v>0$, because, otherwise, we would have $v=d_1=q_0d_0=0$, where $q=(q_0)$ would be the generating vector of the sequence.

Let us suppose that it is a homogeneous linear 2−recurrence. Let

$$d\in Rol[\mathbb{R}][2],\ q=(q_0,q_1)\in G[\mathbb{R}][2](d).$$

We obtain the relations

$$2v=d_2=q_0d_1+q_1d_0=q_0v,$$

$$3v=d_3=q_0d_2+q_1d_1=2q_0v+q_1v=(2q_0+q_1)v,$$

which yields $q_0=2$ and $q_1=-1$. By verifying this solution, we obtain

$$nv=d_n=q_0d_{n-1}+q_1d_{n-2}=2d_{n-1}-d_{n-2}=$$

$$=2(n-1)v-(n-2)v=nv,\ \forall n\geq 2,$$

which is true. Thus, $d\in Rol[\mathbb{R}][2]$, $q=(2,-1)\in G[\mathbb{R}][2](d)$ and $I_2^{[d]}=(0,v)$. □

1.2 Direct formula for homogeneous linear recurrences

Next, we obtain the direct formulas for the generating function and the terms of the sequence. Also, we present the conditions that a function must satisfy to be a generating function for a homogeneous linear recurrence.

Definition 1.8 *A fraction* $\frac{A(z)}{B(z)}$ *is called rational if* $A(z)\in\mathbb{C}[z]$ *and* $B(z)\in\mathbb{C}[z]$.

Definition 1.9 *A rational fraction* $\frac{A(z)}{B(z)}$ *is called regular if* $\deg(A(z))<\deg(B(z))$, *where* $\deg(X(z))$ *is the degree of an arbitrary polynomial* $X(z)\in\mathbb{C}[z]$.

Theorem 1.1 *Let* $a\in Rol[K][m]$ *and* $q=(q_k)_{k=0}^{m-1}\in G[K][m](a)$. *The generating function* $G^{[a]}(z)$ *is a regular rational fraction given by the formula*

$$G^{[a]}(z)=\frac{G_m^{[a]}(z)-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Proof. We have

$$R_m^{[a]}(z)=\sum_{n=m}^{\infty}a_nz^n=\sum_{n=m}^{\infty}z^n\sum_{k=0}^{m-1}q_ka_{n-1-k}=z\sum_{k=0}^{m-1}q_kz^k\sum_{n=m}^{\infty}a_{n-1-k}z^{n-1-k}=$$

$$=z\sum_{k=0}^{m-1}q_kz^kR_{m-1-k}^{[a]}(z)=zG^{[a]}(z)\sum_{k=0}^{m-1}q_kz^k-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z)=$$

$$=zG_m^{[q]}(z)G^{[a]}(z)-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z),$$

which implies

$$G^{[a]}(z)=G_m^{[a]}(z)+zG_m^{[q]}(z)G^{[a]}(z)-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z)\ .$$

By grouping the terms containing $G^{[a]}(z)$ on the left side of the equality and dividing the obtained equality by the polynomial $H_m^{[q]}(z)$, we obtain the formula:

$$G^{[a]}(z)=\frac{G_m^{[a]}(z)-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}$$

From the following inequalities, it follows that $G^{[a]}(z)$ is a regular rational fraction:

$$\deg\left(G_m^{[a]}(z)-z\sum_{k=0}^{m-1}q_kz^kG_{m-1-k}^{[a]}(z)\right)\le$$

$$\le\max\left\{\deg(G_m^{[a]}(z)),\deg\left(\sum_{k=0}^{m-1}q_kz^{k+1}G_{m-1-k}^{[a]}(z)\right)\right\}=$$

$$=\max\left\{m-1,\deg\left(\sum_{k=0}^{m-2}q_kz^{k+1}G_{m-1-k}^{[a]}(z)\right)\right\}=m-1<m=\deg(H_m^{[q]}(z)).\qquad\square$$

Theorem 1.2 *Let* $G^{[a]}(z)=\dfrac{A(z)}{B(z)}$ *be a regular rational fraction. The direct formula for the terms of the sequence* $a\subseteq\mathbb{C}$ *is*

$$a_n=I_m^{[a]}((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T,\ \forall n\in\mathbb{N},$$

where $I_m^{[a]} = (a_n)_{n=\overline{0,\,m-1}}$ *and* $z_0, z_1, \ldots, z_{p-1}$ *are all the distinct roots, with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$ *of the polynomial* $B(z)$ *of degree* m,

$$\beta_i^{[a]} = \left(\frac{i^j}{z_k^i} \right)_{k=\overline{0,\,p-1},\;\; j=\overline{0,\,s_k-1}}, \;\; \forall i \in \mathbb{N},$$

and $B^{[a]} = (\beta_i^{[a]})_{i=\overline{0,\,m-1}}$, *with the convention* $0^0 = 1$.

Proof. By writing the fraction $G^{[a]}(z)$ as a sum of simple rational fractions

$$G^{[a]}(z) = \sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} (z - z_k)^{-j},$$

and computing the first n derivatives after that, we obtain

$$G^{[a]\,(n)}(z) = \left[\sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} (z - z_k)^{-j} \right]^{(n)} =$$

$$= \sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} (-1)^n \frac{(j+n-1)!}{(j-1)!} (z - z_k)^{-j-n}.$$

By Property 1.3, for $\forall n \geq 0$, we have

$$a_n = \frac{1}{n!} G^{[a]\,(n)}(0) = \sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} (-1)^n C_{j+n-1}^n (-z_k)^{-j-n} = \sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} \gamma_{n,\,j}^{[a]\,(k)},$$

where

$$\gamma_{i,\,j}^{[a]\,(k)} = (-1)^j C_{i+j-1}^i \frac{1}{z_k^{i+j}}, \;\; \gamma_i^{[a]} = \left(\gamma_{i,\,j}^{[a]\,(k)} \right)_{\substack{k=\overline{0,\,p-1},\\ j=\overline{1,\,s_k}}}, \;\; \forall i \in \mathbb{N}$$

and $\Gamma^{[a]} = (\gamma_i^{[a]})_{i=\overline{0,\,m-1}}$. Consequently, we have

$$a_n = \sum_{k=0}^{p-1} \sum_{j=1}^{s_k} \alpha_{k,\,j} \gamma_{n,\,j}^{[a]\,(k)} = \gamma_n^{[a]} y^T, \;\; \forall n \in \mathbb{N},$$

where $y = (\alpha_{k,\,j})_{\substack{k=\overline{0,\,p-1},\\ j=\overline{1,\,s_k}}}$. Therefore, we obtain $\gamma_n^{[a]} y^T = a_n$, $n = \overline{0, m-1}$, equivalent to $\Gamma^{[a]} y^T = (I_m^{[a]})^T$, which involves $y = I_m^{[a]} ((\Gamma^{[a]})^T)^{-1}$ (provided that the matrix

$\Gamma^{[a]}$ is invertible). The terms of the sequence a satisfy the following formula:

$$a_n = \gamma_n^{[a]} y^T = y \cdot (\gamma_n^{[a]})^T = I_m^{[a]} \cdot ((\Gamma^{[a]})^T)^{-1} \cdot (\gamma_n^{[a]})^T, \ \forall n \in \mathbb{N}.$$

This represents a direct formula for the homogeneous linear recurrence a, but we will continue to simplify it. We will use the fact that

$$P_j(n) = C_{j+n-1}^n = \frac{(n+1)(n+2)\ldots(n+j-1)}{(j-1)!}$$

represents a polynomial of degree $(j-1)$ with the variable n. Let

$$\mu_{k,j} = \alpha_{k,j} \frac{(-1)^j}{z_k^j}, \ k = \overline{0, p-1}, \ j = \overline{1, s_k}.$$

We have the equality

$$a_n = \sum_{k=0}^{p-1} \frac{1}{z_k^n} \sum_{j=1}^{s_k} \mu_{k,j} P_j(n), \ \forall n \in \mathbb{N},$$

where $\sum_{j=1}^{s_k} \mu_{k,j} P_j(n)$ is a polynomial of degree $(s_k - 1)$. If

$$\sum_{j=1}^{s_k} \mu_{k,j} P_j(n) = \sum_{j=0}^{s_k-1} A_{k,j} n^j, \ k = \overline{0, p-1},$$

then the formula

$$a_n = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{k,j} \frac{n^j}{z_k^n} = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{k,j} \beta_{n,j}^{[a](k)} = \beta_n^{[a]} x^T, \ \forall n \in \mathbb{N},$$

holds, where $x = (A_{k,j})_{k=\overline{0,p-1}, j=\overline{0,s_k-1}}$. We obtained

$$\beta_n^{[a]} x^T = a_n, \ n = \overline{0, m-1},$$

equivalent to $B^{[a]} x^T = (I_m^{[a]})^T$, which, assuming that $B^{[a]}$ is invertible, involves $x = I_m^{[a]} ((B^{[a]})^T)^{-1}$. In this way, we obtained the direct formula

$$a_n = \beta_n^{[a]} x^T = x \cdot (\beta_n^{[a]})^T = I_m^{[a]} \cdot ((B^{[a]})^T)^{-1} \cdot (\beta_n^{[a]})^T, \ \forall n \in \mathbb{N}. \qquad \square$$

The converse also holds. If a sequence has a regular rational generating function, then it satisfies a homogeneous linear recurrence.

Theorem 1.3 *Assume that the function* $G^{[a]}(z)$ *is a regular rational fraction. Let* $A(z)=\sum_{k=0}^{m-1}\lambda_k z^k \in \mathbb{C}[z]$ *be its numerator and let* $B(z)=\beta\left(1-\sum_{k=0}^{m-1}q_k z^{k+1}\right)\in K[z]$ *be its denominator, where* $\beta\neq 0$ *and* $m=\deg(B(z))$. *Then*

$$a\in Rol[K][m],\ B(z)\in H[K][m](a).$$

Also, the initial state $I_m^{[a]}$ *can be determined using the formula*

$$a_k=\sum_{j=0}^{k-1}q_j a_{k-1-j}+\lambda_k,\ k=\overline{0,m-1}.$$

Proof. From Theorem 1.2, we have

$$a_n=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta_n^{[a]})^T,\ \forall n\in\mathbb{N}.$$

Consider the sequence $b=(b_n)_{n=0}^{\infty}\in Rol[K][m]$ with the characteristic polynomial $B(z)\in H[K][m](b)$ and the initial state $I_m^{[b]}=I_m^{[a]}$. From Theorem 1.1, we have that $G^{[b]}(z)$ is a regular rational fraction with denominator $B(z)$.

Therefore, the fractions $G^{[a]}(z)$ and $G^{[b]}(z)$ have the denominator $B(z)$ and, from Theorem 1.2, we have

$$b_n=I_m^{[b]}\cdot((B^{[b]})^T)^{-1}\cdot(\beta_n^{[b]})^T=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta_n^{[a]})^T=a_n,\ \forall n\in\mathbb{N},$$

which implies $a=b$, i.e. $a\in Rol[K][m]$ and $B(z)\in H[K][m](a)$. The formula

$$a_k=\sum_{j=0}^{k-1}q_j a_{k-1-j}+\lambda_k,\ k=\overline{0,m-1},$$

follows from the relation $A(z)=G^{[a]}(z)\cdot B(z)$ by matching the first m terms. □

Theorem 1.4 *If the function* $G^{[a]}(z)=\dfrac{A(z)}{B(z)}$ *is a non-regular rational fraction with* $B(z)=1-z\sum_{k=0}^{m-1}q_k z^k$ *and* $q_k\in K$, $k=\overline{0,m-1}$, *then* $a\in Rol^*[K][t+1]$ *with characteristic polynomial* $B(z)\in H^*[K][t+1](a)$, *where* $t=\deg(A(z))$.

Proof. The fraction $G^{[a]}(z)$ can be uniquely written in the form

$$G^{[a]}(z) = T(z) + \frac{S(z)}{B(z)},$$

where $\frac{S(z)}{B(z)}$ is a regular rational fraction. Let $T(z) = \sum_{k=0}^{r} s_k z^k$, where $r = \deg(T(z))$.

We build the sequence $b = (b_n)_{n=0}^{\infty}$ in the following way:

$$b_n = a_n - s_n,\ n = \overline{0,r},$$

$$b_n = a_n,\ \forall n > r.$$

We have that

$$G^{[b]}(z) = G^{[a]}(z) - T(z) = \frac{S(z)}{B(z)}$$

is a regular rational fraction. Hence

$$b \in Rol[K][m],\ B(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k,$$

which involves $q = (q_k)_{k=0}^{m-1} \in G[K][m](b)$, i.e.

$$b_n = \sum_{k=0}^{m-1} q_k b_{n-1-k},\ \forall n \geq m.$$

This relation, in particular, yields the recurrence relation

$$b_n = \sum_{k=0}^{m-1} q_k b_{n-1-k},\ \forall n \geq r+m+1,$$

equivalent to

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq r+m+1.$$

If we consider $q_k = 0$, $k = \overline{m, m+r}$, we obtain the recurrence relation

$$a_n = \sum_{k=0}^{r+m} q_k a_{n-1-k},\ \forall n \geq r+m+1,$$

i.e. $a \in Rol^*[K][t+1]$ and $B(z) \in H^*[K][t+1](a)$, where $t = m + r = \deg(A(z))$. □

Example 1.3 *For the Fibonacci sequence, determine:*

- *the generating function;*
- *the direct formula;*
- *an approximate formula that allows us to calculate the terms exactly.*

Solution. In Example 1.1, we obtained that the solutions of the characteristic equation are $z_{0,1} = \frac{-1 \mp \sqrt{5}}{2}$. We have

$$F_n = I_m^{[F]}((B^{[F]})^T)^{-1}(\beta_n^{[F]})^T =$$

$$= (0,1)\left(\begin{pmatrix} 1 & 1 \\ z_0^{-1} & z_1^{-1} \end{pmatrix}^T\right)^{-1}\begin{pmatrix} z_0^{-n} \\ z_1^{-n} \end{pmatrix} =$$

$$= \begin{pmatrix} (z_0^{-1} - z_1^{-1})^{-1} \\ (z_1^{-1} - z_0^{-1})^{-1} \end{pmatrix}\begin{pmatrix} z_0^{-n} \\ z_1^{-n} \end{pmatrix} = \begin{pmatrix} (z_0 - z_1)^{-1} \\ (z_1 - z_0)^{-1} \end{pmatrix}\begin{pmatrix} z_0^{-n} \\ z_1^{-n} \end{pmatrix} =$$

$$= \begin{pmatrix} (-\sqrt{5})^{-1} \\ (\sqrt{5})^{-1} \end{pmatrix}\begin{pmatrix} z_0^{-n} \\ z_1^{-n} \end{pmatrix} = \frac{1}{\sqrt{5}}\left(-\frac{1}{z_0^n} + \frac{1}{z_1^n}\right) = \frac{z_0^n - z_1^n}{\sqrt{5}(z_0 z_1)^n} =$$

$$= \frac{(-1)^n(z_0^n - z_1^n)}{\sqrt{5}} = \frac{(-z_0)^n - (-z_1)^n}{\sqrt{5}} = \frac{L^n - (-L)^{-n}}{\sqrt{5}}, \ \forall n \in \mathbb{N},$$

where $L = \frac{1+\sqrt{5}}{2}$. We will prove that $F_n \approx \frac{1}{\sqrt{5}}L^n$ with approximation error $\varepsilon_n < \frac{1}{2}$, $\forall n \in \mathbb{N}$, which will allow us to calculate the integer terms of the sequence exactly.

For $n = 0$, we have

$$\varepsilon_n = \varepsilon_0 = \left|F_0 - \frac{1}{\sqrt{5}}L^0\right| = \frac{1}{\sqrt{5}} < \frac{1}{2}.$$

Suppose that $\varepsilon_k < \frac{1}{2}$, $\forall k \le n$. We have

$$\varepsilon_r = \left|F_r - \frac{1}{\sqrt{5}}L^r\right| = \frac{1}{\sqrt{5}}\left(\frac{1}{L}\right)^r, \ \forall r \in \mathbb{N},$$

which involves the inequality

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{1}{L} = \frac{2}{1+\sqrt{5}} < 1,$$

i.e. $\varepsilon_{n+1} < \varepsilon_n < \frac{1}{2}$. By mathematical induction, we obtain $\varepsilon_n < \frac{1}{2}$, $\forall n \in \mathbb{N}$.

Also, we obtained in Example 1.1 that

$$F \in Rol[\mathbb{N}][2],\ I_2^{[F]} = (0,1) \text{ and } q = (1,1) \in G[\mathbb{N}][2](F).$$

By applying Theorem 1.1, we have the generating function

$$G^{[F]}(z) = \frac{G_2^{[F]}(z) - z\sum_{k=0}^{1} q_k z^k G_{1-k}^{[F]}(z)}{H_2^{[q]}(z)} = \frac{z}{1-z-z^2}.$$ □

Example 1.4 *A space probe continuously moves through interstellar space with constant speed v (m/s). Prove that the dynamics of the motion is a homogeneous linear recurrence and determine its initial state and its generating vector.*

Solution. This example is the same as Example 1.2, but we will be presenting here a standardized solution. We obtained that the state of the motion at discrete time n is $d_n = vn$, $\forall n \in \mathbb{N}$. So, the generating function is

$$G^{[d]}(z) = \sum_{n=0}^{\infty} vnz^n = vz\sum_{n=1}^{\infty} nz^{n-1} = vz\frac{\partial}{\partial z}\sum_{n=1}^{\infty} z^n = vz\frac{\partial}{\partial z}\left(\frac{z}{1-z}\right) = \frac{vz}{1-2z+z^2},$$

i.e. a regular rational fraction. By applying Theorem 1.3, we obtain

$$d \in Rol[\mathbb{R}][2],\ H_2^{[d]}(z) = 1-2z+z^2 \in H[\mathbb{R}][2](d),$$

which involves $q = (2,-1) \in G[\mathbb{R}][2](d)$. □

Example 1.5 *Consider the sequence $a = \left(\frac{1}{n+1}\right)_{n=0}^{\infty}$. Show that the sequence a is not a homogeneous linear recurrence over $\mathbb{C}$.*

Solution. The generating function of the sequence $a = \left(\frac{1}{n+1}\right)_{n=0}^{\infty}$ is

$$G^{[a]}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \frac{1}{z}\sum_{n=0}^{\infty} \int_0^z x^n dx =$$

$$= \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} x^n \right) dx = \frac{1}{z}\int_0^z \frac{dx}{1-x} = -\frac{1}{z}\ln|1-z|,$$

i.e. it is a non-rational fraction. Therefore, the sequence a is not a homogeneous linear recurrence over $\mathbb{C}$. □

1.3 Raising the order of the recurrence relation. Linear combinations

Let us describe a series of fundamental results concerning homogeneous linear recurrences. We present a set of algebraic operations with homogeneous linear recurrences. We consider that K is a subset of $\mathbb{C}$.

One of the main theorems from the theory of homogeneous linear recurrences is the theorem for raising the order of the recurrence (TROR):

Theorem 1.5 *If* $a \in Rol[K][m]$ *and* $P(z) \in H[K][m](a)$, *then* $a \in Rol[K][m+1]$ *and*

$$Q(z) = (z-\alpha)P(z) \in H[K][m+1](a), \ \forall \alpha \in K^*.$$

Proof. This theorem can be proved only by checking that $Q(z)$ is the characteristic polynomial of the sequence a. So, if $\lambda = P(0)$ and $\mu = -\alpha\lambda$, the polynomial $Q(z)$ can be written in the form $Q(z) = \mu R(z)$, where $R(z) = 1 - z\sum_{k=0}^{m} q_k^* z^k$, and the coefficients are given by the relations

$$q_0^* = q_0 + \alpha^{-1};\ q_k^* = q_k - \alpha^{-1}q_{k-1},\ k = \overline{1,m-1};\ q_m^* = -\alpha^{-1}q_{m-1}.$$

Since q_{m-1} is nonzero, q_m^* is also nonzero. It remains to check the recurrence

$$a_n = \sum_{k=0}^{m} q_k^* a_{n-1-k},\ \forall n \geq m+1.$$

Indeed, we have

$$\sum_{k=0}^{m} q_k^* a_{n-1-k} = (q_0 + \alpha^{-1})a_{n-1} + \sum_{k=1}^{m-1}(q_k - \alpha^{-1}q_{k-1})a_{n-1-k} - \alpha^{-1}q_{m-1}a_{n-1-m} =$$

$$= a_n + \alpha^{-1}\left(a_{n-1} - \sum_{k=1}^{m} q_{k-1}a_{n-1-k}\right) = a_n + \alpha^{-1}\left(a_{n-1} - \sum_{k=0}^{m-1} q_k a_{(n-1)-1-k}\right) =$$

$$= a_n + \frac{a_{n-1} - a_{n-1}}{\alpha} = a_n .$$

Therefore, we obtained $a \in Rol[K][m+1]$ and $R(z) \in H[K][m+1](a)$. Since $Q(z) = \mu R(z)$ with $\mu \neq 0$, we conclude that $Q(z) \in H[K][m+1](a)$. □

When the set K is not a field, the difficulty may be due to the fact that the generating vector, corresponding to the characteristic polynomial $Q(z)$, may contain elements outside the set K. In this situation, we can apply this theorem for a field containing the set K, obtaining that the sequence a is a homogeneous linear $(m+1)-$recurrence over the given field with characteristic polynomial $Q(z)$.

Corollary 1.1 *The set* $Rol[K]$ *represents the limit of the sequence* $\{Rol[K][m]\}_{m=1}^{\infty}$ *and* $Rol[K][1] \subseteq Rol[K][2] \subseteq \ldots \subseteq Rol[K][m] \subseteq \ldots \subseteq Rol[K]$.

Proof. Let $a \in Rol[K][r]$. According to TROR, we have $a \in Rol[K][r+1]$. By applying TROR again, we obtain $a \in Rol[K][r+2]$. Repeatedly applying the same theorem, we obtain

$$a \in Rol[K][s], \ \forall s \geq r .$$

Therefore, we have the following sequence of inclusions:

$$Rol[K][1] \subseteq Rol[K][2] \subseteq \ldots \subseteq Rol[K][m] \subseteq \ldots .$$

Also, it is known from Property 1.1 that $Rol[K] = \bigcup_{m=1}^{\infty} Rol[K][m]$. Then,

$$Rol[K][s] \subseteq Rol[K], \ \forall s \in \mathbb{N}^*,$$

which implies

$$Rol[K][1] \subseteq Rol[K][2] \subseteq \ldots \subseteq Rol[K][m] \subseteq \ldots \subseteq Rol[K] = \lim_{s \to \infty} Rol[K][s] .$$ □

Corollary 1.2 *Let* $a \in Rol[K]$ *and* $P(z) \in H[K](a)$. *If* $Q(z) \vdots P(z)$, $Q(z) \in K[z]$ *and* $Q(0) \neq 0$, *then* $Q(z) \in H[K](a)$.

Proof. Since $P(z) \in H[K][a]$ and $K \subseteq \mathbb{C}$, we have $P(z) \in H[\mathbb{C}][a]$. Also, since $Q(z) \vdots P(z)$, the polynomial $Q(z)$ can be written in the form $Q(z) = P(z)T(z)$, where $T(z) \in \mathbb{C}[z]$. By consecutively applying TROR for each root α of $T(z)$, we obtain $a \in Rol[\mathbb{C}]$ and $Q(z) \in H[\mathbb{C}](a)$. Since $Q(z) \in K[z]$ and K is a field, we conclude that $Q(z) \in H[K](a)$. □

Corollary 1.3 *The sets* $Rol[\mathbb{C}]$ *and* $Rol[\mathbb{R}]$ *are equal.*

Proof. We will show that $Rol[\mathbb{C}] = Rol[\mathbb{R}]$, i.e., that both $Rol[\mathbb{C}] \subseteq Rol[\mathbb{R}]$ and $Rol[\mathbb{R}] \subseteq Rol[\mathbb{C}]$ hold. Consider $\forall a \in Rol[\mathbb{R}]$. According to the definition, $\exists m \in \mathbb{N}^*$ such that $a \in Rol[\mathbb{R}][m]$. Let $q \in G[\mathbb{R}][m](a)$. Since $\mathbb{R} \subseteq \mathbb{C}$, we have $q \in \mathbb{R}^m \subseteq \mathbb{C}^m$, which implies $a \in Rol[\mathbb{C}][m]$ and $q \in G[\mathbb{C}][m](a)$. Therefore, $Rol[\mathbb{R}] \subseteq Rol[\mathbb{C}]$.

We consider $\forall b \in Rol[\mathbb{C}]$. Let $P(z) \in H[\mathbb{C}](b)$ be a characteristic polynomial of b, and let $z_1, z_2, \ldots, z_s$ be all the distinct roots of the polynomial $P(z)$ with corresponding multiplicities $m_1, m_2, \ldots, m_s$. Then, $P(z) = \alpha \prod_{r=1}^{s} (z - z_r)^{m_r}$. Since

$$Q(z) = \alpha\overline{\alpha} \prod_{r=1}^{s} [(z - z_r)(z - \overline{z_r})]^{m_r} = |\alpha|^2 \prod_{r=1}^{s} [z^2 - 2(\operatorname{Re} z_r)z + |z_r|^2]^{m_r} \in \mathbb{R}[z]$$

satisfies the conditions $Q(z) \vdots P(z)$ and $Q(0) \neq 0$, by applying Corollary 1.2 we obtain $b \in Rol[\mathbb{R}]$ and $Q(z) \in H[\mathbb{R}](b)$, i.e. $Rol[\mathbb{C}] \subseteq Rol[\mathbb{R}]$. Therefore, the statement of this corollary is true. □

Theorem 1.6 *Consider* $a \in Rol[K][m]$ *and* $P(z) \in H[K][m](a)$. *If* $b_n = G_n^{[a]}(1)$, $n = \overline{0, \infty}$, *then* $b = (b_n)_{n=0}^{\infty} \in Rol[K][m+1]$ *and* $Q(z) = (z-1)P(z) \in H[K][m+1](b)$.

Proof. Let $P(z) = H_{m,\lambda}^{[q]}(z)$. According to the proof of the theorem TROR, we have $\lambda = P(0)$ and $Q(z) = -\lambda R(z)$, where $R(z) = 1 - z \sum_{k=0}^{m} q_k^* z^k$ and

$$q_0^* = q_0 + 1,\ q_k^* = q_k - q_{k-1},\ k = \overline{1, m-1} \text{ and } q_m^* = -q_{m-1}.$$

For $\forall n \geq m+1$ we obtain

$$\sum_{k=0}^{m} q_k^* b_{n-1-k} = (q_0+1)b_{n-1} - q_{m-1}b_{n-1-m} + \sum_{k=1}^{m-1}(q_k - q_{k-1})b_{n-1-k} =$$

$$= b_{n-1} + \sum_{k=0}^{m-1} q_k (b_{n-1-k} - b_{n-2-k}) = b_{n-1} + \sum_{k=0}^{m-1} q_k a_{n-2-k} = b_{n-1} + a_{n-1} = b_n .$$

In conclusion, we obtain $b \in Rol[K][m+1]$ and $Q(z) \in H[K][m+1](b)$. □

Definition 1.10 *An operation is called algebraic over a set if, when applied to any elements of the set, its result also belongs to the set.*

Next, we will be presenting a set of algebraic operations over the set $Rol[\mathbb{C}]$, i.e. operations for homogeneous linear recurrences over $\mathbb{C}$, whose results are also homogeneous linear recurrences over $\mathbb{C}$. We will also consider the corresponding operations, defined over the set $Rol[K]$, where K is a subfield of the field $\mathbb{C}$, taking into account that every homogeneous linear recurrence over a subset is also a homogeneous linear recurrence over the corresponding set.

Lemma 1.1 *Consider the linear recurrence* $a \in Rol[K][m]$ *with generating vector* $q \in G[K][m](a)$. *Then, we have* $\alpha a \in Rol[K][m]$ *and* $q \in G[K][m](\alpha a)$, $\forall \alpha \in \mathbb{C}$.

Proof. Let $a \in Rol[K][m]$, $q \in G[K][m](a)$ and $\forall \alpha \in \mathbb{C}$. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} , \ \forall n \geq m .$$

Therefore, it follows that

$$(\alpha a)_n = \alpha a_n = \alpha \sum_{k=0}^{m-1} q_k a_{n-1-k} = \sum_{k=0}^{m-1} q_k (\alpha a)_{n-1-k} , \ \forall n \geq m ,$$

which implies $\alpha a \in Rol[K][m]$ and $q \in G[K][m](\alpha a)$. □

Lemma 1.2 *Consider the linear recurrences* $a, b \in Rol[K][m]$ *with generating vector* $q \in G[K][m](a) \cap G[K][m](b)$. *Then* $a+b \in Rol[K][m]$ *and* $q \in G[K][m](a+b)$.

Proof. Let $a, b \in Rol[K][m]$ and $q \in G[K][m](a) \cap G[K][m](b)$. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} , \ b_n = \sum_{k=0}^{m-1} q_k b_{n-1-k} , \ \forall n \geq m ,$$

which implies

$$(a+b)_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} + \sum_{k=0}^{m-1} q_k b_{n-1-k} =$$

$$= \sum_{k=0}^{m-1} q_k (a_{n-1-k} + b_{n-1-k}) = \sum_{k=0}^{m-1} q_k (a+b)_{n-1-k}\ ,\ \forall n \geq m\ .$$

In conclusion, $a+b \in Rol[K][m]$ with generating vector $q \in G[K][m](a+b)$. □

Lemma 1.3 *Consider the sequences* $a,b \in Rol[K]$ *with characteristic polynomials* $P_1(z) \in H[K](a)$ *and* $P_2(z) \in H[K](b)$. *Then* $a+b \in Rol[K]$ *and*

$$P(z) = LCM(P_1(z), P_2(z)) \in H[K](a+b).$$

Proof. Consider the sequences $a,b \in Rol[K]$ with characteristic polynomials

$$P_1(z) \in H[K](a)\ ,\ P_2(z) \in H[K](b)\ .$$

Also, we consider the polynomial

$$P(z) = LCM(P_1(z), P_2(z))\ ,$$

which is the least common multiple of the polynomials $P_1(z)$ and $P_2(z)$. Since

$$P(z) \in K[z]\ ,\ P(z) \vdots P_1(z)\ ,\ P(z) \vdots P_2(z)\ ,$$

by applying Corollary 1.2, we obtain

$$P(z) \in H[K](a)\ ,\ P(z) \in H[K](b)\ .$$

Next, we write the polynomial $P(z)$ in the form

$$P(z) = \alpha \left(1 - z \sum_{k=0}^{m-1} q_k z^k \right),$$

where $\alpha = P(0) \neq 0$. It follows that

$$q = (q_k)_{k=0}^{m-1} \in G[K][m](a) \cap G[K][m](b)\ .$$

By applying Lemma 1.2, we obtain

$$a+b \in Rol[K][m]\ ,\ q \in G[K][m](a+b)\ ,$$

which implies $a+b \in Rol[K]$ and $P(z) \in H[K](a+b)$. □

For linear combinations, conjugates, real parts and imaginary parts, the following theorems hold:

Theorem 1.7 *Consider the linear recurrences* $a^{(j)} \in Rol[K]$ *with characteristic polynomials* $P_j(z) \in H[K](a^{(j)})$ *and the arbitrary coefficients* $\alpha_j \in \mathbb{C}$, $j = \overline{1,t}$. *Then* $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ *with the characteristic polynomial*

$$P(z) = LCM(P_1(z),\ P_2(z),\ \ldots,\ P_t(z)) \in H[K](a).$$

Proof. For $t = 1$ we have $a = \alpha_1 a^{(1)}$, $a^{(1)} \in Rol[K]$ and $P_1(z) \in H[K](a^{(1)})$. From Lemma 1.1, we obtain $a \in Rol[K]$ and $P(z) = P_1(z) \in H[K](a)$. Therefore, for $t = 1$, the statement is true.

Let the statement be true for $\forall t \le r$. Let us prove it for $t = r+1$. We have

$$a = \sum_{k=1}^{r+1} \alpha_k a^{(k)} = \left(\sum_{k=1}^{r} \alpha_k a^{(k)} \right) + (\alpha_{r+1} a^{(r+1)}) = b + d,$$

where $b = \sum_{k=1}^{r} \alpha_k a^{(k)}$ and $d = \alpha_{r+1} a^{(r+1)}$. According to hypothesis, $b \in Rol[K]$ and

$$T(z) = LCM(P_1(z), P_2(z), \ldots, P_r(z)) \in H[K](b).$$

By Lemma 1.1, we obtain $d \in Rol[K]$ and $P_{r+1}(z) \in H[K](d)$. From Lemma 1.3, we have $a \in Rol[K]$ and

$$P(z) = LCM(P_1(z), P_2(z), \ldots, P_{r+1}(z)) = LCM(T(z), P_{r+1}(z)) \in H[K](a).$$

Therefore, the statement is also true for $t = r+1$. According to the mathematical induction method, the statement is true for $\forall t \in \mathbb{N}^*$. □

Theorem 1.8 *Consider* $a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ *with characteristic polynomial* $P(z) \in H[\mathbb{C}][m](a)$. *Then* $\overline{a} = (\overline{a}_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ *and* $\overline{P(\overline{z})} \in H[\mathbb{C}][m](\overline{a})$.

Proof. Let $P(z) = H_{m,\alpha}^{[q]}(z)$. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \ge m,$$

which implies

$$\overline{a}_n = \overline{\sum_{k=0}^{m-1} q_k a_{n-1-k}} = \sum_{k=0}^{m-1} \overline{q_k a_{n-1-k}} = \sum_{k=0}^{m-1} \overline{q_k} \cdot \overline{a_{n-1-k}} = \sum_{k=0}^{m-1} \overline{q_k} \cdot \overline{a}_{n-1-k}\ ,\ \forall n \geq m\ .$$

It follows that $\overline{a} \in Rol[\mathbb{C}][m]$ with the characteristic polynomial

$$H_m^{[\overline{q}]}(z) = 1 - z\sum_{k=0}^{m-1} \overline{q_k} z^k = \overline{1} - \sum_{k=0}^{m-1} \overline{q_k} \cdot \overline{\overline{z^{k+1}}} = \overline{1} - \overline{\sum_{k=0}^{m-1} q_k \overline{z}^{k+1}} =$$

$$= \overline{1 - \sum_{k=0}^{m-1} q_k \overline{z}^{k+1}} = \overline{1 - \overline{z}\sum_{k=0}^{m-1} q_k \overline{z}^k} = \overline{\overline{1 - \overline{z} G_m^{[q]}(\overline{z})}} = \overline{P(\overline{z})} \in H[\mathbb{C}][m](\overline{a})\ .$$ □

Theorem 1.9 *Consider* $a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ *and* $P(z) \in H[\mathbb{C}][m](a)$. *Then*

$$\mathrm{Re}(a) = (\mathrm{Re}(a_n))_{n=0}^{\infty} \in Rol[\mathbb{C}],\ \mathrm{Im}(a) = (\mathrm{Im}(a_n))_{n=0}^{\infty} \in Rol[\mathbb{C}],$$

with the common characteristic polynomial

$$Q(z) = LCM(P(z), \overline{P(\overline{z})}) \in H[\mathbb{C}](\mathrm{Re}(a)) \cap H[\mathbb{C}](\mathrm{Im}(a))\ .$$

Proof. The proof is based on the relations

$$\mathrm{Re}(a) = \frac{1}{2}(a + \overline{a})\ ,\ \mathrm{Im}(a) = \frac{1}{2i}(a - \overline{a})\ .$$

Since the conjugate and the linear combination are algebraic operations over $Rol[\mathbb{C}]$, the statement of the theorem is true. □

Example 1.6 *Show that the sequence* $a = (\cos(\alpha n))_{n=0}^{\infty}$ *is a homogeneous linear recurrence over* $\mathbb{R}$ *for* $\forall \alpha \in \mathbb{R}$ *and determine a generating vector of the sequence* a.

Solution. Consider $\forall \alpha \in \mathbb{R}$. We have

$$a_n = \cos(\alpha n) = \frac{1}{2} a_n^{(1)} + \frac{1}{2} a_n^{(2)}\ ,$$

where

$$a_n^{(1)} = e^{i\alpha n}\ ,\ a_n^{(2)} = e^{-i\alpha n}\ ,\ \forall n \in \mathbb{N}\ .$$

Therefore, the sequence a can be written in the form

$$a = \frac{1}{2} a^{(1)} + \frac{1}{2} a^{(2)}\ ,$$

where $a^{(1)} = (a_n^{(1)})_{n=0}^{\infty}$ and $a^{(2)} = (a_n^{(2)})_{n=0}^{\infty}$.

Since the sequence $a^{(1)}$ satisfies the recurrence

$$a_n^{(1)} = e^{i\alpha n} = e^{i\alpha} \cdot e^{i\alpha(n-1)} = e^{i\alpha} \cdot a_{n-1}^{(1)},\ \forall n \geq 1,$$

we have

$$a^{(1)} \in Rol[\mathbb{C}][1],\ (e^{i\alpha}) \in G[\mathbb{C}](a^{(1)}),$$

i.e. $P_1(z) = 1 - e^{i\alpha} z \in H[\mathbb{C}](a^{(1)})$. Similarly, since $a^{(2)}$ satisfies the recurrence

$$a_n^{(2)} = e^{-i\alpha} \cdot a_{n-1}^{(2)},\ \forall n \geq 1,$$

we have

$$a^{(2)} \in Rol[\mathbb{C}][1],\ (e^{-i\alpha}) \in G[\mathbb{C}](a^{(2)}),$$

i.e. $P_2(z) = 1 - e^{-i\alpha} z \in H[\mathbb{C}](a^{(2)})$. Thus, $a \in Rol[\mathbb{C}]$ with characteristic polynomial

$$P(z) = LCM(1 - e^{i\alpha} z, 1 - e^{-i\alpha} z) = (1 - e^{i\alpha} z)(1 - e^{-i\alpha} z) = 1 - 2(\cos\alpha) z + z^2.$$

From this relation, we obtain

$$a \in Rol[\mathbb{C}],\ q = (2\cos\alpha, -1) \in G[\mathbb{C}](a).$$

Since $q \in \mathbb{R}^2$, we have $a \in Rol[\mathbb{R}][2]$ and $q = (2\cos\alpha, -1) \in G[\mathbb{R}][2](a)$. □

Example 1.7 *Show that the dynamics of the process, determined by the position of a physical body continuously moving with constant speed on a circle, is a homogeneous linear recurrence over* $\mathbb{R}$. *Determine the initial state and the generating vector.*

Solution. We assume that the circle has its center at the origin of the polar coordinate system, and we represent the position of the physical body by a point in the plane and by the complex number corresponding to that point. The equation of the motion is $y(\varphi_n) = r(\cos\varphi_n + i\sin\varphi_n)$, where r is the radius of the circle and φ_n is the measure (in radians) of the angle subtended by the circular arc already traversed at discrete time $n \in \mathbb{N}$. Since the speed is constant, we have $\varphi_n = n\varphi_1$ at every discrete time n. We showed in Example 1.4 that

$$\varphi = (\varphi_n)_{n=0}^{\infty} \in Rol[\mathbb{R}][2],\ q = (2, -1) \in G[\mathbb{R}][2](\varphi).$$

Also, in Example 1.6, we proved that

$$a = (\cos\varphi_n)_{n=0}^{\infty} \in Rol[\mathbb{R}][2],\ q = (2\cos\varphi_1, -1) \in G[\mathbb{R}][2](a).$$

Similarly, we obtain

$$b = (\sin \varphi_n)_{n=0}^{\infty} \in Rol[\mathbb{R}][2],\ q = (2\cos \varphi_1, -1) \in G[\mathbb{R}][2](b).$$

By applying Theorem 1.7, we have

$$y = (y(\varphi_n))_{n=0}^{\infty} = ra + irb \in Rol[\mathbb{R}][2],\ q = (2\cos \varphi_1, -1) \in G[\mathbb{R}][2](y).$$

The initial state is $I_2^{[y]} = (r, r(\cos \varphi_1 + i \sin \varphi_1)) \in \mathbb{C}^2$. □

1.4 Extensions of homogeneous linear recurrences over $\mathbb{Z}$

Let $a \in Rol[K][m]$ and $q = (q_k)_{k=0}^{m-1} \in G[K][m](a)$. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

Next, we consider the extension of the sequence $a = (a_n)_{n=0}^{\infty}$ over $\mathbb{Z}$:

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \in \mathbb{Z}.$$

Definition 1.11 *The bidirectional sequence $\tilde{a} = (a_n)_{n=-\infty}^{\infty}$ is called an extension of the homogeneous linear recurrence $a \in Rol[K]$ over $\mathbb{Z}$, and the sequence a is called the restriction of the sequence $\tilde{a}$ to the set $\mathbb{N}$ (Notation: $\tilde{a}\big|_{\mathbb{N}} = a$).*

Definition 1.12 *The sequence $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$ is called a sequence of finite rank $r \in \mathbb{N}$ if $a_r \neq 0$ and $a_n = 0,\ \forall n > r$ (Notation: $rank(a) = r$).*

Theorem 1.10 *No sequence of finite rank belongs to the set $Rol[K]$.*

Proof. Let $a = (a_n)_{n=0}^{\infty}$ be a sequence of finite rank $t \in \mathbb{N}$. We have

$$a_t \neq 0 \text{ and } a_n = 0,\ \forall n > t.$$

Suppose, to the contrary, that we have $a \in Rol[K]$. It follows that $\exists m \in N^*$ such that $a \in Rol[K][m]$. Let $q \in G[K][m](a)$. Therefore,

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

which, in particular, implies

$$0 = a_{t+m} = \sum_{k=0}^{m-1} q_k a_{t+m-1-k} = q_{m-1} a_t .$$

This leads to a contradiction, since $q_{m-1} \neq 0$ and $a_t \neq 0$. Thus, $a \notin Rol[K]$. □

The theorem for the existence and uniqueness of the extension (TEUE) over $\mathbb{Z}$ holds.

Theorem 1.11 *For each $a \in Rol[K]$, there exists a unique extension $\tilde{a}$ over $\mathbb{Z}$.*

Proof. Let $a \in Rol[K][m]$, $q \in G[K][m](a)$ and $\tilde{a}\big|_{\mathbb{N}} = a$. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} \ , \ \forall n \in \mathbb{Z}.$$

It follows that

$$q_{m-1} a_{n-m} = a_n - \sum_{k=0}^{m-2} q_k a_{n-1-k} \ , \ \forall n \in \mathbb{Z},$$

and, since $q_{m-1} \neq 0$, we obtain the formula

$$a_{n-m} = \frac{1}{q_{m-1}} a_n - \sum_{k=0}^{m-2} \frac{q_k}{q_{m-1}} a_{n-1-k} \ , \ \forall n \in \mathbb{Z},$$

equivalent to

$$a_n = \frac{1}{q_{m-1}} a_{n+m} - \sum_{k=0}^{m-2} \frac{q_k}{q_{m-1}} a_{n+m-1-k} \ , \ \forall n \in \mathbb{Z}.$$

Therefore, knowing the vectors $I_m^{[a]}$ and q, we can uniquely compute, according to the relation above, first a_{-1}, then $a_{-2}, a_{-3}, \ldots$ It follows that the extension $\tilde{a}$ exists and this extension of the homogeneous linear recurrence $a \in Rol[K]$ over $\mathbb{Z}$, based on the generating vector q, is unique.

Let us prove that the extension does not depend on the generating vector, on the basis of which the extension over $\mathbb{Z}$ is performed. Consider $\forall p, r \in G[K](a)$. Suppose, to the contrary, that the extensions related to these vectors are different. Let these extensions be $b = (b_n)_{n=-\infty}^{\infty}$ and $c = (c_n)_{n=-\infty}^{\infty}$. It follows that

$$b_n = c_n = a_n \ , \ \forall n \in \mathbb{N}.$$

Let $k \in \mathbb{Z}$ be such that $b_k \neq c_k$ and $b_n = c_n$, $\forall n > k$. We consider the sequence $f = (f_n)_{n=0}^{\infty}$ with $f_n = b_{n+k}$, $n = \overline{0,\infty}$ and the sequence $g = (g_n)_{n=0}^{\infty}$ with $g_n = c_{n+k}$, $n = \overline{0,\infty}$. We obtain $f, g \in Rol[K]$ (as subsequences of the sequences b and c). If $d = f - g$, then $d_0 \neq 0$ and $d_n = 0$, $\forall n > 0$, i.e. the sequence d has finite rank. By applying Theorem 1.10, we obtain $d \notin Rol[K]$. But, since $f, g \in Rol[K]$, from Theorem 1.7 we have $d = f - g \in Rol[K]$, which is a contradiction. Thus, the extensions b and c coincide, i.e. the extension of the homogeneous linear recurrence over $\mathbb{Z}$ exists, it is unique and it does not depend on the generating vector, based on which the extension is performed. □

Let $\tilde{R}ol[K][m]$ be the set of all extensions over $\mathbb{Z}$ of the homogeneous linear recurrences from the set $Rol[K][m]$, and $\tilde{R}ol[K]$ − the set of all extensions over $\mathbb{Z}$ of the homogeneous linear recurrences from $Rol[K]$.

Corollary 1.4 *If* $a \in Rol[K]$ *and there exists* $q \in K^{m-1} \times K^*$ *such that*

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}, \ \forall n \geq t \geq m,$$

then $q \in G[K](a)$.

Proof. Consider the sequence $b = (b_n)_{n=0}^{\infty}$, $b_n = a_{n+t-m}$, $\forall n \in \mathbb{N}$. We have

$$b_n = a_{n+t-m} = \sum_{k=0}^{m-1} q_k a_{(n-1-k)+t-m} = \sum_{k=0}^{m-1} q_k b_{n-1-k}, \ \forall n \geq m,$$

which implies that $b \in Rol[K][m]$ with generating vector $q \in G[K][m](b)$.

We extend a and b over $\mathbb{Z}$ and we obtain $\tilde{a}$, respectively $\tilde{b}$. The sequence b is a subsequence of the sequence a, so, $\tilde{b}$ is an extension of the sequence a. According to TEUE (Theorem 1.11), we have $\tilde{a}$ and $\tilde{b}$ coincide up to a shift.

Since $b \in Rol[K][m]$ and $q \in G[K][m](b)$, it follows that $\tilde{a} = \tilde{b} \in \tilde{R}ol[K][m]$ and $q \in G[K][m](\tilde{a})$. Finally, we obtain that $a \in Rol[K][m]$ and $q \in G[K][m](a)$,

i.e. $q \in G[K](a)$. □

Theorem 1.12 *Let* $G^{[a]}(z) = \dfrac{A(z)}{B(z)}$ *be a regular rational fraction. The direct formula for the terms of the extension over* $\mathbb{Z}$ *of the homogeneous linear recurrence* $a \subseteq \mathbb{C}$ *is*

$$a_n = I_m^{[a]}((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T, \ \forall n \in \mathbb{Z},$$

where $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ *and* $z_0, z_1, \ldots, z_{p-1}$ *are all the distinct roots, with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$, *of the polynomial* $B(z)$ *of degree* m,

$$\beta_i^{[a]} = \left(\frac{i^j}{z_k^i} \right)_{k=\overline{0,\ p-1},\ j=\overline{0,\ s_k-1}}, \ \forall i \in \mathbb{N}$$

and $B^{[a]} = (\beta_i^{[a]})_{i=0}^{m-1}$ *(with the convention* $0^0 = 1$*).*

Proof. Consider the sequence $(\gamma_n^{[a]})_{n=0}^{\infty}$, defined by formula

$$\gamma_n^{[a]} = \left((-1)^j C_{n+j-1}^{j-1} \frac{1}{z_k^{n+j}} \right)_{\substack{k=\overline{0,\ p-1,} \\ j=\overline{1,\ s_k}}},$$

where $z_0, z_1, \ldots, z_{p-1}$ are all the distinct roots of the characteristic polynomial of the sequence $a = (a_n)_{n=0}^{\infty} \in Rol[K][m]$, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-1}$. The terms of this sequence were used as part of the proof of Theorem 1.2. We will extend the sequence $(\gamma_n^{[a]})_{n=0}^{\infty}$ over $\mathbb{Z}$ using the same formula

$$\gamma_n^{[a]} = \left((-1)^j C_{n+j-1}^{j-1} \frac{1}{z_k^{n+j}} \right)_{\substack{k=\overline{0,\ p-1,} \\ j=\overline{1,\ s_k}}}, \ \forall n \in \mathbb{Z},$$

where

$$C_n^k = \frac{(n-k+1)(n-k+2)\ldots n}{k!}, \ \forall n \in \mathbb{Z}, \ \forall k \in \mathbb{N}.$$

Firstly, we will prove that this extension is a homogeneous linear recurrence, i.e., if $a \in Rol[K]$ and $q \in G[K](a)$, then

$$\tilde{\gamma}^{[a]} = (\gamma_n^{[a]})_{n=-\infty}^{\infty} \in \tilde{Rol}[K],\ q \in G[K](\tilde{\gamma}^{[a]}).$$

Let $a \in Rol[K][m]$ and $q = (q_k)_{k=0}^{m-1} \in G[K][m](a)$. Next, we introduce the sequences $\tilde{S} = (S_n)_{n=-\infty}^{\infty}$ and $S_n = \left(S_{n,j}^{(k)}\right)_{\substack{k=\overline{0,\,p-1},\\ j=\overline{1,\,s_k}}}$ by using the following relations:

$$S_n = \sum_{t=0}^{m-1} q_t \gamma_{n-1-t}^{[a]},\ \forall n \in \mathbb{Z},$$

$$S_{n,j}^{(k)} = \sum_{t=0}^{m-1} q_t \gamma_{n-1-t,j}^{[a](k)},\ k = \overline{0,\,p-1},\ j = \overline{1,\,s_k},\ \forall n \in \mathbb{Z}.$$

Therefore,

$$S_{n,j}^{(k)} = \frac{(-1)^j}{z_k^{n+j}} \sum_{t=0}^{m-1} q_t C_{n-t+j-2}^{j-1} z_k^{t+1} =$$

$$= \frac{(-1)^j}{(j-1)!} \sum_{t=0}^{m-1} q_t [(n-t)(n-t+1)\ldots(n-t+j-2)]\, z_k^{t-n-j+1} =$$

$$= -\frac{1}{(j-1)!} \sum_{t=0}^{m-1} q_t [(t-n)(t-n-1)\ldots(t-n-(j-2))]\, z_k^{t-n-j+1} =$$

$$= -\frac{1}{(j-1)!} \cdot \left(\sum_{t=0}^{m-1} q_t z^{t-n} \right)^{(j-1)} \Bigg|_{z=z_k} = -\frac{1}{(j-1)!} \cdot (z^{-n-1} \cdot zG_m^{[q]}(z))^{(j-1)} \Bigg|_{z=z_k}.$$

Consider $\forall k \in \{0, 1, \ldots, p-1\}$. The value z_k is a root of multiplicity s_k of the characteristic polynomial $H_m^{[q]}(z)$, thus $H_m^{[q](j)}(z_k) = 0$, $j = \overline{0,\,s_k - 1}$. We obtain

$$(z^{-n-1} \cdot zG_m^{[q]}(z))^{(j-1)} = \sum_{t=0}^{j-1} C_{j-1}^t (z^{-n-1})^{(j-1-t)} \cdot (zG_m^{[q]}(z))^{(t)} =$$

$$= (z^{-n-1})^{(j-1)} \cdot zG_m^{[q]}(z) + \sum_{t=1}^{j-1} C_{j-1}^t (z^{-n-1})^{(j-1-t)} \cdot (zG_m^{[q]}(z) - 1)^{(t)} =$$

$$= z^{-n-j}[(-n-1)(-n-2)\ldots(-n-j+1)] \cdot zG_m^{[q]}(z) - \sum_{t=1}^{j-1} C_{j-1}^t (z^{-n-1})^{(j-1-t)} H_m^{[q](t)}(z).$$

It follows that

$$(z^{-n-1}\cdot zG_m^{[q]}(z))^{(j-1)}\Big|_{z=z_k} = z_k^{-n-j}[(-n-1)(-n-2)\dots(-n-j+1)]\cdot z_k G_m^{[q]}(z_k) =$$

$$= z_k^{-n-j}(-1)^{j-1}\cdot[(n+1)(n+2)\dots(n+j-1)]\cdot(1-H_m^{[q]}(z_k)) =$$

$$= z_k^{-n-j}(-1)^{j-1}\cdot[(n+1)(n+2)\dots(n+j-1)].$$

Therefore, for $k=\overline{0,\,p-1}$, $j=\overline{1,\,s_k}$ and $\forall n\in\mathbb{Z}$, we have

$$S_{n,\,j}^{(k)} = -\frac{1}{(j-1)!}\cdot z_k^{-n-j}(-1)^{j-1}\cdot[(n+1)(n+2)\dots(n+j-1)] = \frac{(-1)^j}{z_k^{n+j}}C_{n+j-1}^{j-1} = \gamma_{n,\,j}^{[a](k)},$$

which implies $S_n=\gamma_n^{[a]}$, $\forall n\in\mathbb{Z}$, i.e.

$$\gamma_n^{[a]} = \sum_{t=0}^{m-1} q_t\gamma_{n-1-t}^{[a]},\ \forall n\in\mathbb{Z}.$$

Consequently, we have the following relations:

$$\tilde{\gamma}^{[a]} = (\gamma_n^{[a]})_{n=-\infty}^{\infty}\in\tilde{R}ol[K],\ q\in G[K](\tilde{\gamma}^{[a]}).$$

We consider the bidirectional sequence

$$\Delta = (d_n)_{n=-\infty}^{\infty},\ d_n = I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot(\gamma_n^{[a]})^T,\ \forall n\in\mathbb{Z},$$

where $\Gamma^{[a]}=(\gamma_i^{[a]})_{i=0}^{m-1}$.

According to the first part of the proof of Theorem 1.2, we have $d_n=a_n$, $n=\overline{0,\,\infty}$. Consider $\forall n\in\mathbb{Z}$. From the first part of this proof, we obtain:

$$\sum_{k=0}^{m-1} q_k d_{n-1-k} = \sum_{k=0}^{m-1} q_k\cdot I_m^{[a]}((\Gamma^{[a]})^T)^{-1}(\gamma_{n-1-k}^{[a]})^T =$$

$$= I_m^{[a]}((\Gamma^{[a]})^T)^{-1}\cdot\sum_{k=0}^{m-1} q_k(\gamma_{n-1-k}^{[a]})^T = I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot(\gamma_n^{[a]})^T = d_n,$$

which implies $\Delta\in\tilde{R}ol[K][m]$, $q\in G[K][m](\Delta)$, i.e. $\tilde{d}=\Delta\in\tilde{R}ol[K]$ is an extension of the homogeneous linear recurrence $d=a$. According to TEUE, we have $\tilde{d}=\tilde{a}$. Therefore,

$$a_n = I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot(\gamma_n^{[a]})^T,\ \forall n\in\mathbb{Z}.$$

Similarly performing the second part of the proof of Theorem 1.2, we obtain

$$a_n = I_m^{[a]}((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T,\ \forall n \in \mathbb{Z}.$$ □

Example 1.8 *Show that the extension of the Fibonacci sequence over* $\mathbb{Z}$ *satisfies the formula* $F_n = \dfrac{L^n - (-L)^{-n}}{\sqrt{5}},\ \forall n \in \mathbb{Z}$, *where* $L = \dfrac{1+\sqrt{5}}{2}$.

Solution. The proof directly follows from Theorem 1.12 and Example 1.3. Therefore, the given formula also allows us to determine the terms with negative index. □

Example 1.9 *Show that the terms of two distinct homogeneous linear recurrences over a field* K *do not fully coincide starting with a given index* t.

Solution. Consider two distinct homogeneous linear recurrences $a^{(1)} \in Rol[K]$ and $a^{(2)} \in Rol[K]$. Let $P_1(z) \in H[K](a^{(1)})$ and $P_2(z) \in H[K](a^{(2)})$. From Theorem 1.7, we have $a = a^{(1)} - a^{(2)} \in Rol[K]$ and $P(z) = LCM(P_1(z), P_2(z)) \in H[K](a)$.

Suppose, to the contrary, that the terms of the linear recurrences $a^{(1)}$ and $a^{(2)}$ coincide starting with a given index t. We have $a_r^{(1)} = a_r^{(2)},\ \forall r \geq t$ and $a_{t-1}^{(1)} \neq a_{t-1}^{(2)}$. Therefore, $a_r = 0,\ \forall r \geq t$ and $a_{t-1} \neq 0$, i.e. a has the finite rank $t-1$, which implies that $a \notin Rol[K]$. As a result, this contradicts the first part of the proof. Hence, the statement of the theorem is true. □

1.5 Homogeneous linear recurrence functions. Reversibility and continuity

We extend the results described above to the homogeneous linear recurrence functions. Also, we will consider K as a subfield of the field $\mathbb{C}$.

Definition 1.13 *The function* $f : \mathbb{R} \to \mathbb{C}$ *is called a homogeneous linear* $m-$*recurrence over* K *if there exists* $q = (q_k)_{k=0}^{m-1} \in K^{m-1} \times K^*$ *such that*

$$f(x) = \sum_{k=0}^{m-1} q_k f(x-1-k),\ \forall x \in \mathbb{R}.$$

Definition 1.14 *The function* $f:\mathbb{R}\to\mathbb{C}$ *is called a homogeneous linear recurrence over* K *if* $\exists m\in\mathbb{N}$ *such that* f *is also a homogeneous linear* $m-$*recurrence over* K.

Denote:

- $Rolf[K][m]$ – the set of homogeneous linear $m-$recurrence functions over K;
- $Rolf[K]$ – the set of all homogeneous linear recurrence functions over K.

Theorem 1.13 *An extension over the set* $\mathbb{R}$ *of the homogeneous linear recurrence* $a\in Rol[K][m]$ *is the function* $\varphi:\mathbb{R}\to\mathbb{C}$,

$$\varphi(x)=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta^{[a]}(x))^T,$$

where $z_0,z_1,\ldots,z_{p-1}$ *are all the distinct roots of the characteristic polynomial of the sequence* a, *with corresponding multiplicities* $s_0,s_1,\ldots,s_{p-1}$,

$$\beta^{[a]}(x)=\left(\frac{x^j}{z_k^x}\right)_{\substack{k=\overline{0,\,p-1,}\\ j=\overline{0,\,s_k-1}}},\ \forall x\in\mathbb{R}$$

and $B^{[a]}=(\beta^{[a]}(i))_{i=0}^{m-1}$ *(with the convention* $0^0=1$*).*

Proof. Let $a\in Rol[K][m]$ and $q\in G[K][m](a)$. We have

$$\varphi(n)=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta^{[a]}(n))^T=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta_n^{[a]})^T=a_n,\ \forall n\in\mathbb{N}.$$

Using the same approach as for extensions over $\mathbb{Z}$, we obtain

$$\sum_{k=0}^{m-1}q_k\varphi(x-1-k)=\sum_{k=0}^{m-1}q_k\cdot I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta^{[a]}(x-1-k))^T=$$

$$=\sum_{k=0}^{m-1}q_k\cdot I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot(\gamma^{[a]}(x-1-k))^T=$$

$$=I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot\sum_{k=0}^{m-1}q_k(\gamma^{[a]}(x-1-k))^T=$$

$$=I_m^{[a]}\cdot((\Gamma^{[a]})^T)^{-1}\cdot(\gamma^{[a]}(x))^T=I_m^{[a]}\cdot((B^{[a]})^T)^{-1}\cdot(\beta^{[a]}(x))^T=\varphi(x),\ \forall x\in\mathbb{R}.$$

Thus, $\varphi:\mathbb{R}\to\mathbb{C}$ is an extension over $\mathbb{R}$ of the homogeneous linear recurrence a. □

The theorem for raising the order of the recurrence for homogeneous linear recurrence functions is:

Theorem 1.14 *Consider the function* $f \in Rolf[K][m]$ *with characteristic polynomial* $P(z) \in H[K][m](f)$. *We have* $f \in Rolf[K][m+1]$ *and*

$$Q(z) = (z-\alpha)P(z) \in H[K][m+1](f),\ \forall \alpha \in K^*.$$

Proof. Let $f \in Rolf[K][m]$, $P(z) = H_m^{[q]}(z) \in H[K][m](f)$ and $\forall x \in \mathbb{R}$. Consider

$$a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C},\ a_n = f(x-m-1+n),\ \forall n \in \mathbb{N}.$$

Since $P(z) = H_m^{[q]}(z) \in H[K][m](f)$, we have

$$f(y) = \sum_{k=0}^{m-1} q_k f(y-1-k),\ \forall y \in \mathbb{R},$$

which implies

$$f(x-m-1+n) = \sum_{k=0}^{m-1} q_k f((x-m-1)+(n-1-k)),\ \forall n \in \mathbb{N}.$$

Therefore, the recurrence relation

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

holds, i.e. $a \in Rol[K][m]$ and $P(z) \in H[K][m](a)$. According to TROR, we obtain

$$a \in Rol[K][m+1],\ Q(z) \in H[K][m+1](a).$$

Let $Q(z) = H_{m+1}^{[p]}(z)$. We have

$$a_n = \sum_{k=0}^{m} p_k a_{n-1-k},\ \forall n \geq m+1.$$

In particular, $a_{m+1} = \sum_{k=0}^{m} p_k a_{m-k}$, which leads to the recurrence relation

$$f(x) = \sum_{k=0}^{m} p_k f(x-1-k),\ \forall x \in \mathbb{R}.$$

Therefore, $f \in Rolf[K][m+1]$ and $Q(z) \in H[K][m+1](f)$. □

Definition 1.15 *The function* $f : \mathbb{R} \to \mathbb{C}$ *is called a homogeneous linear* $m-$*recurrence over* K *with inhomogeneity* $c \in \mathbb{C}$ *if there exists* $q \in K^{m-1} \times K^*$ *such that*

$$f(x)=\sum_{k=0}^{m-1} q_k f(x-1-k)+c,\ \forall x\in\mathbb{R}.$$

Theorem 1.15 *Let f be a homogeneous linear $m-$recurrence function over K, with inhomogeneity c, generated by the vector q. Then $f\in Rolf[K][m+1]$ and*

$$Q(z)=(z-1)H_m^{[q]}(z)\in H[K][m+1](f).$$

Proof. From the proof of TROR for $\alpha=1$, we had $Q(z)=\mu R(z)$ with $\mu\in K$ and

$$R(z)=1-z\left[(q_0+1)+\sum_{k=1}^{m-1}(q_k-q_{k-1})z^k-q_{m-1}z^m\right]=1-z\sum_{k=0}^{m}q_k^* z^k\in K[z].$$

We have

$$\sum_{k=0}^{m} q_k^* f(x-1-k)=(q_0+1)f(x-1)+\sum_{k=1}^{m-1}(q_k-q_{k-1})f(x-1-k)-q_{m-1}f(x-1-m)=$$

$$=\left(\sum_{k=0}^{m-1} q_k f(x-1-k)+c\right)+f(x-1)-\left(\sum_{k=1}^{m} q_{k-1} f(x-1-k)+c\right)=$$

$$=f(x)+f(x-1)-\left(\sum_{k=0}^{m-1} q_k f((x-1)-1-k)+c\right)=$$

$$=f(x)+f(x-1)-f(x-1)=f(x),\ \forall x\in\mathbb{R}.$$

Therefore, $f\in Rolf[K][m+1]$ and $Q(z)\in H[K][m+1](f)$. □

Definition 1.16 *Let $f\in Rolf[K][m]$. The function $\overleftarrow{f}:\mathbb{R}\to\mathbb{C}$, $\overleftarrow{f}(x)=f(-x+m-1)$, is called the reverse of the function f.*

Theorem 1.16 *Let $f\in Rolf[K][m]$. Then $\overleftarrow{\overleftarrow{f}}=f$.*

Proof. We have $\overleftarrow{\overleftarrow{f}}(x)=\overleftarrow{f}(-x+m-1)=f(-(-x+m-1)+m-1)=f(x)$, $\forall x\in\mathbb{R}$. □

Theorem 1.17 *Assume that $f\in Rolf[K][m]$ is a recurrence function with generating vector $q=(q_k)_{k=0}^{m-1}\in G[K][m](f)$. Then $\overleftarrow{f}\in Rolf[K][m]$ and*

$$q^*=\left(-\frac{q_{m-2}}{q_{m-1}},-\frac{q_{m-3}}{q_{m-1}},\ldots,-\frac{q_0}{q_{m-1}},\frac{1}{q_{m-1}}\right)\in G[K][m](\overleftarrow{f}).$$

Proof. Let $f\in Rolf[K][m]$ and $q\in G[K][m](f)$. Therefore,

$$f(x)=\sum_{k=0}^{m-1} q_k f(x-1-k),\ \forall x\in\mathbb{R}.$$

Since $q_{m-1}\neq 0$, we obtain the backward recurrence

$$f(x-m)=\frac{1}{q_{m-1}}f(x)-\sum_{k=0}^{m-2}\frac{q_k}{q_{m-1}}f(x-1-k),\ \forall x\in\mathbb{R},$$

equivalent to the backward recurrence relation

$$f(x)=\frac{1}{q_{m-1}}f(x+m)-\sum_{k=0}^{m-2}\frac{q_k}{q_{m-1}}f(x+m-1-k),\ \forall x\in\mathbb{R}.$$

Consider the vector

$$q^*=\left(-\frac{q_{m-2}}{q_{m-1}},-\frac{q_{m-3}}{q_{m-1}},\ldots,-\frac{q_0}{q_{m-1}},\frac{1}{q_{m-1}}\right)\in K^{m-1}\times K^*.$$

The recurrence relation

$$f(x)=\sum_{k=0}^{m-1} q_k^* f(x+1+k),\ \forall x\in\mathbb{R},$$

holds, which leads to

$$\overleftarrow{f}(x)=f(-x+m-1)=\sum_{k=0}^{m-1} q_k^* f(-(x-1-k)+m-1)=\sum_{k=0}^{m-1} q_k^*\overleftarrow{f}(x-1-k),\ \ \forall x\in\mathbb{R},$$

i.e. $\overleftarrow{f}\in Rolf[K][m]$ and $q^*\in G[K][m](\overleftarrow{f})$. □

Theorem 1.18 *Consider the function* $f\in Rolf[K][m]$ *with characteristic polynomial* $P(z)\in H[K][m](f)$. *Then* $z^m P(z^{-1})\in H[K][m](\overleftarrow{f})$.

Proof. Let $P(z)=H_m^{[q]}(z)$ and $q^*\in G[K][m](\overleftarrow{f})$, obtained as part of the proof of the Theorem 1.17. It follows that the generating function is

$$G_m^{[q^*]}(z)=\sum_{k=0}^{m-1} q_k^* z^k=-\frac{1}{q_{m-1}}\left(\sum_{k=0}^{m-2} q_{m-2-k}z^k-z^{m-1}\right),$$

from which we obtain

$$H_m^{[q^*]}(z)=1-zG_m^{[q^*]}(z)=1+\frac{1}{q_{m-1}}\left(\sum_{k=0}^{m-2} q_{m-2-k}z^{k+1}-z^m\right)=$$

$$= \frac{1}{q_{m-1}}\left(q_{m-1} + \sum_{k=1}^{m-1} q_{m-1-k} z^k - z^m \right) = \frac{1}{q_{m-1}}\left(\sum_{k=0}^{m-1} q_{m-1-k} z^k - z^m \right) =$$

$$= \frac{1}{q_{m-1}}\left(\sum_{k=0}^{m-1} q_k z^{m-1-k} - z^m \right) = \frac{1}{q_{m-1}}\left(-z^m + z^{m-1} \sum_{k=0}^{m-1} q_k \left(\frac{1}{z}\right)^k \right) =$$

$$= -\frac{z^m}{q_{m-1}}\left(1 - \frac{1}{z} G_m^{[q]}\left(\frac{1}{z}\right)\right) = -\frac{z^m}{q_{m-1}} H_m^{[q]}(z^{-1}) = -\frac{z^m}{q_{m-1}} P(z^{-1}) \in H[K][m](\overleftarrow{f}),$$

which implies $z^m P(z^{-1}) \in H[K][m](\overleftarrow{f})$. □

Lemma 1.4 *Let* $f \in Rolf[K][m]$. *If* f *is continuous on the interval* $[0, m)$, *then* f *is also continuous on the interval* $[0, +\infty)$.

Proof. Using the mathematical induction, we will prove that the function f is continuous on $[0, n)$, $\forall n \in \mathbb{N}$. Since f is continuous on $[0, m)$, then f is continuous on $[0, n)$, $\forall n \le m$.

Let the statement be true for $\forall n \le t$, $t \ge m$. This means that the function f is continuous on $[0, t)$. For $\forall x \in [t, t+1)$, the recurrence

$$f(x) = \sum_{k=0}^{m-1} q_k f(x-1-k)$$

holds, where $q \in G[K][m](f)$. Since we have the inequalities

$$0 \le x - m \le x - 1 - k \le x - 1 < t$$

for every k satisfying $0 \le k \le m-1$, we obtain

$$(x-1-k) \in [0, t),\ \forall k \in [0, m-1],$$

which implies that the function f is continuous at $(x-1-k)$, $k = \overline{0, m-1}$. Therefore,

$$f(x-1-k-0) = f(x-1-k) = f(x-1-k+0),\ k = \overline{0, m-1},$$

from which we obtain the left-hand limit

$$f(x-0) = \sum_{k=0}^{m-1} q_k f(x-1-k-0) = \sum_{k=0}^{m-1} q_k f(x-1-k) = f(x)$$

and the right-hand limit

$$f(x+0)=\sum_{k=0}^{m-1} q_k f(x-1-k+0)=\sum_{k=0}^{m-1} q_k f(x-1-k)=f(x).$$

It follows that

$$f(x-0)=f(x)=f(x+0),$$

hence f is continuous at x, $\forall x\in[t,t+1)$, i.e. f is continuous on $[t,t+1)$. Consequently, the function f is continuous on $[0,t+1)$. Therefore, the statement is true for $n=t+1$. By mathematical induction, the statement is true for $\forall n\in\mathbb{N}$, i.e. the function f is continuous on $[0,+\infty)$. □

Theorem 1.19 *Let $f\in Rolf[K][m]$. If the function f is continuous on an arbitrary interval $[u,u+m)$, where $u\in\mathbb{R}$, then it is also continuous on the set $\mathbb{R}$.*

Proof. Let $f\in Rolf[K][m]$, $q\in G[K][m](f)$ and suppose that f is continuous on the interval $[u,u+m)$. Consider the translation $h(x)=f(x-u)$. Therefore, the function h is continuous on $[0,m)$. From Lemma 1.4, we have that the function h is continuous on $[0,+\infty)$.

Since $h\in Rolf[K][m]$, from Theorem 1.17 we obtain $\overleftarrow{h}\in Rolf[K][m]$, where

$$\overleftarrow{h}(x)=h(-x+m-1),\ \forall x\in\mathbb{R}.$$

We consider the translation $g(x)=\overleftarrow{h}(x-1)$, $\forall x\in\mathbb{R}$, i.e.

$$g(x)=h(-x+m),\ \forall x\in\mathbb{R}.$$

It follows that $g\in Rolf[K][m]$ and $q^*\in G[K][m](g)$.

Consider $\forall x\in[0,m)$. We have $0<-x+m\le m$. Since the function h is continuous on $[0,+\infty)$, it follows that h is continuous on $(0,m]$. Thus, the function h is continuous at $(-x+m)$, i.e. the function g is continuous at x, $\forall x\in[0,m)$. By applying Lemma 1.4, we obtain that the function g is continuous on $[0,+\infty)$. It follows that the function h is continuous on $[-\infty,m]$. In particular, h is continuous on $[-\infty,0]$. Therefore, h is continuous on the interval $[-\infty,0]$ and on the interval $[0,+\infty)$, i.e. h is continuous on $\mathbb{R}$. Hence, the translated function $f(x)=h(x+u)$ is

also continuous on $\mathbb{R}$. □

Example 1.10 *Prove that the hyperbolic spiral in polar coordinates is not a homogeneous linear recurrence with respect to the angle.*

Solution. The equation of the hyperbolic spiral in polar coordinates is $r = f(\theta)$, where $f(\theta) = \frac{a}{\theta}$. Suppose, to the contrary, that $f \in Rolf[K][m]$, where $m \in \mathbb{N}^*$. Since f is continuous on the interval $[1,1+m)$, from Theorem 1.19 it follows that the function f is continuous on $\mathbb{R}$, in contradiction to the fact that the function f is discontinuous at $\theta = 0$. □

Example 1.11 *Prove that the homothety of the Archimedean spiral in polar coordinates is a homogeneous linear recurrence with respect to the angle.*

Solution. The equation of the homothety of the Archimedean spiral in polar coordinates is $r = f(\theta)$, where $f(\theta) = a\theta + b$. Inspired by the previous examples (Example 1.2 and Example 1.4), we have that the function $g(\theta) = \theta$, satisfies the recurrence relation

$$g(\theta) = 2g(\theta - 1) - g(\theta - 2),\ \forall \theta \in \mathbb{R}.$$

It follows that $g \in Rolf[\mathbb{R}][2]$ and $(2,-1) \in G[\mathbb{R}][2](g)$. Also, the constant function $h(\theta) = b$ satisfies the recurrence relation

$$h(\theta) = h(\theta - 1),\ \forall \theta \in \mathbb{R},$$

hence

$$h \in Rolf[\mathbb{R}][1],\ (1) \in G[\mathbb{R}][1](h).$$

The characteristic polynomials of g and h are

$$H_2^{[g]}(z) = 1 - 2z + z^2 = (1-z)^2 \in H[\mathbb{R}][2](g),$$

$$H_1^{[h]}(z) = 1 - z \in H[\mathbb{R}][1](h).$$

The function f is a linear combination of g and h, hence $f \in Rolf[\mathbb{R}]$ and

$$Q(z) = LCM(H_2^{[g]}(z), H_1^{[h]}(z)) = (1-z)^2 \in H[\mathbb{R}](f).$$

Since $\deg(Q(z)) = 2$, we have $f \in Rolf[\mathbb{R}][2]$ and $Q(z) = (1-z)^2 \in H[\mathbb{R}][2](f)$. □

1.6 Algebraic operations with homogeneous linear recurrences. Polynomials

Section 1.3 presented several algebraic operations over the set $Rol[K]$: linear combination, conjugate, real and imaginary parts of the homogeneous linear recurrences. It is obvious that these operations are also algebraic over the set $Rolf[K]$. This fact follows from the application of the theorems from that section to all sequences $a(x)=(f(x+n))_{n=0}^{\infty}$, $\forall x\in\mathbb{R}$. Next, we will enlarge the class of algebraic operations over $Rolf[\mathbb{C}]$ with three new operations: differentiation, integration and multiplication. Additionally, it will be shown that the multiplication is also an algebraic operation over $Rol[\mathbb{C}]$.

Theorem 1.20 *Let f be a differentiable function over $\mathbb{R}$, $f\in Rolf[K][m]$ and $P(z)\in H[K][m](f)$. Then $f'\in Rolf[K][m]$ and $P(z)\in H[K][m](f')$.*

Proof. Consider $P(z)=H_{m,\alpha}^{[q]}(z)$. We have

$$f(x)=\sum_{k=0}^{m-1}q_k f(x-1-k),\ \forall x\in\mathbb{R},$$

which implies

$$f'(x)=\sum_{k=0}^{m-1}q_k f'(x-1-k),\ \forall x\in\mathbb{R},$$

i.e. $f'\in Rolf[K][m]$ and $P(z)\in H[K][m](f')$. □

Theorem 1.21 *Let f be an integrable function over $\mathbb{R}$, for which $f\in Rolf[K][m]$, $P(z)\in H[K][m](f)$ and $P(1)\neq 0$. If $F(x)$ is an arbitrary antiderivative of the function $f(x)$, then there exists a unique antiderivative $\Phi:\mathbb{R}\to\mathbb{C}$,*

$$\Phi(x)=F(x)+\frac{F(0)-\sum_{k=0}^{m-1}q_k F(-1-k)}{P(1)},$$

which satisfies the relations $\Phi\in Rolf[K][m]$ and $P(z)\in H[K][m](\Phi)$.

Proof. Since the functions $F(x)$ and $\Phi(x)$ are antiderivatives of the function $f(x)$, we have $\Phi(x) = F(x) + c$, where $c \in \mathbb{C}$. Consider the function $f \in Rolf[K][m]$ with characteristic polynomial $P(z) = H_{m,\alpha}^{[q]}(z) \in H[K][m](f)$. We have

$$f(x) = \sum_{k=0}^{m-1} q_k f(x-1-k), \ \forall x \in \mathbb{R},$$

which means that the following relation holds:

$$\int_0^x f(u)du = \int_0^x \sum_{k=0}^{m-1} q_k f(u-1-k)du =$$

$$= \sum_{k=0}^{m-1} q_k \int_0^x f(u-1-k)d(u-1-k) = \sum_{k=0}^{m-1} q_k \int_{-1-k}^{x-1-k} f(y)d(y).$$

It follows that

$$\Phi(x) - \Phi(0) = \sum_{k=0}^{m-1} q_k (\Phi(x-1-k) - \Phi(-1-k)),$$

equivalent to

$$\Phi(x) - \sum_{k=0}^{m-1} q_k \Phi(x-1-k) = \Phi(0) - \sum_{k=0}^{m-1} q_k \Phi(-1-k) =$$

$$= F(0) + c - \sum_{k=0}^{m-1} q_k (F(-1-k) + c) = F(0) - \sum_{k=0}^{m-1} q_k F(-1-k) + c\left(1 - \sum_{k=0}^{m-1} q_k\right) =$$

$$= F(0) - \sum_{k=0}^{m-1} q_k F(-1-k) + cP(1), \ \forall x \in \mathbb{R}.$$

Therefore, $\Phi \in Rolf[K][m]$ and $P(z) \in H[K][m](\Phi)$ if and only if

$$\Phi(x) - \sum_{k=0}^{m-1} q_k \Phi(x-1-k) = 0, \ \forall x \in \mathbb{R},$$

equivalent to

$$F(0) - \sum_{k=0}^{m-1} q_k F(-1-k) + cP(1) = 0.$$

This equality holds if and only if

$$c = \frac{F(0) - \sum_{k=0}^{m-1} q_k F(-1-k)}{P(1)},$$

i.e. the antiderivative $\Phi(x)$ satisfies the relation

$$\Phi(x) = F(x) + \frac{F(0) - \sum_{k=0}^{m-1} q_k F(-1-k)}{P(1)}, \ \forall x \in \mathbb{R}. \qquad \square$$

Theorem 1.22 *Let* Φ *be an antiderivative of the integrable function* $f \in Rolf[K][m]$ *with characteristic polynomial* $P(z) \in H[K][m](f)$. *Then* $\Phi \in Rolf[K][m+1]$ *and* $Q(z) = (z-1)P(z) \in H[K](\Phi)$.

Proof. Consider the characteristic polynomial $P(z) = H_{m,\alpha}^{[q]}(z)$. Let us fix an arbitrary antiderivative F of the function f. We consider an antiderivative Φ of f. From the proof of Theorem 1.21, we have:

$$\Phi(x) - \sum_{k=0}^{m-1} q_k \Phi(x-1-k) = F(0) - \sum_{k=0}^{m-1} q_k F(-1-k) + cP(1) = \lambda, \ \forall x \in \mathbb{R},$$

where $c = \Phi(x) - F(x)$ and λ are constants. Therefore,

$$\Phi(x) = \sum_{k=0}^{m-1} q_k \Phi(x-1-k) + \lambda, \ \forall x \in \mathbb{R},$$

i.e. Φ is a homogeneous linear $m-$recurrence over K with the inhomogeneity λ, generated by q. Hence, $\Phi \in Rolf[K][m+1]$ and $Q(z) \in H[K][m+1](\Phi)$. $\square$

Theorem 1.23 *Let* $P(X) \in \mathbb{C}[X]$ *be a polynomial of degree* $\deg(P(X)) = m$. *Then* $P \in Rolf[\mathbb{Z}][m+1]$ *and* $Q(z) = (1-z)^{m+1} \in H[\mathbb{Z}][m+1](P)$.

Proof. If $m = 0$, then $P(X) = \alpha$ is constant, i.e. $P(x) = P(x-1), \ \forall x \in \mathbb{R}$. It follows that $P \in Rolf[\mathbb{Z}][1]$ and $(1-z) \in H[\mathbb{Z}][1](P)$. So, for $m = 0$ the statement is true.

Let the statement be true for each polynomial of degree less than r, $r \geq 2$. Let $m = r$. We consider the polynomial $D(X) = P'(X)$ of degree

$$\deg(D(X)) = \deg(P(X)) - 1 = r - 1 < r.$$

Therefore,

$$D \in Rolf[\mathbb{Z}][r],\ (1-z)^r \in H[\mathbb{Z}][r](D),$$

which implies

$$D \in Rolf[\mathbb{Z}][m],\ (1-z)^m \in H[\mathbb{Z}][m](D).$$

Since $P(X)$ is an antiderivative of $D(X)$, from Theorem 1.22 it follows that

$$P \in Rolf[\mathbb{Z}][m+1],\ (1-z)^{m+1} \in H[\mathbb{Z}][m+1](P).$$

By mathematical induction, the statement is true for each $P(X) \in \mathbb{C}[X]$. □

Corollary 1.5 *Every polynomial* $P(X) \in \mathbb{C}[X]$ *of degree* m *satisfies the relation:*

$$\sum_{k=0}^{m+1} (-1)^k C_{m+1}^k P(x-k) = 0,\ \forall x \in \mathbb{R}.$$

Proof. From Theorem 1.23, we have

$$P \in Rolf[\mathbb{Z}][m+1],\ (1-z)^{m+1} \in H[\mathbb{Z}][m+1](P).$$

Since

$$(1-z)^{m+1} = \sum_{k=0}^{m+1} C_{m+1}^k (-z)^k = 1 - z\sum_{k=1}^{m+1} C_{m+1}^k (-z)^{k-1} = 1 - z\sum_{k=0}^{m} C_{m+1}^{k+1} (-1)^k z^k,$$

it follows that

$$q = ((-1)^k C_{m+1}^{k+1})_{k=0}^m \in G[\mathbb{Z}](P).$$

Therefore,

$$P(x) = \sum_{k=0}^{m} (-1)^k C_{m+1}^{k+1} P(x-1-k) = -\sum_{k=1}^{m+1} (-1)^k C_{m+1}^k P(x-k),$$

which implies

$$\sum_{k=0}^{m+1} (-1)^k C_{m+1}^k P(x-k) = 0,\ \forall x \in \mathbb{R}.$$ □

Theorem 1.24 *Consider the sequence* $a \in Rol[K][m]$ *with characteristic polynomial* $P(z) \in H[K][m](a)$ *and the linear recurrence* $b \in Rol[K][1]$ *with generating vector* $(q_0) \in G[K][1](b)$. *Then* $ab = (a_n b_n)_{n=0}^{\infty} \in Rol[K][m]$ *and* $P(q_0 z) \in H[K](ab)$.

Proof. Let $P(z) = H_{m,\alpha}^{[p]}(z) \in H[K][m](a)$ and $(q_0) \in G[K][1](b)$. We have

$$b_n = q_0 b_{n-1},\ \forall n \geq 1,\ a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k},\ \forall n \geq m.$$

Consequently, we obtain the recurrence relations

$$b_n = q_0^k b_{n-k},\ \forall n \geq k,\ a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k},\ \forall n \geq m,$$

which imply the recurrence relation for the product:

$$(ab)_n = a_n b_n = \sum_{k=0}^{m-1} p_k a_{n-1-k} q_0^{k+1} b_{n-1-k} = \sum_{k=0}^{m-1} p_k q_0^{k+1} (ab)_{n-1-k},\ \forall n \geq m.$$

Therefore, we obtain

$$ab \in Rol[K][m],\ w = (p_k q_0^{k+1})_{k=0}^{m-1} \in G[K](ab).$$

The generating function is

$$G_m^{[w]}(z) = \sum_{k=0}^{m-1} w_k z^k = \sum_{k=0}^{m-1} p_k q_0^{k+1} z^k,$$

which leads to the following characteristic polynomial:

$$H_m^{[w]}(z) = 1 - zG_m^{[w]}(z) = 1 - z\sum_{k=0}^{m-1} p_k q_0^{k+1} z^k = 1 - q_0 z G_m^{[p]}(q_0 z) = P(q_0 z) \in K[z].$$

In conclusion, $ab \in Rol[K][m]$ and $P(q_0 z) \in H[K][m](ab)$. □

Theorem 1.25 *Consider the linear recurrences* $a \in Rol[\mathbb{C}][m_1]$ *and* $b \in Rol[\mathbb{C}][m_2]$ *with generating vectors* $u \in G[\mathbb{C}][m_1](a)$ *and* $v \in G[\mathbb{C}][m_2](b)$. *Let* $z_0, z_1, \ldots, z_{p-1}$ *be all the distinct roots, with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$, *of the polynomial* $H_{m_1}^{[u]}(z)$ *and let* $z_0^*, z_1^*, \ldots, z_{p^*-1}^*$ *be all the distinct roots, with corresponding multiplicities* $s_0^*, s_1^*, \ldots, s_{p^*-1}^*$, *of the polynomial* $H_{m_2}^{[v]}(z)$. *Then,* $ab \in Rol[\mathbb{C}]$ *and*

$$P(z) = LCM(\{(z - z_k z_r^*)^{s_k + s_r^* - 1} \mid k = \overline{0, p-1}, r = \overline{0, p^* - 1}\}) \in H[\mathbb{C}](ab).$$

Proof. We consider the decompositions

$$x^{[a]} = I_{m_1}^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{\substack{k=\overline{0,p-1} \\ j=0,s_k-1}},$$

$$x^{[b]} = I_{m_2}^{[b]}((B^{[b]})^T)^{-1} = (A_{k,j}^*)_{\substack{k=\overline{0,p^*-1} \\ j=\overline{0,s_k^*-1}}}.$$

For $\forall n \in \mathbb{N}$, it follows that

$$a_n = x^{[a]}(\beta_n^{[a]})^T = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{n^j}{z_k^n},$$

$$b_n = x^{[b]}(\beta_n^{[b]})^T = \sum_{k=0}^{p^*-1}\sum_{j=0}^{s_k^*-1} A_{k,j}^*\frac{n^j}{(z_k^*)^n}.$$

By multiplying the terms of the sequences a and b, we obtain

$$(ab)_n = a_n b_n = \left(\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{n^j}{z_k^n}\right)\cdot\left(\sum_{k=0}^{p^*-1}\sum_{j=0}^{s_k^*-1} A_{k,j}^*\frac{n^j}{(z_k^*)^n}\right) =$$

$$= \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1}\sum_{r=0}^{p^*-1}\sum_{i=0}^{s_r^*-1} A_{k,j}\frac{n^j}{z_k^n}A_{r,i}^*\frac{n^i}{(z_r^*)^n} = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1}\sum_{r=0}^{p^*-1}\sum_{i=0}^{s_r^*-1}\alpha_{ijrk}h_{ijrk}(n),$$

where, for $k=\overline{0,p-1}$, $j=\overline{0,s_k-1}$, $r=\overline{0,p^*-1}$, $i=\overline{0,s_r^*-1}$,

$$\alpha_{ijrk} = A_{k,j}A_{r,i}^*, \quad h_{ijrk}(n) = \frac{n^{i+j}}{(z_k z_r^*)^n}.$$

Next, from Theorem 1.23, we have

$$h_{i,j} = (n^{i+j})_{n=0}^{\infty} \in Rol[\mathbb{C}][i+j+1], \quad (1-z)^{i+j+1} \in H[C](h_{i,j}).$$

Since

$$g_{k,r} = \left(\frac{1}{(z_k z_r^*)^n}\right)_{n=\overline{0,\infty}} \in Rol[\mathbb{C}][1], \quad \left(\frac{1}{z_k z_r^*}\right) \in G[\mathbb{C}](g_{k,r}),$$

by applying Theorem 1.24 we obtain

$$h_{ijrk} = (h_{ijrk}(n))_{n=\overline{0,\infty}} \in Rol[\mathbb{C}][i+j+1],$$

$$\left(1-\frac{z}{z_k z_r^*}\right)^{i+j+1} \in H[\mathbb{C}](h_{ijrk}).$$

Further, from Theorem 1.7, we have

$$f_{rk}=\sum_{j=0}^{s_k-1}\sum_{i=0}^{s_r^*-1}\alpha_{ijrk}h_{ijrk}\in Rol[\mathbb{C}][s_k+s_r^*-1],$$

$$LCM\left(\left\{\left(1-\frac{z}{z_k z_r^*}\right)^{i+j+1}\,\middle|\, j=\overline{0,s_k-1}, i=\overline{0,s_r^*-1}\right\}\right)=\left(1-\frac{z}{z_k z_r^*}\right)^{s_k+s_r^*-1}\in H[\mathbb{C}](f_{rk}).$$

From the relation

$$(ab)_n=\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1}\sum_{r=0}^{p^*-1}\sum_{i=0}^{s_r^*-1}\alpha_{ijrk}h_{ijrk}(n)=\sum_{k=0}^{p-1}\sum_{r=0}^{p^*-1}f_{rk}(n),$$

with $f_{rk}=(f_{rk}(n))_{n=0}^{\infty}$, by applying Theorem 1.7 again, we obtain $ab\in Rol[\mathbb{C}]$ and

$$LCM\left(\left\{\left(1-\frac{z}{z_k z_r^*}\right)^{s_k+s_r^*-1}\,\middle|\, k=\overline{0,p-1}, r=\overline{0,p^*-1}\right\}\right)\in H[\mathbb{C}](ab).$$

Therefore, $ab\in Rol[\mathbb{C}]$ and

$$LCM(\{(z-z_k z_r^*)^{s_k+s_r^*-1}\,|\,k=\overline{0,p-1}, r=\overline{0,p^*-1}\})\in H[\mathbb{C}](ab). \qquad \square$$

Example 1.12 *Show that uniformly accelerated motion is a homogeneous linear recurrence over the set of integers and determine the generating vector.*

Solution. The motion state at time $t\in\mathbb{R}^+$ is

$$x(t)=x_0+v_0t+\frac{1}{2}at^2,$$

where x_0 is the initial state, v_0 is the initial speed and a is the acceleration. The equation of the motion represents a polynomial of degree 2, hence

$$x\in Rolf[\mathbb{Z}][3],\ Q(z)=(1-z)^3\in H[\mathbb{Z}][3](x).$$

Since $Q(z)=1-z(3-3z+z^2)\in H[\mathbb{Z}][3](x)$, we have $q=(3,-3,1)\in G[\mathbb{Z}][3](x)$, i.e. the following homogeneous linear recurrence holds:

$$x(t)=3x(t-1)-3x(t-2)+x(t-3),\ \forall t\geq 3. \qquad \square$$

Example 1.13 *Prove that the harmonic oscillatory motion and the potential, kinetic and total energy of the oscillator are homogeneous linear recurrences and determine the generating vectors of these recurrences.*

Solution. The harmonic oscillatory motion $x(t)$ and its kinetic energy $E_c(t)$, potential energy $U(t)$ and total energy $E(t)$ are given by the following equations:

$$x(t) = A\cos(\omega t + \varphi),$$

$$E_c(t) = \frac{k}{2}A^2\cos^2(\omega t + \varphi) = \frac{k}{2}x^2(t),$$

$$E(t) = \frac{k}{2}A^2 \text{ (constant with respect to } t),$$

$$U(t) = \frac{k}{2}A^2\sin^2(\omega t + \varphi) = E(t) - E_c(t).$$

For the harmonic oscillatory motion, we have

$$x(t) = A\cos(\omega t + \varphi) = \frac{A}{2}(e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}) = a_1u_1(t) + a_2u_2(t),$$

where

$$a_1 = \frac{A}{2}e^{i\varphi},\ a_2 = \frac{A}{2}e^{-i\varphi},\ u_1(t) = (e^{i\omega})^t \text{ and } u_2(t) = (e^{-i\omega})^t.$$

Also, since

$$u_1 \in Rolf[\mathbb{C}][1],\ (e^{i\omega}) \in G[\mathbb{C}][1](u_1),$$

$$u_2 \in Rolf[\mathbb{C}][1],\ (e^{-i\omega}) \in G[\mathbb{C}][1](u_2),$$

it follows that

$$u_1 \in Rolf[\mathbb{C}][1],\ 1 - e^{i\omega}z \in H[\mathbb{C}][1](u_1),$$

$$u_2 \in Rolf[\mathbb{C}][1],\ 1 - e^{-i\omega}z \in H[\mathbb{C}][1](u_2),$$

which implies

$$x \in Rolf[\mathbb{R}][2],\ (1 - e^{i\omega}z)(1 - e^{-i\omega}z) = 1 - (2\cos\omega)z + z^2 \in H[\mathbb{R}][2](x),$$

hence

$$x \in Rolf[\mathbb{R}][2],\ q(x) = (2\cos\omega, -1) \in G[\mathbb{R}][2](x).$$

For the kinetic energy, we have

$$E_c(t) = \frac{k}{2}x^2(t) = \frac{k}{2}x(t)x(t),$$

where

$$x \in Rolf[\mathbb{R}][2] \text{ and } (1-e^{i\omega}z)(1-e^{-i\omega}z) \in H[\mathbb{R}][2](x).$$

Since $z_0 = e^{-i\omega}$ and $z_1 = e^{i\omega}$ are all the roots, with multiplicities $s_0 = s_1 = 1$, of the characteristic polynomial, it follows that $E_c \in Rolf[\mathbb{R}]$ and

$$P(z) = (z-1)(z-e^{-2i\omega})(z-e^{2i\omega}) = (z-1)(z^2-(2\cos(2\omega))z+1) =$$

$$= z^3-(1+2\cos(2\omega))z^2+(1+2\cos(2\omega))z-1 = -(1-z(\alpha-\alpha z+z^2)) \in H[\mathbb{R}](E_c),$$

where $\alpha = 1+2\cos(2\omega)$. Therefore,

$$E_c \in Rolf[\mathbb{R}][3] \text{ and } q(E_c) = (\alpha,-\alpha,1) \in G[\mathbb{R}][3](E_c).$$

Since the total energy E is constant, it follows that

$$E \in Rolf[\mathbb{R}][1] \text{ and } q(E) = (1) \in G[\mathbb{R}][1](E).$$

Also, for the potential energy, we obtain $U = E - E_c$. Therefore

$$U \in Rolf[\mathbb{R}], \quad LCM(1-z, P(z)) = P(z) \in H[\mathbb{R}](U),$$

which implies $U \in Rolf[\mathbb{R}][3]$ and $(\alpha,-\alpha,1) \in G[\mathbb{R}][3](U)$. □

Example 1.14 *Consider the process that describes the evolution of the total surface area of a structure shaped as a regular quadrilateral prism. Prove that, if the side of the base and the height vary according to homogeneous linear recurrences, then the given process is also a homogeneous linear recurrence.*

Solution. Let $a(t)$, $h(t)$ and $s(t)$ be the side of the base, the height and the total surface area of the regular quadrilateral prism at time $t \in \mathbb{R}^+$. We have

$$s(t) = 2a^2(t) + 4a(t)h(t).$$

Since $a(t) \in Rolf[\mathbb{R}]$ and $h(t) \in Rolf[\mathbb{R}]$, it follows that

$$a^2(t) = a(t)\cdot a(t) \in Rolf[\mathbb{R}], \quad a(t)\cdot h(t) \in Rolf[\mathbb{R}],$$

which implies that $s(t) \in Rolf[\mathbb{R}]$. □

1.7 Linear recurrences with homogeneous linear recurrence inhomogeneity

Next, we study the inhomogeneous linear recurrences with homogeneous linear recurrence inhomogeneity. We show that these sequences are homogeneous linear recurrences and we determine their characteristic polynomials.

Definition 1.17 *Let* $b=(b_n)_{n=0}^{\infty}\in Rol[K]$. *The sequence* $a=(a_n)_{n=0}^{\infty}$, *which satisfies the recurrence relation*

$$a_n=\sum_{k=0}^{m-1}q_k a_{n-1-k}+b_n,\ \forall n\geq m,$$

where $q=(q_k)_{k=0}^{m-1}\in K^{m-1}\times K^*$, *is called an inhomogeneous linear recurrence over* K *with homogeneous linear recurrence inhomogeneity* b.

Definition 1.18 *The sequence* $\xi=(\xi_n)_{n=0}^{\infty}$, *which satisfies the recurrence*

$$\xi_n=\sum_{k=0}^{m-1}q_k\xi_{n-1-k},\ \forall n\geq m,$$

is called the homogeneous linear recurrence associated to the inhomogeneous linear recurrence a *generated by the vector* q.

Theorem 1.26 *Let* K *be a subfield of* $\mathbb{C}$. *Then, the inhomogeneous linear recurrence* a *over* K *with homogeneous linear recurrence inhomogeneity* b *is a homogeneous linear recurrence over* K. *If* ξ *is the homogeneous linear recurrence associated to the sequence* a, $Q(z)\in H[K](\xi)$ *and* $P(z)\in H[K](b)$, *then* $P(z)Q(z)\in H[K](a)$.

Proof. Let $Q(z)=H_m^{[q]}(z)$ and $P(z)=H_r^{[p]}(z)$. Then, we have

$$a_n=\sum_{k=0}^{m-1}q_k a_{n-1-k}+b_n,\ \forall n\geq m,$$

$$b_n=\sum_{j=0}^{r-1}p_j b_{n-1-j},\ \forall n\geq r.$$

We extend the homogeneous linear recurrence b over the set $\mathbb{Z}$, using the generating vector p, obtaining the extension $\tilde{b}=(b_n)_{n=-\infty}^{\infty}$. Next, we extend the inhomogeneous

linear recurrence a, using the generating vector q and the inhomogeneity $\tilde{b}$, obtaining the extension $\tilde{a} = (a_n)_{n=-\infty}^{\infty}$. They exist and are unique. We have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} + b_n, \quad b_n = \sum_{j=0}^{r-1} p_j b_{n-1-j}, \ \forall n \in \mathbb{Z}.$$

From the relation for $(a_n)_{n=-\infty}^{+\infty}$, it follows that

$$b_n = a_n - \sum_{k=0}^{m-1} q_k a_{n-1-k}, \ \forall n \in \mathbb{Z},$$

which implies

$$b_{n-1-j} = a_{n-1-j} - \sum_{k=0}^{m-1} q_k a_{n-2-j-k}, \ \forall n \in \mathbb{Z}.$$

Also, from the relation for $(b_n)_{n=-\infty}^{+\infty}$, we have

$$a_n - \sum_{k=0}^{m-1} q_k a_{n-1-k} = \sum_{j=0}^{r-1} p_j \left[a_{n-1-j} - \sum_{k=0}^{m-1} q_k a_{n-2-j-k} \right], \ \forall n \in \mathbb{Z},$$

i.e. the following recurrence relation holds:

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} + \sum_{j=0}^{r-1} p_j a_{n-1-j} - \sum_{j=0}^{r-1} \sum_{k=0}^{m-1} p_j q_k a_{n-2-j-k}, \ \forall n \in \mathbb{Z}.$$

Further, by grouping like terms, we obtain

$$a_n = \sum_{t=0}^{m+r-1} w_t a_{n-1-t}, \ \forall n \in \mathbb{Z},$$

where w_t, $t = \overline{0, m+r-1}$, are the coefficients of the polynomial

$$G_{m+r}^{[w]}(z) = \sum_{t=0}^{m+r-1} w_t z^t$$

and $w_{m+r-1} = p_{r-1} q_{m-1} \neq 0$. Therefore, $\tilde{a} \in \tilde{Rol}[K][m+r]$ (the set of all extensions over $\mathbb{Z}$ of the non-degenerate homogeneous linear $(m+r)-$recurrences over K).

We observe that

$$G_{m+r}^{[w]}(z) = G_m^{[q]}(z) + G_r^{[p]}(z) - z G_m^{[q]}(z) G_r^{[p]}(z),$$

which implies

$$H_{m+r}^{[w]}(z)=1-zG_{m+r}^{[w]}(z)=1-zG_m^{[q]}(z)-zG_r^{[p]}(z)+z^2G_m^{[q]}(z)G_r^{[p]}(z)=$$

$$=(1-zG_m^{[q]}(z))(1-zG_r^{[p]}(z))=H_m^{[q]}(z)H_r^{[p]}(z)=P(z)Q(z).$$

In conclusion, $a\in Rol[K]$ and $P(z)Q(z)\in H[K](a)$. □

Example 1.15 *Consider the sequence* $(a_n)_{n=0}^{\infty}\subseteq\mathbb{C}$, $a_n=2a_{n-1}-a_{n-2}+2^n$, $\forall n\geq 2$. *Determine the homogeneous linear recurrence and the direct formula.*

Solution. We have

$$P(z)=1-z(2-z)=z^2-2z+1=(z-1)^2\in H[\mathbb{C}](\xi),$$

$$Q(z)=1-2z\in H[\mathbb{C}](b).$$

It follows that

$$P(z)Q(z)=(z-1)^2(1-2z)=1-z(4-5z+2z^2)\in H[\mathbb{C}](a),$$

i.e. the following recurrence holds:

$$a_2=2a_1-a_0+4,\ a_n=4a_{n-1}-5a_{n-2}+2a_{n-3},\ \forall n\geq 3.$$

The roots of the polynomial $P(z)Q(z)$ are $z_0=1$ of multiplicity $s_0=2$ and $z_1=\frac{1}{2}$ of multiplicity $s_1=1$. We obtain $\beta_n^{[a]}=(1,n,2^n)$, hence

$$(B^{[a]})^T=\begin{pmatrix}1&1&1\\0&1&2\\1&2&4\end{pmatrix}\text{ and }((B^{[a]})^T)^{-1}=\begin{pmatrix}0&-2&1\\2&3&-2\\-1&-1&1\end{pmatrix},$$

which implies

$$((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T=\begin{pmatrix}0&-2&1\\2&3&-2\\-1&-1&1\end{pmatrix}\begin{pmatrix}1\\n\\2^n\end{pmatrix}=\begin{pmatrix}2^n-2n\\2+3n-2^{n+1}\\2^n-n-1\end{pmatrix}.$$

Therefore, we obtain the direct formula

$$a_n=(2^n-2n)a_0+(2+3n-2^{n+1})a_1+(2^n-n-1)a_2=$$

$$=(2^n-2n)a_0+(2+3n-2^{n+1})a_1+(2^n-n-1)(2a_1-a_0+4)=$$

$$= (1-n)a_0 + na_1 + 4(2^n - n - 1).$$ □

Example 1.16 *A snail moves along a path from point A to point B according to the following rule. In each unit of time it covers half of the remaining distance. Prove that the motion process is a homogeneous linear recurrence and determine the distance traveled by the snail in n units of time, given that the distance between points A and B is δ.*

Solution. Let d_n be the distance traveled by the snail in n units of time. It is easy to observe that the sequence $d = (d_n)_{n=0}^{\infty}$ satisfies the recurrence

$$d_n = d_{n-1} + \frac{\delta}{2^n}, \ \forall n \geq 1,$$

where $d_0 = 0$, i.e. d is an inhomogeneous linear recurrence with homogeneous linear recurrence inhomogeneity. By applying Theorem 1.26, we obtain $d \in Rol[\mathbb{R}]$ and

$$(1-z)(1-\frac{1}{2}z) = 1 - \frac{3}{2}z + \frac{1}{2}z^2 \in H[\mathbb{R}][d].$$

Therefore, $d \in Rol[\mathbb{R}][2]$, $I_2^{[d]} = \left(0, \frac{\delta}{2}\right)$ and $q = \left(\frac{3}{2}, -\frac{1}{2}\right) \in G[\mathbb{R}][2](d)$.

The roots of the characteristic polynomial are $z_0 = 1$ and $z_1 = 2$ and they are simple. We obtain $\beta_n^{[d]} = \left(1, \frac{1}{2^n}\right)$, hence

$$(B^{[d]})^T = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}, \ ((B^{[d]})^T)^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix},$$

which implies

$$d_n = I_2^{[d]}((B^{[d]})^T)^{-1}(\beta_n^{[d]})^T = \left(0, \frac{\delta}{2}\right)\begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{2^n} \end{pmatrix} = \frac{\delta}{2}\left(2 - \frac{2}{2^n}\right) = \left(1 - \frac{1}{2^n}\right)\delta.$$

Since $\lim_{n\to\infty} d_n = \delta$ and $d_n < \delta$, $\forall n \geq 0$, we can note that the snail will never reach the destination; rather, it approaches it asymptotically. □

1.8 Homogeneous linear recurrences over matrix sets

Consider the homogeneous linear recurrences over $M_t(K)$. The set $M_t(K)$ consists of all square matrices of size $t \times t$ with entries in $K \subseteq \mathbb{C}$.

Definition 1.19 *Consider the sequence $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}^t$. This sequence is called a homogeneous linear $m-$recurrence over the set $M_t(K)$ if there exists a vector*

$$q = (q^{(0)}, q^{(1)}, \ldots, q^{(m-1)}) \in (M_t(K))^m$$

such that

$$a_n = \sum_{k=0}^{m-1} q^{(k)} a_{n-1-k}, \ \forall n \geq m.$$

If $\left|q^{(m-1)}\right| \neq 0$, then the sequence a is called non-degenerate, otherwise degenerate.

Theorem 1.27 *Consider the sequence $a \subseteq \mathbb{C}^t$ such that $a \in Rol[M_t(K)][m]$ with generating vector $q \in G[M_t(K)][m](a)$. Then $a \in Rol[K][mt]$ and*

$$P(z) = \left|I - zG_m^{[q]}(z)\right| \in H[K][mt](a).$$

Proof. Consider the sequence a such that

$$a_n = (a_{n,j})_{j=0}^{t-1} \in \mathbb{C}^t, \ \forall n \in \mathbb{N},$$

$$a \in Rol[M_t(K)][m], \ q \in G[M_t(K)][m](a).$$

We have

$$a_n = \sum_{k=0}^{m-1} q^{(k)} a_{n-1-k}, \ \forall n \geq m,$$

which implies

$$a_{n,i} = \sum_{k=0}^{m-1} q_i^{(k)} a_{n-1-k} = \sum_{k=0}^{m-1} \sum_{j=0}^{t-1} q_{i,j}^{(k)} a_{n-1-k,j}, \ i = \overline{0, t-1}, \ \forall n \geq m.$$

Next, let us consider the notation

$$a^{(i)} = (a_{n,i})_{n=0}^{\infty}, \ i = \overline{0, t-1}.$$

For $\forall i \in \{0,1,\ldots,t-1\}$, we have

$$G^{[a^{(i)}]}(z)=\sum_{n=0}^{\infty}a_{n,\,i}z^n=\sum_{n=m}^{\infty}z^n\sum_{k=0}^{m-1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}a_{n-1-k,\,j}+\sum_{n=0}^{m-1}a_{n,\,i}z^n=$$

$$=G_m^{[a^{(i)}]}(z)+\sum_{k=0}^{m-1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}z^{k+1}\sum_{n=m}^{\infty}a_{n-1-k,\,j}z^{n-1-k}=$$

$$=G_m^{[a^{(i)}]}(z)+\sum_{k=0}^{m-1}z^{k+1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}\sum_{n=m-1-k}^{\infty}a_{n,\,j}z^n=$$

$$=G_m^{[a^{(i)}]}(z)+\sum_{k=0}^{m-1}z^{k+1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}(G^{[a^{(j)}]}(z)-G_{m-1-k}^{[a^{(j)}]}(z))=$$

$$=\left[G_m^{[a^{(i)}]}(z)-\sum_{k=0}^{m-1}z^{k+1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}G_{m-1-k}^{[a^{(j)}]}(z)\right]+z\sum_{j=0}^{t-1}G^{[a^{(j)}]}(z)\sum_{k=0}^{m-1}q_{i,\,j}^{(k)}z^k=$$

$$=F_i(z)+z\sum_{j=0}^{t-1}G_m^{[q_{i,\,j}]}G^{[a^{(j)}]}(z),$$

where

$$F_i(z)=G_m^{[a^{(i)}]}(z)-\sum_{k=0}^{m-1}z^{k+1}\sum_{j=0}^{t-1}q_{i,\,j}^{(k)}G_{m-1-k}^{[a^{(j)}]}(z).$$

We can rewrite the above relation in the following form:

$$G^{[a^{(i)}]}(z)-z\sum_{j=0}^{t-1}G_m^{[q_{i,\,j}]}G^{[a^{(j)}]}(z)=F_i(z),\ i=\overline{0,t-1}.$$

Also, we introduce the notation

$$x_i(z)=G^{[a^{(i)}]}(z),\ i=\overline{0,t-1}.$$

It follows that

$$x_i(z)-z\sum_{j=0}^{t-1}G_m^{[q_{i,\,j}]}x_j(z)=F_i(z),\ i=\overline{0,t-1}.$$

We obtained a system of t linear equations with t unknown variables ($x_i(z)$, $i=\overline{0,t-1}$). Let $x(z)=(x_i(z))_{i=0}^{t-1}$, $F(z)=(F_i(z))_{i=0}^{t-1}$, I – the identity matrix of size t and $W(z)=I-zG_m^{[q]}(z)$. The system takes the form $W(z)x(z)=F(z)$, which implies

$$x(z) = W^{-1}(z)F(z),\ \forall z \in D \setminus F,$$

where D is the domain of convergence of the series $G^{[a]}(z)$, and F is the set of the roots of the polynomial $\left|W(z)\right|$.

Let $V(z) = W^{-1}(z)$. So, $x(z) = V(z)F(z)$, $\forall z \in D \setminus F$. Componentwise, this relation can be written as follows:

$$x_i(z) = V_i(z)F(z) = \sum_{j=0}^{t-1} V_{i,\,j}(z)F(z),\ \forall z \in D \setminus F.$$

From the formula of $W(z)$, we have

$$\left|W(z)\right| \in K[z],\ \deg(\left|W(z)\right|) \le mt.$$

The coefficient of the term z^{mt} of the polynomial $\left|W(z)\right|$ is

$$\left|-q^{(m-1)}\right| = (-1)^t \left|q^{(m-1)}\right| \neq 0,$$

hence $\deg(\left|W(z)\right|) = mt$.

The entries of the matrix $V(z)$ are

$$V_{i,\,j}(z) = \frac{\left|M_{j,i}(z)\right|}{\left|W(z)\right|},$$

where $\left|M_{j,i}(z)\right|$ is the cofactor of the entry $W_{j,i}(z)$ of the matrix $W(z)$, $i, j = \overline{0, t-1}$.

We have $M_{j,i}(z) \in M_{t-1}(K)$, hence

$$\deg(\left|M_{j,i}(z)\right|) \le (t-1)m,\ i, j = \overline{0, t-1}.$$

It is easy to observe that $\deg(F_i(z)) \le m-1 < m$. Thus, the following formula holds:

$$x_i(z) = V_i(z)F(z) = \sum_{j=0}^{t-1} V_{i,\,j}(z)F(z) =$$

$$= \sum_{j=0}^{t-1}\left[\frac{\left|M_{j,\,i}(z)\right|}{\left|W(z)\right|}F(z)\right] = \frac{\sum_{j=0}^{t-1}\left(\left|M_{j,\,i}(z)\right|F(z)\right)}{\left|W(z)\right|},$$

where

$$\deg\left(\sum_{j=0}^{t-1}\left(\left|M_{j,\,i}(z)\right|F(z)\right)\right)<(t-1)m+m=mt=\deg(\left|W(z)\right|).$$

We obtained that $G^{[a^{(i)}]}(z)=x_i(z)$ is a regular rational fraction. Therefore,

$$a^{(i)}\in Rol[K][mt] \text{ and } \left|W(z)\right|\in H[K][mt](a^{(i)}),$$

which implies $a\in Rol[K][mt]$ and $\left|W(z)\right|\in H[K][mt](a)$. □

Theorem 1.28 *Consider the sequence* $a\subseteq\mathbb{C}^t$. *If* $a\in Rol^*[M_t(K)][m]$ *with generating vector* $q\in G^*[M_t(K)][m](a)$, *then* $a\in Rol^*[K][mt]$ *and*

$$P(z)=\left|I-zG_m^{[q]}(z)\right|\in H^*[K][mt](a).$$

Proof. This theorem concerns the degenerate case corresponding to Theorem 1.27. The problem that arises in the logical sequence of the proof in the non-degenerate case is that the rational fractions $G^{[a^{(i)}]}(z)$, $i=\overline{0,t-1}$, may be non-regular. However,

$$G^{[a^{(i)}]}(z)=\frac{A_i(z)}{\left|W(z)\right|},$$

where

$$A_i(z)=\sum_{j=0}^{t-1}\left(\left|M_{j,\,i}(z)\right|F(z)\right),\ \deg A_i(z)<mt\ ,\ i=\overline{0,\,t-1}.$$

Therefore, we have

$$a^{(i)}\in Rol^*[K][mt],\ \left|W(z)\right|\in H^*[K][mt](a^{(i)}),$$

which implies $a\in Rol^*[K][mt]$ and $\left|W(z)\right|\in H^*[K][mt](a)$. □

Example 1.17 *Let* $a=(a_n)_{n=0}^{\infty}$ *be a homogeneous linear recurrence, where*

$$a_0=\begin{pmatrix}1\\3\end{pmatrix},\ a_1=\begin{pmatrix}2\\1\end{pmatrix},\ a_n=\begin{pmatrix}1&3\\3&5\end{pmatrix}a_{n-1}+\begin{pmatrix}1&1\\1&-1\end{pmatrix}a_{n-2},\ \forall n\geq 2.$$

Determine the homogeneous linear recurrence formula over $\mathbb{C}$.

Solution. We have

$$|W(z)| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z\left[\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} + z\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right] \right| = \begin{vmatrix} 1-z-z^2 & -3z-z^2 \\ -3z-z^2 & z^2-5z+1 \end{vmatrix} =$$

$$= 1 - z(6 + 4z + 2z^2 + 2z^3) \in H[\mathbb{C}][4](a).$$

It follows that $q = (6,4,2,2) \in G[\mathbb{C}][4](a)$, i.e. the following recurrence holds:

$$a_n = 6a_{n-1} + 4a_{n-2} + 2a_{n-3} + 2a_{n-4}, \ \forall n \geq 4.$$

Next, we obtain the initial state of the linear recurrence over $\mathbb{C}$. We have

$$a_2 = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} a_1 + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} a_0 = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} a_2 + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} a_1 = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 39 \\ 73 \end{pmatrix}.$$

In conclusion, we have

$$a_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ a_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 9 \\ 9 \end{pmatrix}, \ a_3 = \begin{pmatrix} 39 \\ 73 \end{pmatrix},$$

$$a_n = 6a_{n-1} + 4a_{n-2} + 2a_{n-3} + 2a_{n-4}, \ \forall n \geq 4. \qquad \square$$

1.9 Degenerate homogeneous linear recurrences

In what follows, the results obtained for non-degenerate recurrences are considered in the context of both non-degenerate and degenerate recurrences. Denote:

- $Rol^*[K][m]$ – the set of homogeneous linear $m-$recurrences over K;
- $Rol^*[K]$ – the set of homogeneous linear recurrences over K;
- $G^*[K][m](a)$ – the set of generating vectors of length m of $a \in Rol^*[K][m]$;
- $G^*[K](a)$ – the set of generating vectors of the sequence $a \in Rol^*[K]$;
- $H^*[K][m](a)$ – the set of characteristic polynomials of degree not exceeding m of the sequence $a \in Rol^*[K][m]$;
- $H^*[K](a)$ – the set of characteristic polynomials of $a \in Rol^*[K]$.

The remaining notation retains a similar meaning.

Theorem 1.29 *If* $a \in Rol^*[K][m]$ *and* $q \in G^*[K][m](a)$, *then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Proof. Let $a \in Rol^*[K][m]$ and $q \in G^*[K][m](a)$. As in the case of non-degenerate recurrences, we have

$$G^{[a]}(z) - G_m^{[a]}(z) = \sum_{n=m}^{\infty} a_n z^n = \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} q_k a_{n-1-k} =$$

$$= z \sum_{k=0}^{m-1} q_k z^k \sum_{n=m}^{\infty} a_{n-1-k} z^{n-1-k} = z \sum_{k=0}^{m-1} q_k z^k (G^{[a]}(z) - G_{m-1-k}^{[a]}(z)),$$

from which we obtain the statement of the theorem. □

Theorem 1.30 *If* $a \in Rol^*[K][m]$ *and* $P(z) \in H^*[K][m](a)$, *then* $a \in Rol^*[K][m+1]$ *and* $Q(z) = (z-\alpha)P(z) \in H^*[K][m+1](a)$, $\forall \alpha \in K^*$.

Proof. Let $a \in Rol^*[K][m]$, $P(z) \in H^*[K][m](a)$ and $r = deg(P(z))$. We consider the subsequence $b = (b_n)_{n=0}^{\infty}$ of a, where $b_n = a_{n+m-r}$, for $\forall n \geq 0$. It is easy to observe that $b \in Rol[K][r]$ and $P(z) \in H[K][r](b)$. We obtain $b \in Rol[K][r+1]$ and $Q(z) \in H[K][r+1](b)$, which implies the statement of the theorem. □

Theorem 1.31 *Assume that*

$$a^{(j)} \in Rol^*[K][m_j],\ P_j(z) \in H^*[K][m_j](a^{(j)}),\ \alpha_j \in \mathbb{C},\ j = \overline{1,t}.$$

Then $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol^*[K][m]$ *and*

$$P(z) = LCM((P_j(z))_{j=1}^{t}) \in H^*[K][m](a),$$

where $m = r + s$, $r = deg(P(z))$, $r_j = deg(P_j(z))$, $j = \overline{1,t}$ *and* $s = \max_{j=\overline{1,t}}(m_j - r_j)$.

Proof. Let $a^{(j)} \in Rol^*[K][m_j]$, $P_j(z) \in H^*[K][m_j](a^{(j)})$ and $\alpha_j \in \mathbb{C}$, $j = \overline{1,t}$. We consider the subsequence $b^{(j)} = (b_n^{(j)})_{n=0}^{\infty}$ of the linear recurrence $a^{(j)}$, where

$$b_n^{(j)} = a_{n+m_j-r_j}^{(j)},\ j=\overline{1,t},\ \forall n \geq 0.$$

It is easy to observe that

$$b^{(j)} \in Rol[K][r_j],\ P_j(z) \in H[K][r_j](b^{(j)}),\ j=\overline{1,t}.$$

Next, let $c^{(j)} = (c_n^{(j)})_{n=0}^{\infty}$ be a subsequence of the sequence $b^{(j)}$, where

$$c_n^{(j)} = a_{n+s}^{(j)},\ j=\overline{1,t},\ \forall n \geq 0.$$

It follows that $c^{(j)} \in Rol[K][r_j]$ and $P_j(z) \in H[K][r_j](c^{(j)}),\ j=\overline{1,t}$. Consequently, we obtain $c = (a_{n+s})_{n=0}^{\infty} \in Rol[K][r]$ and $P(z) \in H[K][r](c)$, which implies that $a \in Rol^*[K][r+s]$ and $P(z) \in H^*[K][r+s](a)$. □

Theorem 1.32 *Consider the sequences* $a \in Rol^*[\mathbb{C}][m_1]$ *and* $b \in Rol^*[\mathbb{C}][m_2]$ *with generating vectors* $u \in G^*[\mathbb{C}][m_1](a)$ *and* $v \in G^*[\mathbb{C}][m_2](b)$. *We suppose that all the distinct roots* $z_0, z_1, \ldots, z_{p-1}$, *with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$, *of the polynomial* $H_{m_1}^{[u]}(z)$ *of degree* r_1 *and all the distinct roots* $z_0^*, z_1^*, \ldots, z_{p-1}^*$, *with corresponding multiplicities* $s_0^*, s_1^*, \ldots, s_{p^*-1}^*$, *of the polynomial* $H_{m_2}^{[v]}(z)$ *of degree* r_2, *are known. Then* $ab \in Rol^*[\mathbb{C}][m]$ *and*

$$P(z) = LCM(\{(z - z_k z_l^*)^{s_k + s_l^* - 1} \mid k = \overline{0, p-1}, l = \overline{0, p^* - 1}\}) \in H^*[\mathbb{C}][m](ab),$$

where $m = r + s$, $r = deg(P(z))$ *and* $s = \max\{m_1 - r_1, m_2 - r_2\}$.

Proof. Consider $a \in Rol^*[\mathbb{C}][m_1]$ and $b \in Rol^*[\mathbb{C}][m_2]$ with generating vectors $u \in G^*[\mathbb{C}][m_1](a)$ and $v \in G^*[\mathbb{C}][m_2](b)$. Let $c^{(1)} = (c_n^{(1)})_{n=0}^{\infty}$ be a subsequence of the sequence a and let $c^{(2)} = (c_n^{(2)})_{n=0}^{\infty}$ be a subsequence of the sequence b, where

$$c_n^{(1)} = a_{n+m_1-r_1},\ c_n^{(2)} = b_{n+m_2-r_2},\ \forall n \geq 0.$$

It is easy to observe that

$$c^{(1)} \in Rol[\mathbb{C}][r_1],\ c^{(2)} \in Rol[\mathbb{C}][r_2],$$

$$u \in G[\mathbb{C}][r_1](c^{(1)}),\ v \in G[\mathbb{C}][r_2](c^{(2)}).$$

We consider the subsequence $d^{(j)} = (d_n^{(j)})_{n=0}^{\infty}$ of the sequence $c^{(j)}$, $j = \overline{1,2}$, where

$$d_n^{(1)} = a_{n+s},\ d_n^{(2)} = b_{n+s},\ \forall n \geq 0.$$

We have

$$d^{(1)} \in Rol[\mathbb{C}][r_1],\ d^{(2)} \in Rol[\mathbb{C}][r_2],$$

$$u \in G[\mathbb{C}][r_1](d^{(1)}),\ v \in G[\mathbb{C}][r_2](d^{(2)}),$$

which implies

$$d = d^{(1)}d^{(2)} \in Rol[\mathbb{C}][r],\ P(z) \in H[\mathbb{C}][r](d).$$

Therefore, $ab \in Rol^*[\mathbb{C}][r+s]$ and $P(z) \in H^*[\mathbb{C}][r+s](ab)$. □

Theorem 1.33 *If* $a \in Rol^*[K][m]$, $P(z) \in H^*[K][m](a)$ *and* $b_n = G_n^{[a]}(1)$, $n = \overline{0,\infty}$, *then* $b = (b_n)_{n=0}^{\infty} \in Rol^*[K][m+1]$ *and* $Q(z) = (z-1)P(z) \in H^*[K][m+1](b)$.

Proof. Let $P(z) = H_{m,\lambda}^{[q]}(z)$ with $\lambda = P(0)$. We have $Q(z) = -\lambda R(z)$, where

$$R(z) = 1 - z\sum_{k=0}^{m} q_k^* z^k,$$

$$q_0^* = q_0 + 1,\ q_k^* = q_k - q_{k-1},\ k = \overline{1,m-1},\ q_m^* = -q_{m-1}.$$

It follows that

$$\sum_{k=0}^{m} q_k^* b_{n-1-k} = (q_0+1)b_{n-1} - q_{m-1}b_{n-1-m} + \sum_{k=1}^{m-1}(q_k - q_{k-1})b_{n-1-k} =$$

$$= b_{n-1} + \sum_{k=0}^{m-1} q_k(b_{n-1-k} - b_{n-2-k}) = b_{n-1} + \sum_{k=0}^{m-1} q_k a_{n-2-k} = b_{n-1} + a_{n-1} = b_n,\ \forall n \geq m+1.$$

Therefore, $b \in Rol^*[K][m+1]$ and $Q(z) \in H^*[K][m+1](b)$. □

Example 1.18 *What can be said about the sum and the product of a finite number of non-degenerate homogeneous linear recurrences from which a finite number of terms have been removed at random?*

Solution. We consider t sequences $a^{(k)} = (a_n^{(k)})_{n=0}^{\infty}$, $k = \overline{1,t}$. Let

$$a^{(k)} \in Rol[\mathbb{C}][m_k],\ P_k(z) \in H[\mathbb{C}][m_k](a^{(k)}),\ k = \overline{1,t}.$$

We assume that, from each sequence $a^{(k)}$, s_k terms have been removed at random, obtaining the corresponding sequences $b^{(k)} = (b_n^{(k)})_{n=0}^{\infty}$, $k = \overline{1,t}$.

Let $a_{n_j^{(k)}}^{(k)}$, $j = \overline{1,s_k}$, be the removed terms and $n^{(k)} = \max\limits_{j=\overline{1,s_k}} n_j^{(k)}$, $k = \overline{1,t}$. Since

$$b_{n-s_k}^{(k)} = a_n^{(k)},\ \forall n \geq n^{(k)} + 1,$$

we have

$$b^{(k)} \in Rol^*[\mathbb{C}][m_k + n^{(k)} + 1 - s_k],$$

$$P_k(z) \in H^*[\mathbb{C}][m_k + n^{(k)} + 1 - s_k](b^{(k)}),\ k = \overline{1,t}.$$

By applying Theorem 1.31, we obtain

$$b = \sum_{k=1}^{t} b^{(k)} \in Rol^*[\mathbb{C}][m^{(b)}],$$

$$P(z) = LCM((P_k(z))_{k=1}^{t}) \in H^*[\mathbb{C}][m^{(b)}](b),$$

where

$$m^{(b)} = \deg(P(z)) + \max_{k=\overline{1,t}}(n^{(k)} + 1 - s_k).$$

Similarly, from Theorem 1.32 we have

$$c = \prod_{k=1}^{t} b^{(k)} \in Rol^*[\mathbb{C}][m^{(c)}],$$

where $m^{(c)}$ and the characteristic polynomial are determined according to the same theorem. Therefore, the sum and the product are degenerate homogeneous linear recurrences over $\mathbb{C}$. □

2 ORDER MINIMIZATION METHODS AND DECISION CRITERIA

In this chapter, the concept of minimality is defined, and two order minimization methods are proposed: the rank-based order minimization method and the order minimization method based on eliminating characteristic zeros. In addition, several inequalities for the dimension of homogeneous linear recurrences are formulated and proved, together with several fundamental decision criteria, including the homogeneous linear recurrence criterion and the periodicity and inclusion criteria. Necessary and sufficient conditions for the convergence of homogeneous linear recurrences are analyzed, while the uniqueness of the minimal generating vector and the structure of the set of characteristic polynomials are established.

2.1 Dimension of the homogeneous linear recurrences. Inequalities

In this section, we introduce the definitions of minimality and dimension for the homogeneous linear recurrences. Additionally, we present several important inequalities for them, which will be subsequently applied in the development of minimization methods.

Definition 2.1 *Let* $a \in Rol[K]$ *be a sequence that is not identically zero. The sequence* a *is called* $m-$*minimal over the set* K *if* $a \in Rol[K][m]$ *and* $a \notin Rol[K][t]$, $\forall t < m$. *The number* m *is called the dimension of the sequence a over* K *(denoted by* $\dim[K](a) = m$*). The dimension of the sequence that is identically zero is considered equal to zero.*

Next, we consider that the set K is a subset of the field $\mathbb{C}$. The properties of the dimension are given by the following theorems.

Theorem 2.1 *If* $a \in Rol[K][m]$, *then* $\dim[K](a) \le m$.

Proof. Suppose, to the contrary, that $\dim[K](a) = r > m$. By definition, we have $a \notin Rol[K][t],\ \forall t < r$. Since $m < r$, we obtain that $a \notin Rol[K][m]$, which is a contradiction. Therefore, $\dim[K](a) \le m$. □

Theorem 2.2 *If* $a^{(k)} \in Rol[K],\ \alpha_k \in \mathbb{C},\ k = \overline{1,t}$, *then*

$$\dim[K]\left(\sum_{k=1}^{t} \alpha_k a^{(k)}\right) \le \sum_{k=1}^{t} \dim[K](a^{(k)}).$$

Proof. Let $\dim[K](a^{(k)}) = m_k$ and $P_k(z) \in H[K][m_k],\ k = \overline{1,t}$. This implies

$$\deg(P_k(z)) = \dim[K](a^{(k)}),\ k = \overline{1,t}.$$

From Theorem 1.7 we have $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol[K]$ and

$$P(z) = LCM(P_1(z), P_2(z), \ldots, P_t(z)) \in H[K](a).$$

By applying Theorem 2.1, we obtain

$$\dim[K](a) \le \deg(LCM(P_1(z), P_2(z), \ldots, P_t(z))) \le$$

$$\le \deg\left(\prod_{k=1}^{t} P_k(z)\right) = \sum_{k=1}^{t} m_k = \sum_{k=1}^{t} \dim[K](a^{(k)}),$$

which implies

$$\dim[K]\left(\sum_{k=1}^{t} \alpha_k a^{(k)}\right) \le \sum_{k=1}^{t} \dim[K](a^{(k)}).$$ □

Lemma 2.1 *If* $a \in Rol[\mathbb{C}]$ *and* $b \in Rol[\mathbb{C}]$, *then*

$$\dim[\mathbb{C}](ab) \le \dim[\mathbb{C}](a) \cdot \dim[\mathbb{C}](b).$$

Proof. Let $\dim[\mathbb{C}](a) = m_1$ and $\dim[\mathbb{C}](b) = m_2$. It follows that $a \in Rol[\mathbb{C}][m_1]$ and $b \in Rol[\mathbb{C}][m_2]$. Therefore, $ab \in Rol[\mathbb{C}]$ with the characteristic polynomial

$$P(z) = LCM(\{(z - z_k z_r^*)^{s_k + s_r^* - 1} \mid k = \overline{0, p-1}, r = \overline{0, p^* - 1}\}) \in H[\mathbb{C}](ab).$$

We have

$$Q(z) = \prod_{k=0}^{p-1} \prod_{r=0}^{p^*-1} (z - z_k z_r^*)^{s_k + s_r^* - 1} \vdots P(z),$$

hence $Q(z) \in H[\mathbb{C}](ab)$. We obtain:

$$\deg(Q(z)) = \sum_{k=0}^{p-1}\sum_{r=0}^{p^*-1}(s_k + s_r^* - 1) = \sum_{k=0}^{p-1}\sum_{r=0}^{p^*-1} s_k + \sum_{k=0}^{p-1}\sum_{r=0}^{p^*-1} s_r^* - \sum_{k=0}^{p-1}\sum_{r=0}^{p^*-1} 1 =$$

$$= p^* m_1 + p m_2 - p p^* \overset{Not}{=} f(p, p^*).$$

Consider the rectangular domain $D: 1 \le p \le m_1,\ 1 \le p^* \le m_2$. It is easy to prove that

$$\max_{(p,p^*)\in D} f(p, p^*) = f(m_1, m_2) = m_1 m_2$$

and, consequently, the following series of estimations holds:

$$\dim[\mathbb{C}](ab) \le \deg(Q(z)) = f(p, p^*) \le$$

$$\le \max_{(p,p^*)\in D} f(p, p^*) = m_1 m_2 = \dim[\mathbb{C}](a) \cdot \dim[\mathbb{C}](b). \qquad \square$$

Theorem 2.3 *If* $a^{(k)} \in Rol[\mathbb{C}],\ k = \overline{1,t}$, *then*

$$\dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \le \prod_{k=1}^{t} \dim[\mathbb{C}](a^{(k)}).$$

Proof. Since $\dim[\mathbb{C}](a^{(1)}) \le \dim[\mathbb{C}](a^{(1)})$, the statement of the theorem is true for $t = 1$. Assume that the inequality holds for $\forall t \le r$. Let us prove it for $t = r + 1$. By applying Lemma 2.1, we have

$$\dim[\mathbb{C}]\left(\prod_{k=1}^{r+1} a^{(k)}\right) = \dim[\mathbb{C}]\left(\left(\prod_{k=1}^{r} a^{(k)}\right) a^{(r+1)}\right) \le$$

$$\le \dim[\mathbb{C}]\left(\prod_{k=1}^{r} a^{(k)}\right) \dim[\mathbb{C}](a^{(r+1)}) \le$$

$$\le \left(\prod_{k=1}^{r} \dim[\mathbb{C}](a^{(k)})\right) \cdot \dim[\mathbb{C}](a^{(r+1)}) = \prod_{k=1}^{r+1} \dim[\mathbb{C}](a^{(k)}).$$

By mathematical induction, the statement is true for $\forall t \in \mathbb{N}^*$. $\square$

Theorem 2.4 *If* $K_1 \subseteq K_2$ *and* $a \in Rol[K_1]$, *then* $a \in Rol[K_2]$ *and*

$$\dim[K_2](a) \le \dim[K_1](a).$$

Proof. Consider $a \in Rol[K_1]$. It follows that $\exists m \in \mathbb{N}^*$, such that $a \in Rol[K_1][m]$. Therefore, $\exists q \in K_1^{m-1} \times K_1^*$ which satisfies the recurrence

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}\ ,\ \forall n \geq m\ .$$

Since $K_1 \subseteq K_2$, we have $q \in K_2^{m-1} \times K_2^*$ and $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$, $\forall n \geq m$. Therefore, $a \in Rol[K_2][m]$, i.e. $a \in Rol[K_2]$.

Suppose, to the contrary, that

$$r = \dim[K_2](a) > \dim[K_1](a) = s\ .$$

Therefore, $a \notin Rol[K_2][s]$, which implies $a \notin Rol[K_1][s]$. But, since $\dim[K_1](a) = s$, it follows that $a \in Rol[K_1][s]$, which is a contradiction. In conclusion,

$$\dim[K_2](a) \leq \dim[K_1](a)\ .$$ □

Similar results can be formulated for degenerate homogeneous linear recurrences.

Definition 2.2 *Let $a \in Rol^*[K]$ be a sequence that is not identically zero. The recurrence a is called $m-$minimal over the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, $\forall t < m$. The number m is called the dimension of the sequence a over the set K (denoted by $\dim[K](a) = m$). The dimension of the sequence that is identically zero is defined to be zero.*

Theorem 2.5 *If $a \in Rol^*[K][m]$, then $\dim[K](a) \leq m$.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

Theorem 2.6 *If $a^{(k)} \in Rol^*[K]$, $\alpha_k \in \mathbb{C}$, $k = \overline{1,t}$, then*

$$\dim[K]\left(\sum_{k=1}^{t} \alpha_k a^{(k)}\right) \leq \sum_{k=1}^{t} \dim[K](a^{(k)})\ .$$

Proof. We use the notation from Theorem 1.31. For $\forall a^{(k)} \in Rol^*[K]$ and $\alpha_k \in \mathbb{C}$, $k = \overline{1,t}$, we have the following inequality

$$dim[K]\left(\sum_{k=1}^{t}\alpha_k a^{(k)}\right) \le r+s \le \sum_{k=1}^{t} r_k + \sum_{k=1}^{t}(m_k - r_k) = \sum_{k=1}^{t} m_k = \sum_{k=1}^{t} dim[K](a^{(k)}). \quad \square$$

Theorem 2.7 *If* $a^{(k)} \in Rol^*[\mathbb{C}]$, $k = \overline{1,t}$, *then*

$$\dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \le \prod_{k=1}^{t} \dim[\mathbb{C}](a^{(k)}).$$

Proof. We consider the notation from Theorem 1.32. Next, we use the method of mathematical induction. For $t = 2$ and $\forall a^{(k)} \in Rol^*[\mathbb{C}]$, $k = \overline{1,2}$, we have

$$dim[\mathbb{C}](a^{(1)}a^{(2)}) \le r+s \le r_1 r_2 + (m_1 - r_1)(m_2 - r_2)$$

$$= m_1 m_2 - r_1(m_2 - r_2) - r_2(m_1 - r_1) \le m_1 m_2 = dim[\mathbb{C}](a^{(1)}) dim[\mathbb{C}](a^{(2)}).$$

We consider that the statement is true for $2 \le t \le T$. Therefore,

$$dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \le \prod_{k=1}^{t} dim[\mathbb{C}](a^{(k)}), \forall a^{(k)} \in Rol^*[\mathbb{C}],\ k = \overline{1,t}.$$

For $t = T+1$, we obtain

$$dim[\mathbb{C}]\left(\prod_{k=1}^{T+1} a^{(k)}\right) \le dim[\mathbb{C}]\left(\prod_{k=1}^{T} a^{(k)}\right) dim[\mathbb{C}](a^{(T+1)}) \le \prod_{k=1}^{T+1} dim[\mathbb{C}](a^{(k)}).$$

In conclusion, the property holds for $\forall t \ge 2$. $\square$

Theorem 2.8 *If* $K_1 \subseteq K_2$ *and* $a \in Rol^*[K_1]$, *then* $a \in Rol^*[K_2]$ *and*

$$\dim[K_2](a) \le \dim[K_1](a).$$

Proof. The proof of this theorem is similar to the proof of Theorem 2.4. $\square$

The dimension of the functions from $Rolf[K]$ is defined analogously. Similar theorems hold for the functions from $Rolf[K]$. Let us present several statements specific to the functions.

Theorem 2.9 *Assume that* f *is an integrable function on* $\mathbb{R}$, *such that* $f \in Rolf[K]$ *and* $\dim[K](f) = m$. *Let* F *be an antiderivative of* f. *Then* $\dim[K](F) \in \{m, m+1\}$.

Proof. Since we have $\dim[K](f) = m$, it follows that $f \in Rolf[K][m]$, which implies $F \in Rolf[K][m+1]$, i.e. $\dim[K](F) \le m+1$. Let us prove that $\dim[K](F) \ge m$.

Suppose, to the contrary, that $\dim[K](F) = r < m$. We have $F \in Rolf[K][r]$. Consequently, it follows that $f = F' \in Rolf[K][r]$, i.e.

$$\dim[K](f) \le r < m,$$

which contradicts the initial statement. Therefore, we have $\dim[K](F) \ge m$. In conclusion, we obtain

$$m \le \dim[K](F) \le m+1,$$

hence $\dim[K](F) \in \{m, m+1\}$. □

Corollary 2.1 *Assume that the function* f *is differentiable on* $\mathbb{R}$. *Let* $f \in Rolf[K]$ *and* $\dim[K](f) = m$. *Then* $\dim[K](f') \in \{m, m-1\}$.

Proof. Since $f \in Rolf[K]$, we have $f' \in Rolf[K]$. Let $\dim[K](f') = r$. The function f is an antiderivative of f', thus $m = \dim[K](f) \in \{r, r+1\}$. Therefore,

$$r = \dim[K](f') \in \{m, m-1\}.$$ □

Corollary 2.2 *Assume that the function* f *is differentiable* k *times on* $\mathbb{R}$. *Let* $f \in Rolf[K]$ *and* $\dim[K](f) = m$. *Then* $\max\{1, m-k\} \le \dim[K](f^{(k)}) \le m$.

Proof. By definition, we have

$$\dim[K](F) \ge 1, \ \forall F \in Rolf[K].$$

It follows that $\dim[K](f^{(k)}) \ge 1$. Since $\dim[K](f') \in \{m, m-1\}$, the statement holds for $k = 1$, hence

$$\max\{1, m-1\} \le \dim[K](f') \le m.$$

Let the statement be true for $\forall k \le r$. Let us prove the inequality for $k = r+1$. By the inductive hypothesis, we have

$$\max\{1, m-r\} \le \dim[K](f^{(r)}) \le m,$$

which implies

$$m - r \le s \le m,$$

where $s = \dim[K](f^{(r)})$. If $\dim[K](f^{(r+1)}) = t$, then $t \in \{s, s-1\}$, which implies the inequalities

$$m - (r+1) \le s - 1 \le t \le s \le m.$$

It follows that

$$\max\{1, m-(r+1)\} \le \dim[K](f^{(r+1)}) \le m.$$

By mathematical induction, the statement holds for $\forall k \in \mathbb{N}^*$. □

Theorem 2.10 *For* $\forall f \in Rolf[K]$*, the equality* $\dim[K](\overleftarrow{f}) = \dim[K](f)$ *holds.*

Proof. Consider an arbitrary function $f \in Rolf[K]$ with $\dim[K](f) = m$**.** We have $f \in Rolf[K][m]$, which implies $\overleftarrow{f} \in Rolf[K][m]$, i.e. $\dim[K](\overleftarrow{f}) \le m$. Suppose, to the contrary, that $\dim[K](\overleftarrow{f}) \ne \dim[K](f) = m$. It follows that there exists an integer t, $t < m$, such that $\dim[K](\overleftarrow{f}) = t$, which implies $\overleftarrow{f} \in Rolf[K][t]$. We have $f = \overleftarrow{\overleftarrow{f}} \in Rolf[K][t]$, hence $\dim[K](f) \le t < m$, i.e. a contradiction. Therefore,

$$\dim[K](\overleftarrow{f}) = \dim[K](f).$$ □

Example 2.1 *Consider the function* $f(x) = x2^x \sin^3 x + x^2 3^x \cos^3 x$*. Prove that this function is a homogeneous linear recurrence over* $\mathbb{R}$ *and estimate its dimension.*

Solution. Since the functions x, 2^x, $\sin x$, x^2, 3^x and $\cos x$ belong to the set $Rolf[\mathbb{R}]$ and the product and the linear combination are algebraic operations over this set, it also follows that $f(x) \in Rolf[\mathbb{R}]$. Let us estimate the dimension of the function $f(x)$ over the set $\mathbb{R}$. We have:

$$1 \le \dim[\mathbb{R}](f) \le \dim[\mathbb{R}](x2^x \sin^3 x) + \dim[\mathbb{R}](x^2 3^x \cos^3 x) \le$$

$$\le \dim[\mathbb{R}](x)\dim[\mathbb{R}](2^x)(\dim[\mathbb{R}](\sin x))^3 +$$

$$+ \dim[\mathbb{R}](x^2)\dim[\mathbb{R}](3^x)(\dim[\mathbb{R}](\cos x))^3 \le$$

$$\le 2 \cdot 1 \cdot 2^3 + 3 \cdot 1 \cdot 2^3 = 40.$$ □

Example 2.2 *Consider the function* $f(x) = 2^x((x+1)\sin x + x\cos x)$*. Prove that it is a homogeneous linear recurrence over* $\mathbb{R}$ *and its dimension is a prime number or a perfect square.*

Solution. Since the functions x, $x+1$, 2^x, $\sin x$ and $\cos x$ belong to the set $Rolf[\mathbb{R}]$ and the product and the linear combination are algebraic operations over this set, it

also follows that $f(x) \in Rolf[\mathbb{R}]$. Let us estimate the dimension of the function $f(x)$ over the set $\mathbb{R}$. Since $f(x) = (2^x x \sin x)'$, we have

$$1 \le \dim[\mathbb{R}](f) \le \dim[\mathbb{R}](2^x x \sin x) \le$$

$$\le \dim[\mathbb{R}](2^x) \dim[\mathbb{R}](x) \dim[\mathbb{R}](\sin x) \le 1 \cdot 2 \cdot 2 = 4,$$

Therefore, $\dim[\mathbb{R}](f)$ is a prime number or a perfect square. □

2.2 Minimal generating vectors. The homogeneous linear recurrence criterion

In this section, important results concerning the minimality of homogeneous linear recurrences are presented, such as the uniqueness of the minimal generating vector, the structure of the set $H[K](a)$ and the homogeneous linear recurrence criterion over a subfield of $\mathbb{C}$. The results are theoretically justified.

The theorem regarding the uniqueness of the minimal generating vector is:

Theorem 2.11 *If $a \in Rol[K]$ and it is not identically zero, then*

$$\left|G[K][\dim[K](a)](a)\right| = 1.$$

Proof. Let the sequence $a \in Rol[K]$ be not identically zero and let $\dim[K](a) = m$. We have $a \in Rol[K][m]$, which implies $\left|G[K][m](a)\right| \ge 1$. There are two cases.

Case $m = 1$. Let $q = (q_0) \in G[K][m](a)$. If $a_0 = 0$, then

$$a_n = q_0 a_{n-1} = \ldots = q_0^n a_0 = 0, \ \forall n \in \mathbb{N},$$

which implies that a is identically equal to zero, i.e. it contradicts the initial hypothesis. Therefore, $a_0 \ne 0$ and, since $q_0 \ne 0$, it follows that $a_n = q_0^n a_0 \ne 0$, $\forall n \in \mathbb{N}$. Consequently, we have $q_0 = \dfrac{a_{n+1}}{a_n}$, $\forall n \in \mathbb{N}$. Thus, $q_0 = \dfrac{a_1}{a_0}$, i.e. $q = \left(\dfrac{a_1}{a_0}\right)$ is the unique generating vector of the sequence a, hence $\left|G[K][m](a)\right| = 1$.

Case $m \ge 2$. Suppose, to the contrary, that $\left|G[K][m](a)\right| \ge 2$, i.e. there exist

$p, r \in K^{m-1} \times K^*$, $p \neq r$, such that

$$a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k} = \sum_{k=0}^{m-1} r_k a_{n-1-k}\ ,\ \forall n \geq m\ .$$

It follows that

$$\sum_{k=0}^{m-1} q_k a_{n-1-k} = 0\ ,\ \forall n \geq m\ ,$$

where $q_k = p_k - r_k$, $k = \overline{0, m-1}$. Since $p \neq r$, it follows that $\exists j \in \{0,1,\ldots,m-1\}$ such that $q_j \neq 0$ and $q_k = 0$, $k = \overline{0, j-1}$. Therefore,

$$q_j a_{n-1-j} + \sum_{k=j+1}^{m-1} q_k a_{n-1-k} = 0, \forall n \geq m\ ,$$

equivalent to the recurrence formula

$$a_{n-1-j} = \sum_{k=j+1}^{m-1} \left(-\frac{q_k}{q_j} \right) a_{n-1-k}, \forall n \geq m\ .$$

We obtain the recurrence relation

$$a_n = \sum_{k=j+1}^{m-1} \left(-\frac{q_k}{q_j} \right) a_{n+j-k} = \sum_{k=0}^{m-j-2} \left(-\frac{q_{k+j+1}}{q_j} \right) a_{n-1-k}\ ,\ \forall n \geq m-1-j\ .$$

If $q_{k+j+1} = 0$, $k = \overline{0, m-j-2}$, then a has a finite rank, which is a contradiction.

Consider that $\exists t \in \{0,1,\ldots,m-j-2\}$ such that

$$q_{t+j+1} \neq 0\ ,\ q_{k+j+1} = 0\ ,\ k = \overline{t+1,\ m-j-2}\ .$$

It follows that

$$a_n = \sum_{k=0}^{t} \left(-\frac{q_{k+j+1}}{q_j} \right) a_{n-1-k}\ ,\ \forall n \geq m-1-j \geq t+1\ .$$

From Corollary 1.4, we have $a \in Rol[K][t+1]$, which implies the contradiction:

$$\dim[K](a) \leq t+1 \leq m-1-j \leq m-1 < m\ .$$

Therefore, the minimal generating vector is unique, i.e. $\left| G[K][m](a) \right| = 1$. □

Lemma 2.2 *Assume that the sequence* $a \in Rol[K][m]$ *is not identically zero. Let* $P(z), Q(z) \in H[K][m](a)$ *and* $P(0) \neq Q(0)$. *Then* $T(z) = P(z) - Q(z) \in H[K](a)$.

Proof. Consider the polynomials

$$P(z) = H_{m,\alpha}^{[p]}(z) = \alpha(1 - z\sum_{k=0}^{m-1} p_k z^k),$$

$$Q(z) = H_{m,\beta}^{[q]}(z) = \beta(1 - z\sum_{k=0}^{m-1} q_k z^k).$$

We have

$$T(z) = P(z) - Q(z) = (\alpha - \beta)(1 - z\sum_{k=0}^{m-1} w_k z^k),$$

where

$$w_k = \frac{\alpha p_k - \beta q_k}{\alpha - \beta},\ k = \overline{0, m-1}.$$

When $w_k = 0,\ k = \overline{0, m-1}$, we obtain

$$q_k = \frac{\alpha}{\beta} p_k,\ k = \overline{0, m-1}$$

and, therefore, the relation

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} = \frac{\alpha}{\beta} \sum_{k=0}^{m-1} p_k a_{n-1-k} = \frac{\alpha}{\beta} a_n,\ \forall n \geq m,$$

holds, which implies

$$\left(1 - \frac{\alpha}{\beta}\right) a_n = 0,\ \forall n \geq m.$$

Since $\alpha = P(0) \neq Q(0) = \beta$, we have $a_n = 0,\ \forall n \geq m$, hence the sequence a has a finite rank, i.e. $a \notin Rol[K]$, which contradicts the hypothesis. Therefore, there exists $r \in \{1, 2, \ldots, m\}$ such that $w_{r-1} \neq 0$ and $w_k = 0,\ \forall k \geq r$. Consequently, we have

$$T(z) = (\alpha - \beta)(1 - z\sum_{k=0}^{r-1} w_k z^k) \in K[z].$$

Further, we obtain

$$\sum_{k=0}^{r-1} w_k a_{n-1-k} = \sum_{k=0}^{m-1} w_k a_{n-1-k} = \sum_{k=0}^{m-1} \frac{\alpha p_k - \beta q_k}{\alpha - \beta} a_{n-1-k} =$$

$$= \frac{\alpha}{\alpha - \beta} \sum_{k=0}^{m-1} p_k a_{n-1-k} - \frac{\beta}{\alpha - \beta} \sum_{k=0}^{m-1} q_k a_{n-1-k} = \frac{\alpha}{\alpha - \beta} a_n - \frac{\beta}{\alpha - \beta} a_n = a_n, \ \forall n \geq m .$$

In this way, we obtained $a \in Rol[K]$ and

$$a_n = \sum_{k=0}^{r-1} w_k a_{n-1-k}, \ \forall n \geq m \geq r,$$

where $w \in K^{m-1} \times K^*$. By applying Corollary 1.4, we have $w \in G[K][r](a)$, which implies $T(z) \in H[K](a)$. □

Lemma 2.3 *Let the sequence* $a \in Rol[K][t]$ *be not identically zero,* $\dim[K](a) = m$, $P_t(z) \in H[K][t](a)$ *and* $P_m(z) \in H[K][m](a)$. *Then* $P_t(z) \vdots P_m(z)$.

Proof. If $a \in Rol[K][t]$ and $\dim[K](a) = m$, then $t > m$. Consider the polynomials

$$Q_t(z) = \frac{P_t(z)}{P_t(0)}, \ Q_m(z) = \frac{P_m(z)}{P_m(0)}.$$

We have

$$Q_t(0) = Q_m(0) = 1,$$

$$Q_t(z) \in H[K][t](a), \ Q_m(z) \in H[K][m](a).$$

Also, we consider the decomposition

$$Q_t(z) = Q_m(z)T(z) + R(z),$$

where

$$\deg(R(z)) < \deg(Q_m(z)) = m < t .$$

Let us prove that $R(z) = 0$. Suppose, to the contrary, that $R(z) \neq 0$. Consider that $Q(z) = Q_m(z)T(z)$. We have $R(z) = Q_t(z) - Q(z)$. If $T(0) \notin \{0,1\}$, then

$$Q(0) = Q_m(0)T(0) = T(0) \neq 1 = Q_t(0).$$

Since $Q(z) \vdots Q_m(z)$ and $Q(z) \in K[z]$, we have $Q(z) \in H[K](a)$. From the equality

$$\deg(Q(z)) = \deg(Q_t(z) - R(z)) = t,$$

we obtain $Q_t(z), Q(z) \in H[K][t](a)$ and $Q(0) \neq Q_t(0)$, which implies

$$R(z) = Q_t(z) - Q(z) \in H[K](a),$$

i.e. $\dim[K](a) \le \deg(R(z)) < m$, which is a contradiction.

Therefore, $T(0) \in \{0,1\}$. We have

$$Q_t(z) = U(z) + V(z),$$

where

$$U(z) = 2Q_m(z) + R(z),\ V(z) = Q_m(z)(T(z) - 2).$$

Next, we obtain $V(0) = T(0) - 2 \in \{-2,-1\}$, i.e. $V(0) \neq 0$ and $V(0) \neq 1 = Q_t(0)$. Since $V(z) \vdots Q_m(z)$ and $V(z) \in K[z]$, it follows that $V(z) \in H[K](a)$. Consequently, since $\deg(V(z)) = t$, we have $Q_t(z), V(z) \in H[K][t](a)$ and $V(0) \neq Q_t(0)$, which implies $U(z) = Q_t(z) - V(z) \in H[K](a)$. In this way, since $\deg(U(z)) = m = \dim[K](a)$, we obtain $U(z) \in H[K][\dim[K](a)](a)$. As a result, we have $U(z) = \alpha Q_m(z)$, where $\alpha \in \mathbb{C}$, i.e. $R(z) = (\alpha - 2)Q_m(z)$. Since $\deg(R(z)) < m$ and $\deg(Q_m(z)) = m$, it follows that $\alpha = 2$, i.e. $R(z) = 0$, which contradicts our hypothesis.

Therefore, $R(z) = 0$, i.e. $Q_t(z) \vdots Q_m(z)$. Consequently, since

$$P_t(z) = P_t(0)Q_t(z),\ P_m(z) = P_m(0)Q_m(z),$$

we can conclude that $P_t(z) \vdots P_m(z)$. □

The structure of the set $H[K](a)$ is described by the following theorem:

Theorem 2.12 *Assume that the linear recurrence $a \in Rol[K][t]$ is not identically zero and $P(z) \in H[K][\dim[K](a)](a)$. Then the set of characteristic polynomials is*

$$H[K](a) = \{Q(z) \in K[z] \mid Q(z) \vdots P(z)\}.$$

Proof. According to the results obtained above, the formula $Q(z) \in H[K](a)$ is equivalent to the relations $Q(z) \in K[z]$ and $Q(z) \vdots P(z)$, i.e.

$$H[K](a) = \{Q(z) \in K[z] \mid Q(z) \vdots P(z)\}.$$ □

The following theorem provides us the necessary and sufficient conditions for a sequence to be a homogeneous linear recurrence over a subfield K of $\mathbb{C}$.

Theorem 2.13 *Let* $a \in Rol[\mathbb{C}]$ *and* $P(z) \in H[\mathbb{C}][\dim[\mathbb{C}](a)](a)$. *Then* $a \in Rol[K]$ *if and only if all the roots of the polynomial* $P(z)$ *are algebraic elements over* K. *If*

$$a \in Rol[K],\ Q(z) \in H[K][\dim[K](a)](a),$$

then $Q(z)$ *is a polynomial of minimal degree in* $K[z]$, *which is divisible by* $P(z)$.

Proof. Let $\dim[\mathbb{C}](a) = m$ and $P(z) \in H[\mathbb{C}][m](a)$. We consider that $z_0, z_1, \ldots, z_{p-1}$ are all the distinct roots of $P(z)$, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-1}$.

Next, we consider $a \in Rol[K]$ with $\dim[K](a) = M$. Consequently, we have $a \in Rol[K][M]$. Assume that

$$Q(z) \in H[K][M](a) \subseteq H[\mathbb{C}][M](a).$$

We have $Q(z) \vdots P(z)$, hence

$$Q(z) \vdots (z - z_k),\ k = \overline{0, p-1}.$$

Suppose, to the contrary, that, $\exists z_k$ such that $P(z_k) = 0$ and z_k is transcendental over the set K. It follows that every polynomial, which is divisible by $(z - z_k)$, does not belong to $K[z]$, which implies $Q(z) \notin K[z]$, i.e. it is a contradiction. Therefore, all the roots of the polynomial $P(z)$ are algebraic over the set K. Since

$$Q(z) \vdots P(z),\ Q(z) \in K[z],\ \deg(Q(z)) = \dim[K](a),$$

it follows that $Q(z) \in K[z]$ is a polynomial of minimal degree, divisible by $P(z)$.

Let us prove the converse statement. Consider that z_k is an algebraic element over K, $k = \overline{0, p-1}$. Then, there exists $P_k(z) \in K[z]$, such that

$$P_k(z_k) = 0,\ k = \overline{0, p-1}.$$

We have

$$P_k(z) \vdots (z - z_k),\ k = \overline{0, p-1},$$

which implies $P_k^{s_k}(z) \in K[z]$ and

$$P_k^{s_k}(z) \vdots (z - z_k)^{s_k},\ k = \overline{0, p-1}.$$

Therefore, the polynomial

$$Q(z)=\prod_{k=0}^{p-1}P_k^{s_k}(z)\in K[z]$$

satisfies the property $Q(z)\vdots P(z)$, where $P(z)=\prod_{k=0}^{p-1}(z-z_k)^{s_k}$. In conclusion,

$$a\in Rol[K],\ Q(z)\in H[K](a).$$ □

Example 2.3 *Consider the function* $f(x)=i^x+e^x$. *Is it a homogeneous linear recurrence over* $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ *and* $\mathbb{N}$*? If so, determine the dimension over each set.*

Solution. We can write the function $f(x)$ in the form $f(x)=f_1(x)+f_2(x)$, where

$$f_1(x)=i^x,\ f_2(x)=e^x.$$

Since

$$f_1,f_2\in Rolf[\mathbb{C}][1],\ 1-iz\in H[\mathbb{C}][1](f_1),\ 1-ez\in H[\mathbb{C}][1](f_2),$$

it follows that

$$f\in Rolf[\mathbb{C}][2],\ (1-iz)(1-ez)\in H[\mathbb{C}][2](f).$$

The roots of the characteristic polynomial $P(z)=(1-iz)(1-ez)$ are

$$z_0=i^{-1}=-i,\ z_1=e^{-1}.$$

We have $\dim[\mathbb{C}](f)=2$, since $1\le\dim[\mathbb{C}](f)\le 2$ and $f\notin Rolf[\mathbb{C}][1]$ (otherwise, we would have the equality $\frac{i^2+e^2}{i+e}=\frac{i+e}{i^0+e^0}$, equivalent to $e^2-1=2ie$, which is false).

Moreover, by applying Theorem 2.13, we have

$$f\in Rolf[\mathbb{R}],\ \dim[\mathbb{R}](f)=3,$$

$$Q(z)=(1-iz)(1+iz)(1-ez)=(1+z^2)(1-ez)\in H[\mathbb{R}][3](f).$$

But, since $z_1=e^{-1}$ is a transcendental element over $\mathbb{Q}$, we obtain $f\notin Rolf[\mathbb{Q}]$.

Also, since $\mathbb{Z}\subseteq\mathbb{Q}$ and $\mathbb{N}\subseteq\mathbb{Q}$, we have $f\notin Rolf[\mathbb{Z}]$ and $f\notin Rolf[\mathbb{N}]$. □

2.3 Order minimization method by eliminating the characteristic zeros

With the introduction of the dimension of a homogeneous linear recurrence over a set, the problem of order minimization naturally arises. This problem consists in determining the dimension and the minimal generating vector of the given homogeneous linear recurrence. Next, we present two order minimization methods. The first method is based on the elimination of characteristic zeros, while the second is a rank-based approach. We assume that the set K is a subfield of $\mathbb{C}$.

Lemma 2.4 *Let* $a \in Rol[K][m]$ *and* $P(z) \in H[K][m](a)$, *where* $m \geq 2$. *Assume that the last term of the vector* $x = I_m^{[a]}((B^{[a]})^T)^{-1}$ *is equal to zero. If* $z_0, z_1, \ldots, z_{p-1}$ *are all the distinct roots of the polynomial* $P(z)$, *with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$, *and* $z_{p-1} \in K$, *then* $a \in Rol[K][m-1]$ *and*

$$Q(z) = \frac{P(z)}{z - z_{p-1}} \in H[K][m-1](a).$$

Proof. Let $x = (A_{k,j})_{\substack{k=\overline{0,\,p-1,}\\ j=0,\,s_k-1}}$ and $A_{p-1,\,s_{p-1}-1} = 0$. Denote by x^* the vector obtained from x by eliminating the last term. Similarly, we have

- $I_{m-1}^{[a]} = (a_0, a_1, \ldots, a_{m-2})$;
- B^* − the matrix obtained from $B^{[a]}$ by eliminating the last row and column;
- β_n^* − the vector obtained from $\beta_n^{[a]}$ by eliminating the last term.

We have

$$B^* = ((\beta_k^*)_{k=0}^{m-2})^T,\ x^* = I_{m-1}^{[a]}((B^*)^T)^{-1}.$$

It follows that

$$a_n = I_{m-1}^{[a]}((B^*)^T)^{-1}(\beta_n^*)^T,\ \forall n \in N,$$

which implies $a \in Rol[K][m-1]$. The solutions of the characteristic equation are $z_0, z_1, \ldots, z_{p-2}, z_{p-1}$, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-2}, s_{p-1} - 1$.

Since $z_{p-1} \in K$, we can conclude that $Q(z) \in H[K][m-1](a)$. □

Lemma 2.5 *Let* $a \in Rol[K][m]$ *and* $P(z) \in H[K][m](a)$. *Use the following notation:*

- $x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{\substack{k=\overline{0,\,p-1,} \\ j=0,\,s_k-1}}$;
- t_k – *the number of trailing zeros of the vector* $(A_{k,j})_{j=\overline{0,\,s_k-1}}$, $k = \overline{0, p-1}$;
- $t = \sum\limits_{k:\ z_k \in K} t_k$;

where $z_0, z_1, \ldots, z_{p-1}$ *are all the distinct roots of the polynomial* $P(z)$, *with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$. *Then*

$$a \in Rol[K][m-t]$$

with the characteristic polynomial

$$Q(z) = \frac{P(z)}{\prod\limits_{k:\ z_k \in K} (z - z_k)^{t_k}} \in H[K][m-t](a).$$

Proof. Since the roots z_k, $k = \overline{0, p-1}$, can be reordered, it follows that we can move the vector $(A_{k,j})_{j=\overline{0,\,s_k-1}}$ at the end of the vector x by interchanging the blocks k and $(p-1)$ in the vector $\beta_n^{[a]}$, performing analogous transformations in the matrix $B^{[a]}$. Applying Lemma 2.4, we eliminate the trailing zeros of the vector x and obtain that, if $z_k \in K$, then $a \in Rol[K][m-t_k]$ with characteristic polynomial

$$\frac{P(z)}{(z - z_k)^{t_k}} \in H[K][m-t_k](a).$$

Performing this operation for $k = \overline{0, p-1}$, we obtain

$$a \in Rol[K][m-t],\ Q(z) \in H[K][m-t](a).$$ □

Lemma 2.6 *If* $a \in Rol[\mathbb{C}][m]$ *is not identically zero,*

$$x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,\,j})_{\substack{k=\overline{0,\,p-1} \\ j=0,\,s_k-1}}$$

and $\dim[\mathbb{C}](a) \neq m$, *then there exists* k, $0 \leq k \leq p-1$, *such that* $A_{k,s_k-1} = 0$.

Proof. Since $\dim[\mathbb{C}](a) \neq m$, there exists $t < m$ such that $\dim[\mathbb{C}](a) = t$. Let

$$P_t(z) \in H[\mathbb{C}][t](a),\ P_m(z) \in H[\mathbb{C}][m](a).$$

It follows that $P_m(z) \vdots P_t(z)$, hence $P_m(z) = P_t(z)T(z)$, where $\deg(T(z)) \geq 1$. We consider a root z_k of the polynomial $T(z)$, which implies that it is also a root of $P_m(z)$. Without loss of generality, we assume that z_{p-1} is that root, i.e. $T(z_{p-1}) = 0$.

Let x^*, $I_{m-1}^{[a]}$ and β_n^* be the vectors obtained from the vectors x, $I_m^{[a]}$ and $\beta_n^{[a]}$, correspondingly, by eliminating the last term. Also, let B^* be the matrix obtained from the matrix $B^{[a]}$ by eliminating the last row and the last column. Since $a \in Rol[\mathbb{C}][m]$ and $\dim[\mathbb{C}](a) \neq m$, it follows that $a \in Rol[\mathbb{C}][m-1]$. Therefore, since $\dfrac{P_m(z)}{z - z_{p-1}} \vdots P_t(z)$, we obtain

$$\frac{P_m(z)}{z - z_{p-1}} \in H[\mathbb{C}][m-1](a),$$

i.e. the equality

$$a_n = I_{m-1}^{[a]}((B^*)^T)^{-1}(\beta_n^*)^T,\ \forall n \geq m-1,$$

holds. On the other hand, we have the equality

$$a_n = I_m^{[a]}((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T,\ \forall n \geq m,$$

which implies

$$I_m^{[a]}((B^{[a]})^T)^{-1}(\beta_n^{[a]})^T = I_{m-1}^{[a]}((B^*)^T)^{-1}(\beta_n^*)^T,\ \forall n \geq m,$$

and, consequently,

$$x(\beta_n^{[a]})^T = x^*(\beta_n^*)^T,\ \forall n \geq m,$$

equivalent to

$$\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,\,j}\beta_{n,\,j}^{[a]\,(k)} = \sum_{k=0}^{p-2}\sum_{j=0}^{s_k-1} A_{k,\,j}^*\beta_{n,\,j}^{*(k)} + \sum_{j=0}^{s_{p-1}-2} A_{p-1,\,j}^*\beta_{n,\,j}^{*(p-1)} =$$

$$= \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{k,\,j} \beta_{n,\,j}^{[a]\,(k)} - A_{p-1,\,s_{p-1}-1} \beta_{n,\,s_{p-1}-1}^{[a]\,(p-1)}, \ \forall n \geq m .$$

We obtain

$$A_{p-1,s_{p-1}-1} \beta_{n,s_{p-1}-1}^{[a]\,(p-1)} = 0, \ \forall n \geq m$$

and, since

$$\beta_{n,\,j}^{[a]\,(p-1)} = \frac{n^j}{z_{p-1}^n} \neq 0, \ \forall n \geq m,$$

we have $A_{p-1,\,s_{p-1}-1} = 0$. Generally, this may be a trailing zero for a subvector $(A_{k,\,j})_{j=\overline{0,\,s_k-1}}$, $k \in \{0,1,\ldots,p-1\}$, i.e. $\exists k$, $0 \leq k \leq p-1$, such that $A_{k,s_k-1} = 0$. □

The order minimization method by eliminating the characteristic zeros, over $\mathbb{C}$, is given by the following lemma:

Lemma 2.7 *Let $a \in Rol[\mathbb{C}][m]$ be a sequence which is not identically zero. Use the following notation:*

- $x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,\,j})_{\substack{k=\overline{0,\,p-1} \\ j=\overline{0,\,s_k-1}}}$;
- t_k – *the number of trailing zeros of the vector* $(A_{k,\,j})_{j=\overline{0,\,s_k-1}}$, $k = \overline{0, p-1}$;
- $t = \sum_{k=0}^{p-1} t_k$.

Then $\dim[\mathbb{C}](a) = m - t$ *and*

$$Q(z) = \frac{P(z)}{\prod_{k=0}^{p-1} (z - z_k)^{t_k}} \in H[\mathbb{C}][m-t](a),$$

where z_k, $k = \overline{0, p-1}$, are all distinct roots of the polynomial $P(z) \in H[\mathbb{C}][m](a)$.

Proof. From Lemma 2.5, we have $\dim[\mathbb{C}](a) \leq m - t$. By eliminating all corresponding characteristic zeros, we obtain

$$a \in Rol[\mathbb{C}][m-t], \ Q(z) \in H[\mathbb{C}][m-t](a).$$

Let $z_0^*, z_1^*, \ldots, z_{p^*-1}^*$ be all the distinct roots, with corresponding multiplicities $s_0^*, s_1^*, \ldots, s_{p^*-1}^*$, of the polynomial $Q(z)$, $\sum_{k=0}^{p^*-1} s_k^* = m-t$. We have

$$A_{k,\, s_k^*-1}^* \neq 0, \quad k = \overline{0, p^*-1}.$$

If $\dim[\mathbb{C}](a) \neq m-t$, then, according to Lemma 2.5, there exists k such that $0 \leq k \leq p^*-1$ and $A_{k,s_k^*-1}^* = 0$, which is a contradiction. Thus, $\dim[\mathbb{C}](a) = m-t$ and

$$Q(z) \in H[\mathbb{C}][m-t](a). \qquad \square$$

The order minimization method by eliminating the characteristic zeros, over a subfield K of $\mathbb{C}$, is given by the following theorem:

Theorem 2.14 *Consider the sequence* $a \in Rol[K][m]$ *which is not identically zero and* $x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,\,j})_{\substack{k=\overline{0,\,p-1} \\ j=\overline{0,\,s_k-1}}}$. *Let* z_k, $k = \overline{0, p-1}$, *be all distinct roots of the polynomial* $P(z) \in H[K][m](a)$, t_k – *the number of trailing zeros of the vector* $(A_{k,j})_{j=\overline{0,s_k-1}}$, $k = \overline{0, p-1}$, $t = \sum_{k:\, z_k \in K} t_k$ *and* $T(z) = \prod_{k:\, z_k \notin K} (z-z_k)^{t_k}$. *Let* $R(z) \in K[z]$ *be the polynomial of maximal degree which satisfies the relation* $T(z) \vdots R(z)$. *Then* $\dim[K](a) = m-t-r$ *and a minimal characteristic polynomial is*

$$S(z) = \frac{P(z)}{R(z) \cdot \prod_{k:z_k \in K} (z-z_k)^{t_k}} \in H[K][m-t-r](a),$$

where $r = \deg(R(z))$.

Proof. Consider the polynomial

$$T(z) = \prod_{k:z_k \notin K} (z-z_k)^{t_k}$$

and $\deg(P(z)) = v$. We have

$$\sum_{k:z_k \notin K} t_k = v$$

and the number of trailing zeros corresponding to the roots z_k, $k=\overline{0,p-1}$, is

$$\sum_{k=0}^{p-1} t_k = \sum_{k:\, z_k \notin K} t_k + \sum_{k:\, z_k \in K} t_k = v + t\,.$$

Let $R(z)$ be a polynomial of maximal degree in $K[z]$, which satisfies $T(z) \vdots R(z)$. We have that any nonzero-degree divisor of the polynomial $Q(z) = \dfrac{T(z)}{R(z)}$ does not belong to the set $K[z]$ and $\deg(Q(z)) = v - r$.

Since $a \in Rol[K][m]$, it follows that $a \in Rol[\mathbb{C}][m]$. By applying Lemma 2.5, we eliminate all trailing characteristic zeros corresponding to the roots $z_k \in K$ and those which correspond to the roots of the polynomial $R(z)$. We obtain $\dim[K](a) \le m - t - r$ and $S(z) \in H[K][m-t-r](a)$. By further eliminating the remaining trailing characteristic zeros corresponding to $Q(z)$, from Lemma 2.7 we obtain $\dim[\mathbb{C}](a) = m - t - v$ and the minimal polynomial $W(z)$ (over $\mathbb{C}$).

Let $\dim[K](a) = M$ and $V(z) \in H[K][M](a)$. It follows that

$$S(z) \vdots V(z) \vdots W(z)$$

and, since $S(z), V(z) \in K[z]$, we have $U(z) = \dfrac{S(z)}{V(z)} \in K[z]$. Therefore, we obtain:

$$S(z) = W(z)Q(z) \;\Rightarrow\; U(z) = \frac{W(z)}{V(z)}Q(z) = \frac{Q(z)}{\dfrac{V(z)}{W(z)}} \;\Rightarrow$$

$$\Rightarrow \frac{Q(z)}{U(z)} = \frac{V(z)}{W(z)} \in \mathbb{C}[z] \;\Rightarrow\; Q(z) \vdots U(z),\; U(z) \in K[z] \;\Rightarrow$$

$$\Rightarrow \deg(U(z)) = 0 \;\Rightarrow\; U(z) = \mu = const \;\Rightarrow$$

$$\Rightarrow S(z) = \mu V(z) \;\Rightarrow\; M = \deg(V(z)) = \deg(S(z)) = m - t - r\,.$$

In conclusion, $\dim[K](a) = m - t - r$ and $S(z) \in H[K][m-t-r](a)$. □

Example 2.4 *Minimize the order of the sequence* $a = (a_n)_{n=0}^{\infty}$, *defined by the recurrence* $a_n = 2a_{n-1} - a_{n-3}$, $\forall n \ge 3$ *and the initial conditions* $a_0 = 0$, $a_1 = a_2 = 1$.

Solution. Consider the sequence $a=(a_n)_{n=0}^{\infty}$. By applying the order minimization method by elimination of the characteristic zeros, we have:

$$I_3^{[a]}=(0,1,1),\ q=(2,0,-1)\in G[\mathbb{C}][3](a)\ \Rightarrow$$

$$\Rightarrow\ G_3^{[q]}(z)=2-z^2\ \Rightarrow\ H_3^{[q]}(z)=z^3-2z+1\ \Rightarrow$$

$$\Rightarrow\ z_0=(-1-\sqrt{5})/2,\ z_1=(-1+\sqrt{5})/2 \text{ and } z_2=1\ \Rightarrow$$

$$\Rightarrow\ \beta_n^{[a]}=\left(\frac{n^j}{z_k^n}\right)_{\substack{k=\overline{0,2}\\ j=0,0}}=\left(\frac{1}{z_0^n},\frac{1}{z_1^n},1\right),\ \forall n\in N\ \Rightarrow$$

$$\Rightarrow\ B^{[a]}=\begin{pmatrix}\beta_0^{[a]}\\ \beta_1^{[a]}\\ \beta_2^{[a]}\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ z_0^{-1} & z_1^{-1} & 1\\ z_0^{-2} & z_1^{-2} & 1\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ x=I_3^{[a]}((B^{[a]})^T)^{-1}=\left(-\frac{1}{\sqrt{5}z_1},\frac{1}{\sqrt{5}z_0},0\right).$$

By eliminating the trailing zeros of each block, we have

$$x^*=\left(-\frac{1}{\sqrt{5}z_1},\frac{1}{\sqrt{5}z_0}\right).$$

Therefore, $\dim[\mathbb{C}](a)=2$ and

$$S(z)=\frac{H_3^{[q]}(z)}{z-z_2}=(-1)(1-z(1+z))\in H[\mathbb{C}][2](a).$$

In conclusion, $q_{\min}=(1,1)\in G[\mathbb{C}][2](a)$. □

2.4 Rank-based order minimization method

Further, we present the rank-based method for solving the order minimization problem. For the numerical sequence $a=(a_n)_{n=0}^{\infty}$ we denote $A_n^{[a]}=(a_{i+j})_{i,j=\overline{0,n-1}}$ and $\Delta_n^{[a]}=\left|A_n^{[a]}\right|$ for every positive integer n. We will provide a theoretical founda-

tion for this method step by step, in the same manner as in the case of the minimization method described in Section 2.3.

Lemma 2.8 *If* $a \in Rol[\mathbb{C}][m]$, *then* $\Delta_k^{[a]} = 0,\ \forall k > m$.

Proof. Let $a \in Rol[\mathbb{C}][m]$ and $q \in G[\mathbb{C}][m](a)$. It follows that

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

which implies the recurrence relation

$$a_{i+m} = \sum_{k=0}^{m-1} q_k a_{i+m-1-k},\ \forall i \geq 0.$$

In particular, we have the recurrence formula

$$a_{i+m} = \sum_{k=0}^{m-1} q_k a_{i+m-1-k},\ i = \overline{0,m},$$

i.e. the last column of the matrix $A_{m+1}^{[a]}$ is the linear combination of the other columns, hence $\Delta_{m+1}^{[a]} = \left| A_{m+1}^{[a]} \right| = 0$. By applying TROR, we have

$$a \in Rol[\mathbb{C}][k],\ \forall k \geq m,$$

which implies $\Delta_k^{[a]} = 0,\ \forall k > m$. □

Lemma 2.9 *If* $a \in Rol[\mathbb{C}]$ *and* $\Delta_m^{[a]} \neq 0$, *then* $\dim[\mathbb{C}](a) \geq m$.

Proof. Consider that $\dim[\mathbb{C}](a) < m$. Then $\dim[\mathbb{C}](a) \leq m-1$, i.e. $a \in Rol[\mathbb{C}][m-1]$. From Lemma 2.8, we have the contradiction $\Delta_m^{[a]} = 0$. Thus, $\dim[\mathbb{C}](a) \geq m$. □

Lemma 2.10 *Consider a sequence* $a \in Rol[\mathbb{C}][m]$, *which is not identically zero. If* $\Delta_m^{[a]} = 0$, *then* $a \in Rol[\mathbb{C}][m-1]$.

Proof. Let $q \in G[\mathbb{C}][m](a)$. This means that

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m.$$

In particular, we have

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}\ ,\ n = \overline{m, 2m-1}.$$

It follows that the system

$$\sum_{k=0}^{m-1} x_k a_{n-1-k} = a_n\ ,\ n = \overline{m, 2m-1}, \tag{2.1}$$

has solutions. Since

$$\Delta_m^{[a]} = \begin{vmatrix} a_0 & a_1 & \dots & a_{m-1} \\ a_1 & a_2 & \dots & a_m \\ \dots & \dots & \dots & \dots \\ a_{m-1} & a_m & \dots & a_{2m-2} \end{vmatrix} = 0,$$

it follows that the matrix of the system (2.1) has the determinant equal to zero, i.e. the system (2.1) has infinitely many solutions.

Consider $p \in \mathbb{C}^{m-1} \times \mathbb{C}^*$, $p \neq q$, such that

$$a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k}\ ,\ n = \overline{m, 2m-1}, \tag{2.2}$$

Let us prove the recurrence relation

$$a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k}\ ,\ \forall n \geq m\ . \tag{2.3}$$

It is true for $m \leq n \leq 2m-1$, since the vector p is a solution of the system (2.1). Let the statement be true for every n, $m \leq n < t$, i.e.

$$a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k}\ ,\ n = \overline{m, t-1}\ ,\ t \geq 2m\ .$$

We have

$$a_t = \sum_{k=0}^{m-1} q_k a_{t-1-k} = \sum_{k=0}^{m-1} q_k \sum_{j=0}^{m-1} p_j a_{t-2-k-j} = \sum_{j=0}^{m-1} p_j \sum_{k=0}^{m-1} q_k a_{t-2-j-k} = \sum_{j=0}^{m-1} p_j a_{t-1-j}\ .$$

Therefore, by mathematical induction, the recurrence relation (2.3) holds. Since $a \in Rol[\mathbb{C}][m]$ is not identically zero and $p, q \in G[\mathbb{C}][m](a)$, $p \neq q$, according to the uniqueness theorem, we obtain $a \in Rol[\mathbb{C}][m-1]$.

Let us prove that $\exists p \in \mathbb{C}^{m-1} \times \mathbb{C}^*$ such that the relation (2.2) holds, i.e. $p \neq q$ and $p_{m-1} \neq 0$. Suppose, to the contrary, that, for $\forall p \neq q$, p being a solution of the system (2.1), we have the equality $p_{m-1} = 0$. Since this system has infinitely many solutions, we can choose two different solutions ξ and η in $\mathbb{C}^m$ satisfying the relation $\xi_{m-1} = \eta_{m-1} = 0$.

Consider $p = q + \xi - \eta$. Since $\xi \neq \eta$, it follows that $p \neq q$. We have

$$p_{m-1} = q_{m-1} + \xi_{m-1} - \eta_{m-1} = q_{m-1} + 0 - 0 = q_{m-1} \neq 0$$

and for $n = \overline{m, 2m-1}$ we obtain

$$\sum_{k=0}^{m-1} p_k a_{n-1-k} = \sum_{k=0}^{m-1} (q_k + \xi_k - \eta_k) a_{n-1-k} =$$

$$= \sum_{k=0}^{m-1} q_k a_{n-1-k} + \sum_{k=0}^{m-1} \xi_k a_{n-1-k} - \sum_{k=0}^{m-1} \eta_k a_{n-1-k} = a_n .$$

Therefore, $p \neq q$, $p_{m-1} \neq 0$ and p is a solution of the system (2.1), contradiction with the hypothesis. In conclusion, we can choose p such that (2.2) holds. □

Lemma 2.11 *If the sequence $a \in Rol[\mathbb{C}]$ is not identically zero, then*

$$\dim[\mathbb{C}](a) = rank((\Delta_n^{[a]})_{n=1}^{\infty}).$$

Proof. Let $a \in Rol[\mathbb{C}][m]$ be not identically zero. Consider $r \in \mathbb{N}$ such that

$$\Delta_r^{[a]} \neq 0,\ \Delta_k^{[a]} = 0,\ k = \overline{r+1, m}.$$

From Lemma 2.8, we have $\Delta_k^{[a]} = 0$, $\forall k > m$, hence $\Delta_r^{[a]} \neq 0$ and $\Delta_k^{[a]} = 0$, $\forall k > r$, which implies that $(\Delta_n^{[a]})_{n=1}^{\infty}$ has finite rank and $rank((\Delta_n^{[a]})_{n=1}^{\infty}) = r$.

Since $a \in Rol[\mathbb{C}][m]$ and $\Delta_m^{[a]} = 0$, by applying Lemma 2.10, we have $a \in Rol[\mathbb{C}][m-1]$. If $\Delta_{m-1}^{[a]} = 0$, by Lemma 2.10 we obtain $a \in Rol[\mathbb{C}][m-2]$. Continuing in the same manner, we have $a \in Rol[\mathbb{C}][r]$, i.e. $\dim[\mathbb{C}](a) \leq r$. But, since $\Delta_r^{[a]} \neq 0$, from Lemma 2.9 we obtain $\dim[\mathbb{C}](a) \geq r$, which implies

$$\dim[\mathbb{C}](a) = r = rank((\Delta_n^{[a]})_{n=1}^{\infty}).$$ □

Lemma 2.12 *If the sequence* $a \in Rol[\mathbb{C}][m]$ *is not identically zero, then*

$$\dim[\mathbb{C}](a) = rank(A_m^{[a]}).$$

Proof. Let $R = rank(A_m^{[a]})$ and $r = rank((\Delta_n^{[a]})_{n=1}^{\infty})$. It follows that

$$\Delta_k^{[a]} = 0,\ \forall k > R,\ \Delta_r^{[a]} \neq 0 \text{ and } \Delta_k^{[a]} = 0,\ \forall k > r,$$

i.e. $r \leq R$. From Lemma 2.11, we have $\dim[\mathbb{C}](a) = r$, hence $a \in Rol[\mathbb{C}][r]$, which implies that $\exists\, q \in \mathbb{C}^{r-1} \times \mathbb{C}^*$, such that

$$a_n = \sum_{k=0}^{r-1} q_k a_{n-1-k},\ \forall n \geq r.$$

It follows that all the rows of the matrix $A_m^{[a]}$, except for the first r rows, are linear combinations of the previous r rows. Therefore, eliminating them from the bottom to the top, one by one, we obtain the matrix $A_m^{[a](m-r)}$, composed of the first r rows of the matrix $A_m^{[a]}$, which satisfies

$$rank(A_m^{[a](m-r)}) = rank(A_m^{[a]}) = R.$$

Since $A_m^{[a](m-r)}$ has only r rows, it follows that

$$rank(A_m^{[a](m-r)}) \leq r.$$

Therefore, $R \leq r$, i.e. $r = R$, which implies $\dim[\mathbb{C}](a) = rank(A_m^{[a]})$. □

The rank-based order minimization method is given by the following theorem:

Theorem 2.15 *If the linear recurrence* $a \in Rol[\mathbb{C}][m]$ *is not identically zero, then* $\dim[\mathbb{C}](a) = R = rank(A_m^{[a]})$ *and* $q = (q_0, q_1, \ldots, q_{R-1}) \in G[\mathbb{C}][R](a)$, *where the tuple* $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ *is the unique solution of the system* $A_R^{[a]} x^T = (f_R^{[a]})^T$, *with* $f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1})$ *and* $A_R^{[a]} = (a_{i+j})_{i,j=\overline{0,R-1}}$.

Proof. By applying Lemma 2.12, we have $\dim[\mathbb{C}](a) = R$. Therefore, there exists a unique generating vector $q = (q_k)_{k=0}^{R-1} \in \mathbb{C}^{R-1} \times \mathbb{C}^*$ which satisfies the relation

$$a_n = \sum_{k=0}^{R-1} q_k a_{n-1-k}\ ,\ \forall n \geq R\ .$$

It follows that

$$a_n = \sum_{k=0}^{R-1} q_k a_{n-1-k}\ ,\ n = \overline{R, 2R-1}\ .$$

If we denote $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$, we obtain the system of equalities

$$a_n = \sum_{k=0}^{R-1} x_k a_{n-R+k}\ ,\ n = \overline{R, 2R-1}\ ,$$

i.e. the vector x is a solution of the system of linear equations

$$A_R^{[a]} x^T = (f_R^{[a]})^T\ .$$

Since

$$\left| A_R^{[a]} \right| = \Delta_R^{[a]} = \Delta_r^{[a]} \neq 0\ ,$$

where $r = rank((\Delta_n^{[a]})_{n=1}^{\infty})$, it follows that x is the unique solution of the system. □

The rank-based order minimization can also be applied to homogeneous linear recurrences over a subset K of the set $\mathbb{C}$. In this case, this method is applied to determine the dimension and the minimal characteristic polynomial over the set $\mathbb{C}$. After that, the dimension and the minimal characteristic polynomial over the set K are determined according to the homogeneous linear recurrence criterion.

The rank-based order minimization method has a wider applicability than the order minimization method by eliminating characteristic zeros, since it does not require knowledge of the complex roots of the characteristic polynomial.

Moreover, the rank-based order minimization method is advantageous, as it can be easily extended to degenerate homogeneous linear recurrences.

Theorem 2.16 *If the sequence* $a \in Rol^*[\mathbb{C}][m]$ *is not identically zero, then* $\dim[\mathbb{C}](a) = R = rank(A_m^{[a]})$ *and* $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, *where the tuple*

$x=(q_{R-1},q_{R-2},\ldots,q_0)$ *is the unique solution of the system* $A_R^{[a]}x^T=(f_R^{[a]})^T$, *with* $f_R^{[a]}=(a_R,a_{R+1},\ldots,a_{2R-1})$ *and* $A_R^{[a]}=(a_{i+j})_{i,j=\overline{0,R-1}}$.

Proof. The proof is similar to that of Theorem 2.15. □

Example 2.5 *Minimize the order of the sequence* $a=(a_n)_{n=0}^{\infty}$, *defined by the recurrence* $a_n=2a_{n-1}-a_{n-3}$, $\forall n\geq 3$ *and the initial conditions* $a_0=0$, $a_1=a_2=1$.

Solution. Consider the sequence $a=(a_n)_{n=0}^{\infty}$. We apply the rank-based order minimization method. We have

$$a_0=0,\ a_1=1,\ a_2=1,\ a_3=2a_2-a_0=2,\ a_4=2a_3-a_1=3.$$

It follows that the formula

$$A_3^{[a]}=\begin{pmatrix} a_0 & a_1 & a_2\\ a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\end{pmatrix}=\begin{pmatrix} 0 & 1 & 1\\ 1 & 1 & 2\\ 1 & 2 & 3\end{pmatrix}$$

holds. Therefore,

$$\dim[\mathbb{C}](a)=rank(A_3^{[a]})=2.$$

We determine the submatrix

$$A_2^{[a]}=\begin{pmatrix} a_0 & a_1\\ a_1 & a_2\end{pmatrix}=\begin{pmatrix} 0 & 1\\ 1 & 1\end{pmatrix}$$

and the vector $f_2^{[a]}=(a_2,a_3)=(1,2)$. By solving the system

$$A_2^{[a]}x^T=(f_2^{[a]})^T,$$

we obtain the solution $x=(1,1)$, which implies that the minimal generating vector is

$$q_{\min}=(1,1)\in G[\mathbb{C}][2](a).$$

Therefore, we obtain that the sequence $a=(a_n)_{n=0}^{\infty}$ satisfies the recurrence

$$a_n=a_{n-1}+a_{n-2},\ \forall n\geq 2$$

and the initial conditions $a_0=0$ and $a_1=1$, i.e. it is the Fibonacci sequence. □

Example 2.6 *Consider the sequence* $a = (\sin(\alpha n))_{n=0}^{\infty}$, *where* $\alpha \in \mathbb{R}$. *Prove that it is a homogeneous linear recurrence over* $\mathbb{R}$ *and determine its dimension and the minimal generating vector.*

Solution. Similarly to Example 1.6, we can easily obtain the relations

$$a \in Rol[\mathbb{C}][2], \quad q = (2\cos\alpha, -1) \in G[\mathbb{C}][2](a).$$

Therefore, the inequality $\dim[\mathbb{C}](a) \le 2$ holds.

We apply the rank-based order minimization method. We note that the sequence is identically equal to zero if and only if $\alpha = 2k\pi$, $k \in \mathbb{Z}$, in which case we have $\dim[\mathbb{R}](a) = 0$ by definition. In other cases, we have

$$A_2^{[a]} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} = \begin{pmatrix} \sin 0 & \sin\alpha \\ \sin\alpha & \sin(2\alpha) \end{pmatrix} = \begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & \sin(2\alpha) \end{pmatrix} \sim \begin{pmatrix} \sin\alpha & \sin(2\alpha) \\ 0 & \sin\alpha \end{pmatrix},$$

which implies $\dim[\mathbb{C}](a) = rank(A_2^{[a]}) = 2$.

Let us solve the system

$$A_2^{[a]} x^T = (f_2^{[a]})^T,$$

which can be written in the form

$$\begin{pmatrix} 0 & \sin\alpha \\ \sin\alpha & \sin(2\alpha) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \sin(2\alpha) \\ \sin(3\alpha) \end{pmatrix}.$$

This system is equivalent to the following one:

$$\begin{cases} x_1 \sin\alpha = \sin(2\alpha) \\ x_0 \sin\alpha + x_1 \sin(2\alpha) = \sin(3\alpha) \end{cases}.$$

Further, we obtain

$$x_1 = \frac{\sin(2\alpha)}{\sin\alpha} = 2\cos\alpha$$

and

$$x_0 = \frac{\sin(3\alpha) - x_1 \sin(2\alpha)}{\sin\alpha} = \frac{\sin(2\alpha)\cos\alpha + \sin\alpha\cos(2\alpha) - 2\cos\alpha \cdot \sin(2\alpha)}{\sin\alpha} =$$

$$= \frac{\sin\alpha\cos(2\alpha) - \cos\alpha \cdot \sin(2\alpha)}{\sin\alpha} = \frac{\sin\alpha\cos(2\alpha) - 2\sin\alpha \cdot \cos^2\alpha}{\sin\alpha} =$$

$$= \cos(2\alpha) - 2\cos^2\alpha = \cos^2\alpha - \sin^2\alpha - 2\cos^2\alpha = -(\cos^2\alpha + \sin^2\alpha) = -1.$$

Therefore, $x = (-1, 2\cos\alpha)$ is the solution of the system

$$A_2^{[a]} x^T = (f_2^{[a]})^T.$$

Consequently, we obtain the minimal generating vector

$$q_{\min} = (2\cos\alpha, -1) \in G[\mathbb{C}][2](a).$$

Since $q_{\min} \in \mathbb{R}^2$ and $\mathbb{R} \subseteq \mathbb{C}$, it follows that $a \in Rol[\mathbb{R}]$, $\dim[\mathbb{R}](a) = 2$ and $q_{\min} = (2\cos\alpha, -1) \in G[\mathbb{R}][2](a)$. □

2.5 Homogeneous linear recurrence criterion for rational fractions

Lemma 2.13 *For* $\forall n \geq 1$ *and* $\forall \alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup \mathbb{N}^*$, *the following formula holds:*

$$\Delta_{n+1}(\alpha) \overset{def}{=} \left| \left(\frac{1}{i+j+\alpha} \right)_{i,j=\overline{0,n}} \right| = \frac{(n!)^2}{(\alpha+2n) \cdot \left(\prod_{i=0}^{n-1} (\alpha+n+i) \right)^2} \Delta_n(\alpha).$$

Proof. For $\forall n \geq 1$ and $\forall \alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup \mathbb{N}^*$ we multiply each row $i = \overline{0, n-1}$ of

$$\Delta_{n+1}(\alpha) \overset{def}{=} \left| \left(\frac{1}{i+j+\alpha} \right)_{i,j=\overline{0,n}} \right|$$

with the coefficient $k_i = -\dfrac{\alpha+n+i}{\alpha+2n} \neq 0$ and we obtain

$$\Delta_{n+1}(\alpha) = (-1)^n \frac{(2n+\alpha)^n}{\prod_{i=0}^{n-1} (\alpha+n+i)} \cdot \begin{vmatrix} \frac{k_0}{\alpha} & \frac{k_0}{\alpha+1} & \cdots & \frac{k_0}{\alpha+n-1} & -\frac{1}{\alpha+2n} \\ \frac{k_1}{\alpha+1} & \frac{k_1}{\alpha+2} & \cdots & \frac{k_1}{\alpha+n} & -\frac{1}{\alpha+2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{k_{n-1}}{\alpha+n-1} & \frac{k_{n-1}}{\alpha+n} & \cdots & \frac{k_{n-1}}{\alpha+2n-2} & -\frac{1}{\alpha+2n} \\ \frac{1}{\alpha+n} & \frac{1}{\alpha+n+1} & \cdots & \frac{1}{\alpha+2n-1} & \frac{1}{\alpha+2n} \end{vmatrix}.$$

By adding the last row to each row $i=\overline{0,n-1}$, we obtain

$$\Delta_{n+1}(\alpha)=(-1)^n\frac{(2n+\alpha)^n}{\prod\limits_{i=0}^{n-1}(\alpha+n+i)}\times$$

$$\times\begin{vmatrix} \frac{k_0}{\alpha}+\frac{1}{\alpha+n} & \frac{k_0}{\alpha+1}+\frac{1}{\alpha+n+1} & \dots & \frac{k_0}{\alpha+n-1}+\frac{1}{\alpha+2n-1} & 0 \\ \frac{k_1}{\alpha+1}+\frac{1}{\alpha+n} & \frac{k_1}{\alpha+2}+\frac{1}{\alpha+n+1} & \dots & \frac{k_1}{\alpha+n}+\frac{1}{\alpha+2n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{k_{n-1}}{\alpha+n-1}+\frac{1}{\alpha+n} & \frac{k_{n-1}}{\alpha+n}+\frac{1}{\alpha+n+1} & \dots & \frac{k_{n-1}}{\alpha+2n-2}+\frac{1}{\alpha+2n-1} & 0 \\ \frac{1}{\alpha+n} & \frac{1}{\alpha+n+1} & \dots & \frac{1}{\alpha+2n-1} & \frac{1}{\alpha+2n} \end{vmatrix}.$$

If we further expand the determinant $\Delta_{n+1}(\alpha)$ along the last column, we obtain

$$\Delta_{n+1}(\alpha)=(-1)^n\frac{(2n+\alpha)^{n-1}}{\prod\limits_{i=0}^{n-1}(\alpha+n+i)}\cdot\left|(b_{ij})_{i,\,j=\overline{0,n-1}}\right|,$$

where, for $i,j=\overline{0,n-1}$, the following equality holds:

$$b_{ij}=\frac{k_i}{\alpha+i+j}+\frac{1}{\alpha+n+j}=-\frac{\alpha+n+i}{(\alpha+2n)(\alpha+i+j)}+\frac{1}{\alpha+n+j}=$$

$$=-\frac{(n-i)(n-j)}{(\alpha+2n)(\alpha+n+j)(\alpha+i+j)}.$$

Next, for $i,j=\overline{0,n-1}$, we denote

$$c_{ij}=\frac{(n-i)(n-j)}{(\alpha+n+j)(\alpha+i+j)},\ d_{ij}=\frac{(n-j)}{(\alpha+n+j)(\alpha+i+j)},\ f_{ij}=\frac{1}{\alpha+i+j}$$

and we obtain

$$\left|(f_{ij})_{i,\,j=\overline{0,n-1}}\right|=\left|\left(\frac{1}{\alpha+i+j}\right)_{i,\,j=\overline{0,n-1}}\right|=\Delta_n(\alpha),$$

$$b_{ij}=-\frac{1}{\alpha+2n}c_{ij},\ c_{ij}=(n-i)d_{ij},\ d_{ij}=\frac{n-j}{\alpha+n+j}f_{ij}.$$

Multiplying all entries of this determinant with $(-2n-\alpha)\neq 0$, it follows that

$$\Delta_{n+1}(\alpha)=\frac{\left|(c_{ij})_{i,j=\overline{0,n-1}}\right|}{\prod\limits_{i=0}^{n}(\alpha+n+i)}.$$

Next, by multiplying each row $i=\overline{0,n-1}$ with $\frac{1}{n-i}\neq 0$, we obtain

$$\Delta_{n+1}(\alpha)=\frac{n!}{\prod\limits_{i=0}^{n}(\alpha+n+i)}\left|(d_{ij})_{i,j=\overline{0,n-1}}\right|.$$

In the end, multiplying each column $j=\overline{0,n-1}$ with $\frac{\alpha+n+j}{n-j}\neq 0$, we conclude that

$$\Delta_{n+1}(\alpha)=\frac{n!}{\prod\limits_{i=0}^{n}(\alpha+n+i)}\cdot\prod_{i=0}^{n-1}\frac{n-i}{\alpha+n+i}\cdot\left|(f_{ij})_{i,j=\overline{1,n}}\right|=\frac{(n!)^2\Delta_n(\alpha)}{(\alpha+2n)\cdot\left(\prod\limits_{i=0}^{n-1}(\alpha+n+i)\right)^2}.$$ □

Theorem 2.17 *For* $\forall n>1$ *and* $\forall\alpha\in(\mathbb{C}\setminus\mathbb{Z})\cup\mathbb{N}^*$*, the following formula holds:*

$$\Delta_n(\alpha)\overset{def}{=}\left|\left(\frac{1}{i+j+\alpha}\right)_{i,j=\overline{0,n-1}}\right|=\frac{1}{\alpha}\prod_{k=1}^{n-1}\frac{(k!)^2}{(\alpha+2k)\cdot\left(\prod\limits_{i=0}^{k-1}(\alpha+k+i)\right)^2}.$$

Proof. Consider $\forall n\geq 1$ and $\forall\alpha\in(\mathbb{C}\setminus\mathbb{Z})\cup\mathbb{N}^*$. From Lemma 2.13, we have

$$\Delta_{n+1}(\alpha)=w_n(\alpha)\Delta_n(\alpha),$$

$$w_n(\alpha)=\frac{(n!)^2}{(\alpha+2n)\cdot\left(\prod\limits_{i=0}^{n-1}(\alpha+n+i)\right)^2}.$$

By applying this recurrence relation successively, we obtain

$$\Delta_n(\alpha)=\Delta_1(\alpha)\prod_{k=1}^{n-1}w_k(\alpha)=\frac{1}{\alpha}\prod_{k=1}^{n-1}w_k(\alpha)=\frac{1}{\alpha}\prod_{k=1}^{n-1}\frac{(k!)^2}{(\alpha+2k)\cdot\left(\prod\limits_{i=0}^{k-1}(\alpha+k+i)\right)^2}.$$ □

Corollary 2.3 *For* $\forall n>1$ *and* $\forall \alpha \in \mathbb{N}^*$, *the following formula holds:*

$$\Delta_n(\alpha) \overset{def}{=} \left| \left(\frac{1}{i+j+\alpha} \right)_{i,j=\overline{0,n-1}} \right| = \frac{\alpha!!}{\alpha \cdot (\alpha+2n-2)!! \cdot \left(\prod_{k=1}^{n-1} (C_{\alpha+2k-1}^{k}) \right)^2}.$$

Proof. Consider $\forall \alpha \in \mathbb{N}^*$. By applying Lemma 2.13, we have

$$\Delta_n(\alpha) = \frac{1}{\alpha} \prod_{k=1}^{n-1} \frac{(k!)^2}{(\alpha+2k) \cdot \left(\prod_{i=0}^{k-1} (\alpha+k+i) \right)^2} =$$

$$= \frac{1}{\alpha} \prod_{k=1}^{n-1} \frac{(k!)^2}{(\alpha+2k) \cdot \left(\frac{(\alpha+2k-1)!}{(\alpha+k-1)!} \right)^2} = \frac{1}{\alpha} \cdot \frac{1}{\prod_{k=1}^{n-1} (\alpha+2k)} \cdot \frac{1}{\prod_{k=1}^{n-1} \left(\frac{(\alpha+2k-1)!}{k!(\alpha+k-1)!} \right)^2} =$$

$$= \frac{1}{\alpha} \cdot \frac{\alpha!!}{(\alpha+2n-2)!!} \cdot \frac{1}{\left(\prod_{k=1}^{n-1} C_{\alpha+2k-1}^{k} \right)^2}, \ \forall n>1. \qquad \square$$

Corollary 2.4 *For* $\forall n \geq 1$ *the following formula holds:*

$$(\Delta_n(1))^{-1} \overset{def}{=} \left| \left(\frac{1}{i+j+1} \right)_{i,j=\overline{0,n-1}} \right|^{-1} = (2n-1)!! \left(\prod_{k=1}^{n-1} C_{2k}^{k} \right)^2 \in \mathbb{N}^*.$$

Proof. The proof follows directly from Corollary 2.3 for $\alpha = 1$. $\square$

Lemma 2.14 *For* $\forall \alpha \in \mathbb{C}$ *we have* $a^{(\alpha)} \notin Rol[\mathbb{C}]$, *where* $a^{(\alpha)} = \left(\frac{1}{n+\alpha} \right)_{n=\overline{0,\infty}}$.

Proof. Consider $\forall \alpha \in \mathbb{C}$. If $\alpha \in \mathbb{Z} \setminus \mathbb{N}^*$, we have

$$(-\alpha) \in \mathbb{N}, \ a_{-\alpha}^{(\alpha)} = \frac{1}{0} = \infty, \ \left| a_{\delta}^{(\alpha)} \right| < \infty, \ \forall \delta \neq -\alpha,$$

which implies $a^{(\alpha)} \notin Rol[\mathbb{C}]$.

If $\alpha \in \mathbb{C} \setminus (\mathbb{Z} \setminus \mathbb{N}^*)$, then $\alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup \mathbb{N}^*$ and, based on Theorem 2.17,

$$\Delta_n^{[a^{(\alpha)}]} = \Delta_n(\alpha) \neq 0, \ \forall n \in \mathbb{N}^*,$$

which implies that $(\Delta_n^{[a^{(\alpha)}]})_{n=0}^{\infty}$ does not have finite rank, hence $a^{(\alpha)} \notin Rol[\mathbb{C}]$. □

Theorem 2.18 *Consider the sequence* $a = \left(\frac{F(n)}{G(n)}\right)_{n=\overline{0,\infty}}$ *with* $F(X), G(X) \in \mathbb{C}[X]$ *and* $G(n) \neq 0$, $n = \overline{0,\infty}$. *Then* $a \in Rol[\mathbb{C}]$ *if and only if* $F(X) \vdots G(X)$.

Proof. If $F(X) \vdots G(X)$, then there exists $Q(X) \in \mathbb{C}[X]$ such that

$$F(X) = G(X) \cdot Q(X).$$

We obtain

$$a_n = \frac{F(n)}{G(n)} = Q(n),\ n = \overline{0,\infty}$$

and, since $Q(X) \in Rolf[\mathbb{C}]$, it follows that $a \in Rol[\mathbb{C}]$.

Let us also prove the converse statement. Consider $a \in Rol[\mathbb{C}]$. Suppose, to the contrary, that $F(X)$ is not divisible by $G(X)$. It follows that there exists $\alpha \in \mathbb{C}$ such that $G(\alpha) = 0$ and $F(\alpha) \neq 0$, which implies that $G(X) \vdots (X - \alpha)$.

Consider the polynomial

$$P(X) = \frac{G(X)}{X - \alpha} \in \mathbb{C}[X].$$

Since $(P(n))_{n=0}^{\infty} \in Rol[\mathbb{C}]$, we obtain

$$\left(\frac{F(n)}{n-\alpha}\right)_{n=\overline{0,\infty}} = (a_n P(n))_{n=0}^{\infty} \in Rol[\mathbb{C}].$$

From $F(\alpha) \neq 0$, it follows that there exists $Q(x) \in \mathbb{C}[X]$ and $r \in \mathbb{C}^*$ such that

$$F(X) = Q(X)(X - \alpha) + r,$$

which implies

$$\frac{1}{n-\alpha} = \frac{1}{r} \cdot \frac{F(n)}{n-\alpha} - \frac{1}{r} \cdot Q(n),\ n = \overline{0,\infty}.$$

Since $(Q(n))_{n=0}^{\infty} \in Rol[\mathbb{C}]$, it follows that

$$\left(\frac{1}{n-\alpha}\right)_{n=\overline{0,\infty}} \in Rol[\mathbb{C}],$$

which contradicts Lemma 2.14. Therefore, we obtain $F(X) \vdots G(X)$. □

Example 2.7 *Prove that* $a = \left(\frac{1}{n+1}\right)_{n=0}^{\infty}$ *is not a homogeneous linear recurrence.*

Solution. According to Theorem 2.18, we have $a = (a_n)_{n=0}^{\infty} \notin Rol[\mathbb{C}]$. □

Example 2.8 *Compute the determinant*

$$\Delta = \begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} \\ \frac{1}{9} & \frac{1}{11} & \frac{1}{13} & \frac{1}{15} \end{vmatrix}.$$

Solution. We have

$$\Delta = \begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} \\ \frac{1}{9} & \frac{1}{11} & \frac{1}{13} & \frac{1}{15} \end{vmatrix} = \frac{1}{16} \begin{vmatrix} \frac{1}{1,5} & \frac{1}{2,5} & \frac{1}{3,5} & \frac{1}{4,5} \\ \frac{1}{2,5} & \frac{1}{3,5} & \frac{1}{4,5} & \frac{1}{5,5} \\ \frac{1}{3,5} & \frac{1}{4,5} & \frac{1}{5,5} & \frac{1}{6,5} \\ \frac{1}{4,5} & \frac{1}{5,5} & \frac{1}{6,5} & \frac{1}{7,5} \end{vmatrix} = \frac{1}{16}\Delta_4\left(\frac{3}{2}\right) =$$

$$= \frac{1}{16} \cdot \frac{2}{3} \prod_{k=1}^{3} \frac{(k!)^2}{\left(\frac{3}{2}+2k\right) \cdot \left(\prod_{i=0}^{k-1}\left(\frac{3}{2}+k+i\right)\right)^2} = \frac{65536}{63275987399625}.$$ □

2.6 Convergence of the homogeneous linear recurrences

Further, we present the necessary and sufficient conditions for the convergence of the homogeneous linear recurrences. The convergence theorem is formulated in the following way:

Theorem 2.19 *Consider the sequence* $a \in Rol[\mathbb{C}][m]$, *which is not identically zero. Let* $\dim[\mathbb{C}](a) = m$ *and* $P(z) \in H[\mathbb{C}][m](a)$. *Also, let* $z_0, z_1, \ldots, z_{p-1}$ *be all the distinct roots of the polynomial* $P(z)$, *with corresponding multiplicities* $s_0, s_1, \ldots, s_{p-1}$. *The sequence* a *is convergent if and only if*

$$|z_k| > 1 \text{ or } (z_k = 1 \text{ and } s_k = 1), \text{ for } k = \overline{0, p-1}.$$

If the sequence a *is convergent, then*

- $\lim\limits_{n\to\infty} a_n = 0$ *when* $|z_k| > 1$, $k = \overline{0, p-1}$;
- $\lim\limits_{n\to\infty} a_n = (I_m^{[a]}((B^{[a]})^T)^{-1})_{t0}$ *when* $z_t = 1$ *and* $s_t = 1$.

Proof. Consider the vector

$$x = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{kj})_{\substack{k=\overline{0,p-1} \\ j=0,s_k-1}}.$$

It follows that

$$a_n = x(\beta_n^{[a]})^T, \ \forall n \in \mathbb{N},$$

which implies

$$a_n = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{kj} \beta_{n,j}^{[a](k)}, \ \forall n \in \mathbb{N}.$$

In what follows, we prove the sufficiency. Let

$$|z_k| > 1 \text{ or } (z_k = 1 \text{ and } s_k = 1), \ k = \overline{0, p-1}.$$

Let k be a fixed value. If $|z_k| > 1$, then

$$\lim_{n\to\infty} \beta_{n,j}^{[a](k)} = \lim_{n\to\infty} \frac{n^j}{z_k^n} = 0, \ j = \overline{0, s_k - 1}.$$

If $z_k = 1$ and $s_k = 1$, then $\beta_{n,0}^{[a](k)} = \frac{1}{1} = 1$. Therefore, the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{kj} \beta_{n,j}^{[a](k)} = \sum_{k=0}^{p-1} \sum_{j=0}^{s_k-1} A_{kj} \lim_{n\to\infty} \beta_{n,j}^{[a](k)},$$

existsi.e. the sequence a is convergent.

If $|z_k|>1,\ k=\overline{0,p-1}$, then

$$\lim_{n\to\infty} a_n=\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{kj}\cdot 0=0.$$

If $z_t=1,\ s_t=1$ and $|z_k|>1,\ \forall k\neq t$, then

$$\lim_{n\to\infty} a_n=\sum_{\substack{k=0,\\ k\neq t}}^{p-1}\sum_{j=0}^{s_k-1} A_{kj}\cdot 0+A_{t0}\cdot 1=A_{t0},$$

hence

$$\lim_{n\to\infty} a_n=(I_m^{[a]}((B^{[a]})^T)^{-1})_{t0}.$$

Next, we prove the necessary conditions. Let $a\in Rol[\mathbb{C}][m]$ be a convergent sequence with $\dim[\mathbb{C}](a)=m$. It follows that

$$a_n=\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{kj}\beta_{n,j}^{[a](k)}=\sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{kj}\frac{n^j}{z_k^n}=\sum_{k=0}^{p-1}\frac{1}{z_k^n}\sum_{j=0}^{s_k-1} A_{kj}n^j=\sum_{k=0}^{p-1}\frac{P_k(n)}{z_k^n},\ \forall n\in\mathbb{N},$$

where $P_k(n)$ is a polynomial of degree $\deg(P_k(n))\le s_k-1,\ k=\overline{0,p-1}$. Since we have $\dim[\mathbb{C}](a)=m$, according to the order minimization method by eliminating the characteristic zeros, we obtain

$$A_{k,s_k-1}\neq 0,\ k=\overline{0,p-1}.$$

It follows that

$$\deg(P_k(n))=s_k-1,\ k=\overline{0,p-1}.$$

Suppose, to the contrary, that $\exists k=t,\ 0\le k<p$, such that

$$|z_t|\le 1 \text{ and } (z_t\neq 1 \text{ or } s_t>1).$$

It follows that $\left(\frac{P_t(n)}{z_t^n}\right)_{n=\overline{0,\infty}}$ is divergent. Since the values $z_0,z_1,\ldots,z_{p-1}$ are pairwise distinct, we obtain that the sequence $a=\left(\sum_{k=0}^{p-1}\frac{P_k(n)}{z_k^n}\right)_{n=\overline{0,\infty}}$ is divergent, which is a contradiction. Therefore,

$$|z_k| > 1 \text{ or } (z_k = 1 \text{ and } s_k = 1),\ k = \overline{0, p-1}. \qquad \square$$

In the case when the convergence of the sequence is guaranteed, its limit can be determined more easily. Next, we present several theorems that efficiently determine the limit of a convergent homogeneous linear recurrence.

Theorem 2.20 *If the sequence* $a \in Rol[\mathbb{C}]$ *is convergent and its characteristic polynomial* $P(z) \in H[\mathbb{C}](a)$ *satisfies the condition* $P(1) \neq 0$, *then* $\lim_{n\to\infty} a_n = 0.$

Proof. Let $a \in Rol[\mathbb{C}][m]$ be a convergent sequence with characteristic polynomial $P(z) = H_m^{[q]}(z) \in H[\mathbb{C}][m](a)$, which satisfies the condition $P(1) \neq 0$. Then, we have $\sum_{k=0}^{m-1} q_k \neq 1$ and the limit $\lim_{n\to\infty} a_n = L < \infty$ exists. Also, the recurrence

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

implies

$$L = \lim_{n\to\infty} a_n = \sum_{k=0}^{m-1} q_k \lim_{n\to\infty} a_{n-1-k} = \sum_{k=0}^{m-1} q_k L.$$

The unique solution of this equation is $L = 0$. $\square$

Theorem 2.21 *Assume that* $a \in Rol[\mathbb{C}][m]$, $P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a)$, $P(1) = 0$ *and* $m \geq 2$. *Then the sequence* a *is a linear* $(m-1)-$*recurrence over* $\mathbb{C}$, *generated by the vector* $q = (q_k)_{k=0}^{m-2}$ *and the inhomogeneity*

$$r_{m-1} = a_{m-1} - \sum_{k=0}^{m-2} q_k a_{m-2-k},$$

where $q_k = \sum_{j=0}^{k} p_j - 1,\ k = \overline{0, m-2}.$

Proof. Let $a \in Rol[\mathbb{C}][m]$, $P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a)$, $P(1) = 0$ and $m \geq 2$. Thus,

$$a_n = \sum_{k=0}^{m-1} p_k a_{n-1-k},\ \forall n \geq m.$$

We consider the values q_k, $k=\overline{0,m-2}$, and the sequence $r=(r_n)_{n=m-1}^{\infty}$, where

$$q_k=\sum_{j=0}^{k}p_j-1,\ k=\overline{0,m-2},$$

$$r_n=a_n-\sum_{k=0}^{m-2}q_k a_{n-1-k},\ \forall n\geq m-1.$$

It follows that

$$r_n=a_n-\sum_{k=0}^{m-2}q_k a_{n-1-k}=\sum_{k=0}^{m-1}p_k a_{n-1-k}-\sum_{k=0}^{m-2}q_k a_{n-1-k}=$$

$$=p_{m-1}a_{n-m}+\sum_{k=0}^{m-2}(p_k-q_k)a_{n-1-k}=p_{m-1}a_{n-m}+\sum_{k=0}^{m-2}(p_k-q_k)a_{n-1-k}=$$

$$=p_{m-1}a_{n-m}-\sum_{k=1}^{m-2}q_{k-1}a_{n-1-k}+a_{n-1}=-q_{m-2}a_{n-m}-\sum_{k=1}^{m-2}q_{k-1}a_{n-1-k}+a_{n-1}=$$

$$=a_{n-1}-q_{m-2}a_{n-m}-\sum_{k=0}^{m-3}q_k a_{n-k-2}=a_{n-1}-\sum_{k=0}^{m-2}q_k a_{n-k-2}=r_{n-1},\ \forall n\geq m.$$

Therefore, we can conclude that $r_n=r_{m-1}$ is a constant, $\forall n\geq m-1$, i.e.

$$a_n=\sum_{k=0}^{m-2}q_k a_{n-1-k}+r_{m-1},\ \forall n\geq m-1.$$ □

Theorem 2.22 *Consider a convergent sequence* $a\in Rol[\mathbb{C}][m]$, *which is not identically zero. Let* $P(z)=H_m^{[p]}(z)\in H[\mathbb{C}][m](a)$, *where* $P(1)=0$ *and* $m\geq 2$. *Assume that the sequence* a *is a linear* $(m-1)-$*recurrence, generated by the vector* q *and inhomogeneity* r_{m-1}. *Then* $\lim\limits_{n\to\infty}a_n=\dfrac{r_{m-1}}{1-\sum\limits_{k=0}^{m-2}q_k}$.

Proof. Since the sequence a is convergent, it follows that $\exists\lim\limits_{n\to\infty}a_n=L$ and all the roots of the minimal characteristic polynomial satisfy the conditions

$$|z_k|>1 \text{ or } (z_k=1 \text{ and } s_k=1),\ k=\overline{0,p-1}.$$

Since $a \in Rol[\mathbb{C}][m]$ and $P(1)=0$, it follows that there exists t such that $z_t = 1$ and $s_t = 1$, i.e. $\left[\frac{P(z)}{z-1}\right]\Big|_{z=1} \neq 0$, which implies the inequality $1 - \sum_{k=0}^{m-2} q_k \neq 0$. Passing to the limit in the equality

$$a_n = \sum_{k=0}^{m-2} q_k a_{n-1-k} + r_{m-1}, \ \forall n \geq m-1,$$

we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sum_{k=0}^{m-2} q_k a_{n-1-k} + r_{m-1}\right),$$

hence $L = L\sum_{k=0}^{m-2} q_k + r_{m-1}$. The solution of this equation is

$$\lim_{n\to\infty} a_n = L = \frac{r_{m-1}}{1 - \sum_{k=0}^{m-2} q_k}.$$

□

Theorem 2.23 *Consider the linear recurrence* $a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ *with characteristic polynomial* $P(z) \in H[\mathbb{C}][m](a)$ *and let* $a_n \neq 0$, $\forall n \in \mathbb{N}$. *Assume that the limit* $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = L \neq 0$ *exists. Then* $P(L) = 0$ *and*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+k}} = L^k, \ \forall k \in \mathbb{Z}.$$

Proof. Let $a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$ and $P(z) = H_m^{[q]}(z) \in H[\mathbb{C}][m](a)$. Assume that the limit $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = L$ exists. We have

$$\lim_{n\to\infty} \frac{a_n}{a_{n+k}} = \lim_{n\to\infty}\left(\prod_{j=0}^{k-1} \frac{a_{n+j}}{a_{n+j+1}}\right) = \prod_{j=0}^{k-1} \lim_{n\to\infty} \frac{a_{n+j}}{a_{n+j+1}} =$$

$$= \prod_{j=0}^{k-1} \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \prod_{j=0}^{k-1} L = L^k, \ \forall k \in \mathbb{N}^*.$$

This equality is also valid for $k = 0$, since

$$\lim_{n\to\infty}\frac{a_n}{a_{n+k}}=\lim_{n\to\infty}\frac{a_n}{a_n}=1=L^0=L^k.$$

For $\forall k\in\mathbb{Z}\setminus\mathbb{N}$, we also have

$$\lim_{n\to\infty}\frac{a_n}{a_{n+k}}=\lim_{n\to\infty}\frac{1}{\frac{a_{n+k}}{a_n}}=\frac{1}{\lim\limits_{n\to\infty}\frac{a_{n+k}}{a_n}}=\frac{1}{\lim\limits_{n\to\infty}\frac{a_n}{a_{n+(-k)}}}=\frac{1}{L^{-k}}=L^k.$$

Therefore,

$$\lim_{n\to\infty}\frac{a_n}{a_{n+k}}=L^k,\ \forall k\in\mathbb{Z}.$$

Since

$$\frac{1}{L}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\sum\limits_{k=0}^{m-1}q_k a_{n-k}}{a_n}=\sum_{k=0}^{m-1}q_k\lim_{n\to\infty}\frac{a_{n-k}}{a_n}=$$

$$=\sum_{k=0}^{m-1}q_k\lim_{n\to\infty}\frac{a_n}{a_{n+k}}=\sum_{k=0}^{m-1}q_kL^k=G_m^{[q]}(L),$$

it follows that

$$P(L)=H_m^{[q]}(L)=1-LG_m^{[q]}(L)=L\left(L^{-1}-G_m^{[q]}(L)\right)=0.$$ □

Corollary 2.5 *Assume that* $a\in Rol[\mathbb{C}][m]$, $a\subseteq\mathbb{R}^*$ *and* $z_k\in\mathbb{C}\setminus\mathbb{R}$, $k=\overline{0,p-1}$, *where* $z_0,z_1,\ldots,z_{p-1}$ *are all distinct roots of* $P(z)\in H[\mathbb{C}][m](a)$. *Then the sequence* $\left(\frac{a_n}{a_{n+1}}\right)_{n=\overline{0,\infty}}$ *is divergent.*

Proof. Suppose, to the contrary, that the sequence $\left(\frac{a_n}{a_{n+1}}\right)_{n=\overline{0,\infty}}$ is convergent, i.e. the limit $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=L$ exists. Since $a_n\in\mathbb{R}^*$, $\forall n\in\mathbb{N}$, we have $\frac{a_n}{a_{n+1}}\in\mathbb{R}^*$, $\forall n\in\mathbb{N}$, i.e.

$$L=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\in\mathbb{R}.$$

From Theorem 2.23, we have $P(L)=0$, hence $\exists k\in\{0,1,\ldots,p-1\}$ such that

$z_k = L \notin \mathbb{C} \setminus \mathbb{R}$, which contradicts the initial hypothesis. Therefore, the sequence $\left(\frac{a_n}{a_{n+1}}\right)_{n=\overline{0,\infty}}$ is divergent. □

Example 2.9 *Let* $(a_n)_{n=0}^{\infty}$ *be a sequence defined by the recurrence*

$$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2},\ \forall n \geq 2.$$

Study its convergence. If it is convergent, determine its limit.

Solution. We have

$$a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][2],\ p = \left(\frac{3}{2}, -\frac{1}{2}\right) \in G[\mathbb{C}][2](a).$$

Therefore, the characteristic polynomial is

$$H_2^{[p]}(z) = 1 - \frac{3}{2}z + \frac{1}{2}z^2 = (1-z)(1-\frac{1}{2}z) \in H[\mathbb{C}][2](a),$$

with the simple roots $z_0 = 1$ and $z_1 = 2$. Since the minimal characteristic polynomial $P(z)$ is a divisor of $H_2^{[p]}(z)$, according to the convergence theorem, it follows that the sequence is convergent regardless of the initial state. There are three possibilities:

$$P(z) = H_2^{[p]}(z),\ P(z) = 1 - z \text{ and } P(z) = 1 - \frac{1}{2}z.$$

If $P(z) = 1 - \frac{1}{2}z$, i.e. $a_n = \frac{1}{2}a_{n-1},\ \forall n \geq 1$, then, according to the convergence theorem, we have $\lim_{n\to\infty} a_n = 0$. For $P(z) = 1 - z$ we have $a_n = a_{n-1},\ \forall n \geq 1$, hence $a_n = a_0,\ \forall n \geq 0$, which implies $\lim_{n\to\infty} a_n = a_0$. Lastly, when $P(z) = H_2^{[p]}(z)$, we apply Theorem 2.21 and Theorem 2.22 and we obtain

$$q_0 = p_0 - 1 = \frac{3}{2} - 1 = \frac{1}{2},\ r_1 = a_1 - q_0 a_0 = a_1 - \frac{1}{2}a_0,$$

which allows us to compute the limit using the following formula:

$$\lim_{n\to\infty} a_n = \frac{r_1}{1-q_0} = 2a_1 - a_0.$$ □

2.7 Periodicity and inclusion criteria

In this section, we study the periodicity and the inclusion criteria. The periodicity criterion is given by the following theorem:

Theorem 2.24 *Consider the sequence* $a \in Rol[\mathbb{C}]$ *with* $P(z) \in H[\mathbb{C}][m](a)$, *where* $m = \dim[\mathbb{C}](a)$. *The sequence* a *is periodic if and only if all roots of the polynomial* $P(z)$ *are simple and they are roots of unity.*

Proof. Let us prove the sufficiency. Consider that all roots z_k, $k = \overline{0, p-1}$, of $P(z)$ are simple and they are roots of order r_k of unity. This supposition implies that

$$p = m,\ z_k^{r_k} = 1,\ s_k = 1,\ k = \overline{0, m-1}.$$

Consider $r = LCM(r_0, r_1, \ldots, r_{m-1})$, hence $z_k^r = 1$, $k = \overline{0, m-1}$. This implies that z_k is a root of the polynomial $Q(z) = 1 - z^r$, $k = \overline{0, m-1}$ and, therefore, $Q(z) \vdots P(z)$, which concludes that $Q(z) \in H[\mathbb{C}][r](a)$, i.e. $a_n = a_{n-r}$, $\forall n \geq r$. Thus, a is $r-$periodic.

The necessity is proved relatively easily. Let a be a $r-$periodic sequence. We have $a \in Rol[\mathbb{C}][r]$ and $Q(z) = 1 - z^r \in H[\mathbb{C}][r](a)$, which implies that $Q(z) \vdots P(z)$. Since all the roots of the polynomial $Q(z)$ are simple and they are roots of order r of unity, then all the roots of the polynomial $P(z)$ are also simple and they are roots of order r of unity. □

Next, we study the inclusion of homogeneous linear recurrences into a given set K. Three statements were obtained for different conditions on the set K.

Theorem 2.25 *Let* K *be a set on which the operations of addition and multiplication are defined and are algebraic. If* $a \in Rol[K]$ *and* $\dim[K](a) = m$, *then* $a \subseteq K$ *if and only if* $I_m^{[a]} \in K^m$.

Proof. Let $a \subseteq K$. It follows that $a_n \in K$, $\forall n \in \mathbb{N}$, which implies $I_m^{[a]} \in K^m$. The converse implication is also proved relatively easily. Consider $I_m^{[a]} \in K^m$, i.e. $a_n \in K$, $n = \overline{0, m-1}$. Since $\dim[K](a) = m$, we have $\exists q \in K^m$ such that

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k} \ , \ \forall n \geq m \ .$$

By mathematical induction, we have $a_n \in K$, $\forall n \in \mathbb{N}$. Therefore, $a \subseteq K$. □

Theorem 2.26 *Consider a subfield K of the field $(\mathbb{C}, +, \cdot)$. Let $a \in Rol[\mathbb{C}]$ be a sequence not identically zero, $m = \dim[\mathbb{C}](a)$ and $q \in G[\mathbb{C}][m](a)$. Then $a \subseteq K$ if and only if $q \in K^m$ and $I_m^{[a]} \in K^m$.*

Proof. Consider the sequence $a \in Rol[\mathbb{C}]$ with

$$m = \dim[\mathbb{C}](a) \ , \ q \in G[\mathbb{C}][m](a) \ .$$

Assume that $q \in K^m$ and $I_m^{[a]} \in K^m$. It follows that

$$a \in Rol[K] \ , \ \dim[K](a) = m \ , \ I_m^{[a]} \in K^m \ .$$

By applying Theorem 2.25, we have $a \subseteq K$.

Let us prove the necessity. Consider

$$a \in Rol[\mathbb{C}] \ , \ \dim[\mathbb{C}](a) = m \ , \ q \in G[\mathbb{C}][m](a) \ .$$

Let $a \subseteq K$. Therefore, we have $I_m^{[a]} \in K^m$. Since

$$\dim[\mathbb{C}](a) = m \ , \ q \in G[\mathbb{C}][m](a) \ ,$$

according to the rank-based order minimization method, it follows that

$$A_m^{[a]} x^T = (f_m^{[a]})^T \ ,$$

where

$$x = (q_{m-1}, q_{m-2}, \ldots, q_0) \ , \ f_m^{[a]} = (a_m, a_{m+1}, \ldots, a_{2m-1}) \ , \ A_m^{[a]} = (a_{i+j})_{i, \ j=\overline{0, \ m-1}} \ .$$

Since

$$A_m^{[a]} \in M_m(K) \ , \ f_m^{[a]} \in K^m \ , \ \left| A_m^{[a]} \right| \neq 0 \ ,$$

the inverse matrix $(A_m^{[a]})^{-1} \in M_m(K)$ exists. Therefore,

$$x^T = (A_m^{[a]})^{-1} (f_m^{[a]})^T \in K^m \ ,$$

which implies $q \in K^m$. Consequently, we obtain $q \in K^m$ and $I_m^{[a]} \in K^m$. □

Theorem 2.27 *Let K be a finite set and $a \in Rol[\mathbb{C}]$. Then $a \subseteq K$ if and only if a is a periodic sequence and $I_T^{[a]} \in K^T$, where T is the period of the sequence a.*

Proof. Consider the $T-$periodic sequence a with $I_T^{[a]} \in K^T$. It follows that $a_n = a_{n \bmod T}$, $\forall n \in \mathbb{N}$. Therefore, $a_n \in K$, $\forall n \in \mathbb{N}$, i.e. $a \subseteq K$.

Let us prove the necessity. Consider the finite set K and the sequence $a \in Rol[\mathbb{C}][m]$ with $q \in G[\mathbb{C}][m](a)$. Assume that $a \subseteq K$ and $|K| = r$. We group the terms of the sequence a into groups of length m:

$$a_0, a_1, \ldots, a_{m-1}; a_m, a_{m+1}, \ldots, a_{2m-1}; \ldots.$$

Since we have r^m (a finite number) different values for each group, it follows that among the first r^m+1 groups two identical groups exist, i.e. there exist $u, v \in \{0,1,2,\ldots,r^m\}$, $u < v$, such that

$$(a_{um}, a_{um+1}, \ldots, a_{(u+1)m-1}) = (a_{vm}, a_{vm+1}, \ldots, a_{(v+1)m-1}).$$

We denote $T = (v-u)m$.

Let $\tilde{a} = (a_n)_{n=-\infty}^{\infty}$ be the extension over $\mathbb{Z}$ of the homogeneous linear recurrence a. Consider the subsequence $b_n = a_{n+um}$, $n = \overline{0,\infty}$. It follows that

$$b \in Rol[\mathbb{C}][m],\ q \in G[\mathbb{C}][m](b).$$

Therefore, $\tilde{a}$ is the extension over $\mathbb{Z}$ of the homogeneous linear recurrence b, according to the generating vector q.

Consider the subsequence $d_n = a_{n+vm}$, $n = \overline{0,\infty}$. We have

$$d \in Rol[\mathbb{C}][m],\ q \in G[\mathbb{C}][m](d),\ I_m^{[d]} = I_m^{[b]}.$$

Therefore, $d = b$, i.e.

$$b_n = d_n = a_{n+vm} = a_{n+um+T} = b_{n+T},\ \forall n \in \mathbb{N},$$

which implies that the sequence b is $T-$periodic, i.e.

$$b \in Rol[\mathbb{C}][T],\ p = (0,0,\ldots,0,1) \in G[\mathbb{C}][T](b).$$

Consider the extension $\tilde{b}$ over $\mathbb{Z}$, according to the generating vector p. According to TEUE, we have $\tilde{b} = \tilde{a}$, i.e. $\tilde{a}$ is $T-$periodic, which implies that a is $T-$periodic. Since $a \subseteq K$, we also have $I_T^{[a]} \in K^T$. □

Theorem 2.28 *Let* $a \in Rol[\mathbb{C}]$, $\dim[\mathbb{C}](a) = m$ *and* $q \in G[\mathbb{C}][m](a)$. *Then the smallest field containing the sequence* a *is*

$$\mathbb{Q}(I_m^{[a]}, q) \overset{def}{=} \mathbb{Q}(a_0, a_1, \ldots, a_{m-1}, q_0, q_1, \ldots, q_{m-1}).$$

Proof. Let $a \in Rol[\mathbb{C}]$, $\dim[\mathbb{C}](a) = m$ and $q \in G[\mathbb{C}][m](a)$. Consider the smallest field K which contains the sequence a. Since K is a field, it follows that $\mathbb{Q} \subseteq K$. From Theorem 2.26, we have $a \subseteq K$ if and only if $q \in K^m$ and $I_m^{[a]} \in K^m$. Therefore, the smallest field containing $\mathbb{Q}$, $a_0, a_1, \ldots, a_{m-1}$ and $q_0, q_1, \ldots, q_{m-1}$ is

$$K = \mathbb{Q}(I_m^{[a]}, q) \overset{def}{=} \mathbb{Q}(a_0, a_1, \ldots, a_{m-1}, q_0, q_1, \ldots, q_{m-1}).$$ □

Example 2.10 *Consider the sequence* $a = (a_n)_{n=0}^{\infty}$, *defined by the recurrence*

$$a_n = S(a_{n-1} + a_{n-5}) - (T+2)(a_{n-2} + a_{n-4}) + (S+T)a_{n-3} - a_{n-6}, \ \forall n \geq 6,$$

where $S = 1 + \sqrt{2} + \sqrt{3}$ *and* $T = S + \sqrt{6}$. *Prove that the sequence* a *is periodic and determine its main period (regardless of the initial state of the recurrence) and the set of values that the minimal period can take depending on the initial state.*

Solution. Consider the sequence $a = (a_n)_{n=0}^{\infty}$. We have $a \in Rol[\mathbb{C}][6]$ and

$$q = (S, -(T+2), S+T, -(T+2), S, -1) \in G[\mathbb{C}][6](a).$$

The unit characteristic polynomial is

$$P(z) = 1 - z\sum_{k=0}^{5} q_k z^k = 1 - z(S - (T+2)z + (S+T)z^2 - (T+2)z^3 + Sz^4 - z^5) =$$

$$= z^3((z^3 + z^{-3}) - S(z^2 + z^{-2}) + (T+2)(z + z^{-1}) - (S+T)).$$

Let us determine the roots of the polynomial $P(z) \in H[\mathbb{C}][6](a)$. We need to find the solutions of the equation

$$(z^3 + z^{-3}) - S(z^2 + z^{-2}) + (T+2)(z + z^{-1}) - (S+T) = 0.$$

Denoting $y = z + z^{-1}$, we obtain $y^2 = z^2 + z^{-2} + 2$, which implies

$$z^2 + z^{-2} = y^2 - 2.$$

Since

$$y^3 = z^3 + z^{-3} + 3(z \cdot z^{-1})(z + z^{-1}) = z^3 + z^{-3} + 3y,$$

we have

$$z^3 + z^{-3} = y^3 - 3y.$$

Therefore, the equation $P(z) = 0$ is equivalent to

$$(y^3 - 3y) - S(y^2 - 2) + (T + 2)y - (S + T) = 0,$$

which can be written in the general form

$$y^3 - (1 + \sqrt{2} + \sqrt{3})y^2 + (\sqrt{2} + \sqrt{3} + \sqrt{6})y - \sqrt{6} = 0.$$

This equation is equivalent to

$$(y - 1)(y - \sqrt{2})(y - \sqrt{3}) = 0.$$

Thus, the solutions are $\{1, \sqrt{2}, \sqrt{3}\}$. Next, we find the solutions of the equations

$$y = 1,\ y = \sqrt{2} \text{ and } y = \sqrt{3},$$

taking into account the notation $y = z + z^{-1}$. We obtain the equations

$$z + z^{-1} = 1,\ z + z^{-1} = \sqrt{2} \text{ and } z + z^{-1} = \sqrt{3},$$

equivalent to the quadratic equations

$$z^2 + 1 = z,\ z^2 + 1 = \sqrt{2}z \text{ and } z^2 + 1 = \sqrt{3}z.$$

The set of all solutions of these equations is

$$\{\cos\varphi \pm i\sin\varphi \mid \varphi \in \{\pi/3, \pi/4, \pi/6\}\}.$$

It is easy to observe that all the roots of the polynomial $P(z)$ are simple and they are roots of unity. It follows that the sequence a is periodic.

The orders of the roots are

$$r_{1,2} = 6,\ r_{3,4} = 8,\ r_{5,6} = 12.$$

Thus, $r = LCM(6,8,12) = 24$ is a period of the sequence a. If a is not identically

zero, its minimal period, depending on initial state, belongs to the set

$$\{6,8,12, LCM(6,8), LCM(6,12), LCM(8,12), LCM(6,8,12)\} = \{6,8,12,24\},$$

because a may fail to be 6-minimal, in which case some roots do not contribute. □

Example 2.11 *Consider the sequence* $a = (a_n)_{n=0}^{\infty}$, *defined by the recurrence*

$$a_n = i\, a_{n-1} - 0.25\, a_{n-2} + 0.25 i\, a_{n-3},\ \forall n \geq 3$$

and the initial values

$$a_0 = 4,\ a_1 = 0,\ a_2 = -1.$$

Show that the sequence a *has only rational terms.*

Solution. Consider the sequence $a = (a_n)_{n=0}^{\infty} \in Rol[\mathbb{C}][3]$. We have

$$a_3 = ia_2 - 0.25a_1 + 0.25 i\, a_0 = 0,$$

$$a_4 = ia_3 - 0.25a_2 + 0.25 i\, a_1 = 0.25.$$

These values, together with the given a_0, a_1 and a_2, define the matrix

$$A_3^{[a]} = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0.25 \end{pmatrix}.$$

Therefore, $\dim[\mathbb{C}](a) = rank(A_3^{[a]}) = 2$.

Next, we determine the submatrix $A_2^{[a]}$ and the vector $f_2^{[a]}$:

$$A_2^{[a]} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

$$f_2^{[a]} = (a_2, a_3) = (-1, 0).$$

We solve the system

$$A_2^{[a]} x^T = (f_2^{[a]})^T$$

and find the solution $x = (-0.25, 0)$, which implies

$$q_{\min} = (0, -0.25) \in G[\mathbb{C}][2](a).$$

Therefore, $I_2^{[a]} = (a_0, a_1) = (4,0) \in \mathbb{Q}^2$ and $q_{\min} \in \mathbb{Q}^2$, which implies that $a \subseteq \mathbb{Q}$. □

3 ADVANCED HOMOGENEOUS LINEAR RECURRENCES

In this chapter, the study of homogeneous linear recurrences is extended to several classes of advanced recurrences. The properties of subsequences of homogeneous linear recurrences and of five-dimensional homogeneous linear recurrences are investigated, providing a basis for the subsequent study of games defined on stochastic systems with final sequences of states. Furthermore, the algebraic properties of homogeneous linear recurrences over rings and their sign-based subsets are analyzed, together with Littlewood, Newman, and Borwein homogeneous linear recurrences. Finally, the Jury stability criterion is presented, and various perturbations of homogeneous linear recurrences are investigated.

3.1 Subsequences of homogeneous linear recurrences

Next, we extend the results by establishing new properties of homogeneous linear recurrences. The following two theorems analyze their subsequences.

Theorem 3.1 *If* $a \in Rol[\mathbb{C}][m]$, *then* $b = (a_{cn+t})_{n=0}^{\infty} \in Rol[\mathbb{C}][m]$, $\forall c,t \in \mathbb{N}$, *with a generating vector that does not depend on* t.

Proof. Let $a \in Rol[\mathbb{C}][m]$ with generating vector $u \in G[\mathbb{C}][m](a)$. We consider all the distinct roots $z_0, z_1, \ldots, z_{p-1}$, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-1}$, of the characteristic polynomial $H_m^{[u]}(z)$. Let $b = (a_{cn+t})_{n=0}^{\infty}$, where c and t are two fixed nonnegative integers.

We consider the decomposition

$$x^{[a]} = I_m^{[a]}((B^{[a]})^T)^{-1} = (A_{k,j})_{k=\overline{0,p-1},\, j=\overline{0,s_k-1}}.$$

We have

$$a_n = x^{[a]}(\beta_n^{[a]})^T = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{n^j}{z_k^n},\ n=\overline{0,\infty}$$

which implies

$$b_n = a_{cn+t} = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1} A_{k,j}\frac{(cn+t)^j}{z_k^{cn+t}} = \sum_{k=0}^{p-1}\sum_{j=0}^{s_k-1}\alpha_{kjt}h_{kjtc}(n),$$

where

$$\alpha_{kjt} = \frac{A_{k,j}}{z_k^t},\ h_{kjtc}(n) = \frac{(cn+t)^j}{(z_k^c)^n},\ k=\overline{0,p-1},\ j=\overline{0,s_k-1},\ n\in\mathbb{N}.$$

Since $h_{jtc} = ((cn+t)^j)_{n=0}^{\infty}$ is a sequence of polynomials of degree j, we have

$$h_{jtc}\in Rol[\mathbb{C}][j+1],\ (1-z)^{j+1}\in H[\mathbb{C}](h_{jtc}),\ j=\overline{0,s_k-1}.$$

Also, since $g_{kc} = ((z_k^c)^{-n})_{n=\overline{0,\infty}}$ is a geometric progression with ratio $\frac{1}{z_k^c}$, we have

$$g_{kc}\in Rol[\mathbb{C}][1],\ \left(\frac{1}{z_k^c}\right)\in G[\mathbb{C}](g_{kc}),\ k=\overline{0,p-1}.$$

From these relations, we obtain

$$h_{kjtc} = (h_{kjtc}(n))_{n=\overline{0,\infty}} = h_{jtc}\cdot g_{kc}\in Rol[\mathbb{C}][j+1],$$

$$\left(1-\frac{z}{z_k^c}\right)^{j+1}\in H[\mathbb{C}](h_{kjtc}),\ k=\overline{0,p-1},\ j=\overline{0,s_k-1}.$$

Next, for $k=\overline{0,p-1}$, we have

$$f_{ktc} = \sum_{j=0}^{s_k-1}\alpha_{kjt}h_{kjtc}\in Rol[\mathbb{C}][s_k]$$

with characteristic polynomial

$$LCM\left(\left\{\left(1-\frac{z}{z_k^c}\right)^{j+1}\mid j=\overline{0,s_k-1}\right\}\right) = \left(1-\frac{z}{z_k^c}\right)^{s_k}\in H[\mathbb{C}](f_{ktc}).$$

Since $b_n = \sum_{k=0}^{p-1} f_{ktc}(n)$, where $f_{ktc} = (f_{ktc}(n))_{n=0}^{\infty}$, we obtain $b\in Rol[\mathbb{C}][m]$ with

$$LCM\left(\left\{\left(1-\frac{z}{z_k^c}\right)^{s_k} \mid k=\overline{0,p-1}\right\}\right)=\prod_{k=0}^{p-1}\left(1-\frac{z}{z_k^c}\right)^{s_k}\in H[\mathbb{C}][m](b).$$

We can observe that this characteristic polynomial does not depend on t. Thus, the corresponding generating vector also does not depend on t. In conclusion, we have

$$b=(a_{cn+t})_{n=0}^{\infty}\in Rol[\mathbb{C}][m],\ \forall c,t\in\mathbb{N},$$

with a generating vector that does not depend on t. □

Theorem 3.2 *If* $a\in Rol^{*}[\mathbb{C}][m]$, *then* $b=(a_{cn+t})_{n=0}^{\infty}\in Rol^{*}[\mathbb{C}][m]$, $\forall c,t\in\mathbb{N}$, *with a generating vector that does not depend on* t.

Proof. Let $a\in Rol^{*}[\mathbb{C}][m]$ with generating vector $u=(u_k)_{k=0}^{m-1}\in G^{*}[\mathbb{C}][m](a)$. Let s be the degree of the characteristic polynomial $H_m^{[u]}(z)$.

We consider the subsequence $\alpha=(\alpha_n)_{n=0}^{\infty}$, where

$$\alpha_n=a_{n+m-s},\ \forall n\geq 0.$$

It is easy to observe that $\alpha\in Rol[\mathbb{C}][s]$ with generating vector $u^{(s)}\in G[\mathbb{C}][s](a)$, where $u^{(s)}=(u_k)_{k=0}^{s-1}$. Applying Theorem 3.1, we have

$$\beta=(\beta_n)_{n=0}^{\infty}\in Rol[\mathbb{C}][s],\ \forall c,t\in\mathbb{N},$$

with a generating vector $v^{(s)}=(v_k)_{k=0}^{s-1}\in G[\mathbb{C}][s](\beta)$ that does not depend on t, where

$$\beta_n=\alpha_{cn+t}=a_{cn+t+m-s},\ \forall n\geq 0.$$

From this relation, we obtain that $b=(b_n)_{n=0}^{\infty}\in Rol^{*}[\mathbb{C}][m]$ with a generating vector $v=(v_k)_{k=0}^{m-1}\in G^{*}[\mathbb{C}][m](b)$ that does not depend on t, where

$$b_n=a_{cn+t},\ \forall n\geq 0,\ \forall c,t\in\mathbb{N}$$

and $v_k=0$, $k=\overline{s,m-1}$. □

3.2 Five-dimensional homogeneous linear recurrences

The following theorem analyzes homogeneous linear recurrences over the set of square matrices whose entries are square matrices:

Theorem 3.3 If $a \subseteq (\mathbb{C}^r)^t$ and $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$, then $a \in Rol^*[K][mtr]$.

Proof. Let $a \subseteq (\mathbb{C}^r)^t$, $a \in Rol^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$ and $q \in G^*[\mathcal{M}_t(\mathcal{M}_r(K))][m]$. Then

$$a_n = \sum_{k=0}^{m-1} q^{(k)} a_{n-1-k}, \ \forall n \geq m,$$

where $a = (a_n)_{n=0}^{\infty}$ and $q = (q^{(k)})_{k=0}^{m-1}$.

We consider the set

$$\Lambda(n) = \{0,1,\ldots,n-1\}, \ \forall n \in \mathbb{N}.$$

Let $a_n = (a_{ni})_{i \in \Lambda(t)}$, where $a_{ni} = (a_{nij})_{j \in \Lambda(r)}$, $i \in \Lambda(t)$, $\forall n \in \mathbb{N}$. We obtain

$$a_{ni} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} q_{is}^{(k)} a_{n-1-k,s}, i \in \Lambda(t), \forall n \geq m,$$

where $q^{(k)} = (q_{is}^{(k)})_{i,s \in \Lambda(t)}$, $k \in \Lambda(m)$. This formula implies the recurrence relation

$$a_{nij} = \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell}, i \in \Lambda(t), j \in \Lambda(r),$$

where $q_{is}^{(k)} = (q_{isj\ell}^{(k)})_{j,\ell \in \Lambda(r)}$, $i,s \in \Lambda(t)$.

Consider $a^{(i,j)} = (a_{nij})_{n=0}^{\infty}$ and $q_{isj\ell} = (q_{isj\ell}^{(k)})_{k=0}^{m-1}$, $i,s \in \Lambda(t)$, $j,\ell \in \Lambda(r)$. Let us construct the generating function of the sequence $a^{(i,j)}$, $i \in \Lambda(t)$, $j \in \Lambda(r)$.

$$G^{[a^{(i,j)}]}(z) = \sum_{n=0}^{\infty} a_{nij} z^n = \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} a_{n-1-k,s,\ell} + \sum_{n=0}^{m-1} a_{nij} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} z^{k+1} \sum_{n=m}^{\infty} a_{n-1-k,s,\ell} z^{n-1-k} =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \sum_{n=m-1-k}^{\infty} a_{ns\ell} z^n =$$

$$= G_m^{[a^{(i,j)}]}(z) + \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} \left(G^{[a^{(s,\ell)}]}(z) - G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) =$$

$$= \left(G_m^{[a^{(i,j)}]}(z) - \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z) \right) + z \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} G^{[a^{(s,\ell)}]}(z) \sum_{k=0}^{m-1} q_{isj\ell}^{(k)} z^k =$$

$$= F_{ij}(z) + z \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z),$$

where

$$F_{ij}(z) = G_m^{[a^{(i,j)}]}(z) - \sum_{k=0}^{m-1} z^{k+1} \sum_{s=0}^{t-1} \sum_{\ell=0}^{r-1} q_{isj\ell}^{(k)} G_{m-1-k}^{[a^{(s,\ell)}]}(z).$$

Therefore, for $i \in \Lambda(t)$ and $j \in \Lambda(r)$, we have

$$G^{[a^{(i,j)}]}(z) - z \sum_{s \in \Lambda(t), \ell \in \Lambda(r)} G_m^{[q_{isj\ell}]} G^{[a^{(s,\ell)}]}(z) = F_{ij}(z).$$

If we denote

$$x_{ij} = G^{[a^{(i,j)}]}(z),\ i \in \Lambda(t),\ j \in \Lambda(r),$$

we obtain the following system of tr linear equations with tr unknown variables:

$$x_{ij}(z) - z \sum_{s \in \Lambda(t), \ell \in \Lambda(r)} G_m^{[q_{isj\ell}]} x_{s,\ell}(z) = F_{ij}(z),\ i \in \Lambda(t),\ j \in \Lambda(r).$$

In matrix form, this system can be written as follows:

$$W(z)x(z) = F(z),$$

where

$$x(z) = (x_{ij}(z))_{(i,j) \in \Lambda(t) \times \Lambda(r)},$$

$$F(z) = (F_{ij}(z))_{(i,j) \in \Lambda(t) \times \Lambda(r)},$$

$$Q = ((q_{(i,j),(s,\ell)}^{(k)})_{(i,j),(s,\ell) \in \Lambda(t) \times \Lambda(r)})_{k=0}^{m-1},$$

$$q_{(i,j),(s,\ell)}^{(k)} = q_{isj\ell}^{(k)},\ i,s \in \Lambda_t,\ j,\ell \in \Lambda_r,$$

$$W(z) = I - zG_m^{[Q]}(z).$$

Therefore, we have

$$x(z) = W^{-1}(z)F(z),\ \forall z \in D \setminus F,$$

where D is the domain of convergence of $G^{[a]}(z)$ and F is the set of roots of the polynomial $|W(z)|$. From this relation, we can conclude that $x_{ij}(z)$ are rational

fractions, $\forall i \in \Lambda(t)$, $\forall j \in \Lambda(r)$. Consequently, we have

$$a^{(i,j)} \in Rol^*[K][mtr],\ \forall i \in \Lambda(t),\ \forall j \in \Lambda(r),$$

which also implies $a \in Rol^*[K][mtr]$. □

3.3 Homogeneous linear recurrences over numerical rings

In this section the homogeneous linear recurrences over numerical rings ([80]) are considered. In particular, such a numerical ring can be one of the well-known sets $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or an extension field of $\mathbb{Q}$.

Theorem 3.4 *Assume that K_1 is a set and K_2 is a subset of K_1. If $a \in Rol[K_2][m]$, then $a \in Rol[K_1][m]$ with the same generating vector.*

Proof. Let $a \in Rol[K_2][m]$ and $q \in G[K_2][m](a)$. Then $q \in K_2^m$ and, since $K_2 \subseteq K_1$, we have $q \in K_1^m$. Therefore, we obtain $a \in Rol[K_1][m]$ and $q \in G[K_1][m](a)$. □

Theorem 3.5 *Let K_1 be a subring of $\mathbb{C}$ and K_2 a subring of K_1. Suppose that $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1. If $Q^*(z)$ is the reciprocal polynomial of $Q(z)$ and each complex root of $Q^*(z)$ is integral over K_2, then $a \in Rol[K_2]$ with the reciprocal of the minimal-degree monic multiple $P(z) \in K_2[z]$ of $Q^*(z)$ as its minimal characteristic polynomial. Otherwise, if $Q^*(z)$ has a complex root, which is not integral over K_2, then $a \notin Rol[K_2]$.*

Proof. Let $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1, $Q^*(z)$ the reciprocal polynomial of $Q(z)$ and each complex root of $Q^*(z)$ is integral over K_2. Thus, there exists a minimal-degree monic polynomial $P(z) \in K_2[z]$, such that $P(z) \vdots Q^*(z)$. In conclusion, we obtain that $a \in Rol[K_2]$ with reciprocal polynomial of $P(z)$ as minimal characteristic polynomial. When $Q^*(z)$ has a root, non integral over K_2, then $Q^*(z)$ does not have any multiple in $K_2[z]$, hence $Q(z)$ does not have any multiple in $K_2[z]$, i.e. $a \notin Rol[K_2]$. □

Theorem 3.6 *Let K_1 be a subfield of $\mathbb{C}$ and K_2 a subfield of K_1. If $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1 and each root of $Q(z)$ is algebraic over K_2, then $a \in Rol[K_2]$ with minimal-degree multiple $P(z) \in K_2[z]$ of the polynomial $Q(z)$ as minimal characteristic polynomial. Otherwise, if $Q(z)$ possesses a transcendental root over K_2, then $a \notin Rol[K_2]$.*

Proof. The proof follows immediately from Theorem 3.5. There is no need to use the monic reciprocal polynomial for the characteristic polynomial $Q(z)$, since the fields are closed under division, which is an algebraic operation. This allows us to normalize the constant term of the characteristic polynomial to 1. □

Theorem 3.6 allows us to decide if a general homogeneous linear recurrence (over $\mathbb{C}$) is also a homogeneous linear recurrence over a subfield, like $\mathbb{R}$, $\mathbb{Q}$ or an extension field of $\mathbb{Q}$. So, each homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{R}$, and, if its minimal characteristic polynomial over $\mathbb{C}$ or $\mathbb{R}$ does not have a transcendental root over $\mathbb{Q}$ (respectively over an extension of $\mathbb{Q}$), then it is also a homogeneous linear recurrence over $\mathbb{Q}$ (respectively, over a given extension of $\mathbb{Q}$).

Theorem 3.5 provides a criterion to decide if a homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over a subring, like $\mathbb{Z}$. However, the above remarks and the following theorem allow us to check easier if a homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{Z}$.

Theorem 3.7 *Let $a \in Rol[\mathbb{Q}]$ with minimal characteristic polynomial $Q(z)$. Then $a \in Rol[\mathbb{Z}]$ if and only if $Q(z) \in \mathbb{Z}[z]$.*

Proof. Let $a \in Rol[\mathbb{Q}]$ with minimal characteristic polynomial $Q(z)$ (over $\mathbb{Q}$). If $Q(z) \in \mathbb{Z}[z]$, then it is obvious that $a \in Rol[\mathbb{Z}]$ and $Q(z) \in H[\mathbb{Z}](a)$.

Next, we consider the converse. Let $a \in Rol[\mathbb{Z}]$ with minimal characteristic polynomial $P(z)$ (over $\mathbb{Z}$). Then $a \in Rol[\mathbb{Q}]$ and $P(z) \in H[\mathbb{Q}](a)$, which implies that $P(z) \vdots Q(z)$, i.e. there exists $T(z) \in \mathbb{Q}[z]$ such that $P(z) = Q(z)T(z)$.

Since $Q(z) \in \mathbb{Q}[z]$ and $Q(0) = 1$, there exists $q \in \mathbb{N}^*$ such that $qQ(z) \in \mathbb{Z}[z]$ and $qQ(0) = q$. Similarly, since $T(z) \in \mathbb{Q}[z]$, $P(0) = 1$ and $Q(0) = 1$, there exists $t \in \mathbb{N}^*$ such that $tT(z) \in \mathbb{Z}[z]$ and $tT(0) = t$. So, $qtP(z) = (qQ(z)) \cdot (tT(z))$.

The relation $P(z) \in \mathbb{Z}[z]$ implies that the left side $qtP(z)$ is divisible by qt. Next, because $qQ(z) \in \mathbb{Z}[z]$, any integral divisor r of $qQ(z)$ satisfies the inequality $r \leq qQ(0) = q$. Similarly, since $tT(z) \in \mathbb{Z}[z]$, any integral divisor s of $tT(z)$ satisfies the inequality $s \leq tT(0) = t$. As a result, we have $rs \leq qt$. But, due to the fact that qt is a divisor of $(qQ(z)) \cdot (tT(z))$, this means that $r = q$ and $s = t$, i.e. $qQ(z)$ is divisible by q and $tT(z)$ is divisible by t. In conclusion, $Q(z) \in \mathbb{Z}[z]$. □

3.4 Recurrence criteria over sign-based ring subsets

In the context of this section, we will consider $K \subseteq \mathbb{R}$ and $(K, +, \cdot)$ a ring with standard addition and multiplication arithmetic operations. In particular, K can be one of the well-known sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or an extension field of $\mathbb{Q}$.

Next, we determine under which conditions $a \in Rol[\mathbb{R}]$ is a homogeneous linear recurrence over K or a subset of it: K_-^*, respectively K_+^*.

Lemma 3.1 *The sets* $Rol[K_-]$ *and* $Rol[K_-^*]$ *are equal. Also, if* $b \in Rol[K_-][m]$ *and* $P(z) \in H[K_-][m](b)$, *then* $b \in Rol[K_-^*]$ *and* $(z+1)^{m-1} P(z) \in H[K_-^*](b)$.

Proof. We will show that the sets $Rol[K_-]$ and $Rol[K_-^*]$ are equal, i.e. both inclusions $Rol[K_-] \subseteq Rol[K_-^*]$ and $Rol[K_-^*] \subseteq Rol[K_-]$ hold.

Let $\forall a \in Rol[K_-^*]$. Since $K_-^* \subseteq K_-$, we have $a \in Rol[K_-]$. Consequently, we obtain the inclusion $Rol[K_-^*] \subseteq Rol[K_-]$.

Now, we consider $\forall b \in Rol[K_-]$. Let $P(z) \in H[K_-](b)$ be a characteristic polynomial of degree m of the sequence b. Then $P(z) \in K_+[z]$, which implies that $(z+1)^{m-1} P(z) \in K_+^*[z]$.

Since $(z+1)^{m-1}P(z) \vdots P(z)$, we have $b \in Rol[K_-^*]$ and $(z+1)^{m-1}P(z) \in H[K_-^*](b)$. □

Lemma 3.2 *The relation* $a \in Rol[K_-^*]$ *holds if and only if* $a \in Rol[K]$ *and its minimal characteristic polynomial* $Q(z)$ *over* K *has no positive real roots. Additionally, the characteristic polynomial of* a *over* K_-^* *can be chosen of the form* $(z+1)^n Q(z)$.

Proof. We consider $a \in Rol[K_-^*]$. Since $K_-^* \subseteq K$, we have $a \in Rol[K]$. Let $q = (q_k)_{k=0}^{m-1} \in G[K_-^*](a)$ be a generating vector of the sequence $a \in Rol[K_-^*]$. The corresponding characteristic polynomial is $H_m^{[q]}(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k \in K_+^*[z]$. According to Descartes' rule of signs, the characteristic polynomial $H_m^{[q]}(z)$ has no positive real roots. Since the minimal characteristic polynomial $Q(z)$ over K is a divisor of $H_m^{[q]}(z)$, it also has no positive real roots.

Now we consider that $a \in Rol[K]$ and its minimal characteristic polynomial $Q(z)$ over K has no positive real roots. According to [6] and Lemma 3.1, there exists $n_0 \in \mathbb{N}$ such that, $\forall n > n_0$, the polynomial $(z+1)^n Q(z)$ has all its coefficients of the same sign, i.e. $(z+1)^n Q(z) \in \mathbb{R}_+^*[z]$. Since $(z+1)^n Q(z) \in K_+^*[z]$ and $(z+1)^n Q(z) \vdots Q(z)$, we have $a \in Rol[K_-^*]$ and $H_m^{[q]}(z) = (z+1)^n Q(z) \in H[K_-^*](a)$. □

Lemma 3.3 *If* $a \in Rol[K_-^*][m]$ *and* $q \in G[K_-^*][m](a)$, *then* $a \in Rol[K_+^*][m+1]$ *with characteristic polynomial* $(1-\alpha z)H_m^{[q]}(z)$, $\forall \alpha \in K$, $\alpha > \max\left\{-q_0, \max\limits_{k=0,m-2} \dfrac{q_{k+1}}{q_k}\right\}$.

Proof. Let $a \in Rol[K_-^*][m]$, $q \in G[K_-^*][m](a)$ and $\alpha > \max\left\{-q_0, \max\limits_{k=0,m-2} \dfrac{q_{k+1}}{q_k}\right\}$, where $\alpha \in K$. We have $H_m^{[q]}(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k \in K_+^*[z]$, which implies

$$(1-\alpha z)H_m^{[q]}(z) = 1 - \sum_{k=1}^{m} q_{k-1} z^k - \alpha z + \sum_{k=1}^{m} \alpha q_{k-1} z^{k+1} =$$

$$= 1-(\alpha+q_0)z+\sum_{k=2}^{m} q_{k-2}\left(\alpha-\frac{q_{k-1}}{q_{k-2}}\right)z^k+\alpha q_{m-1}z^{m+1}.$$

From the inequality

$$\alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\} > 0,$$

we obtain

$$-(\alpha+q_0)<0,\ \alpha q_{m-1}<0 \text{ and } q_{k-2}\left(\alpha-\frac{q_{k-1}}{q_{k-2}}\right)<0,\ k=\overline{2,m}.$$

Since $(1-\alpha z)H_m^{[q]}(z) \vdots H_m^{[q]}(z)$, we have $a \in Rol[K_+^*][m+1]$ and $(1-\alpha z)H_m^{[q]}(z)$. □

Theorem 3.8 *Let $a \in Rol[K]$ with minimal characteristic polynomial $Q(z)$ over K and N_Q^+ the number of positive real roots of $Q(z)$ counted with their multiplicity.*

- *If $N_Q^+ = 0$, then $a \in Rol[K_-^*] \cap Rol[K_+^*]$. In this case, $\exists n \in \mathbb{N}$ such that*

  $$H_m^{[q]}(z) = (z+1)^n Q(z) \in H[K_-^*](a)$$

  and

  $$R(z) = (1-\alpha z)H_m^{[q]}(z) \in H[K_+^*](a),$$

  $$\forall \alpha \in K,\ \alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\};$$

- *If $N_Q^+ \geq 2$, then $a \notin Rol[K_-] \cup Rol[K_+]$;*

- *If $N_Q^+ = 1$, then $a \notin Rol[K_-]$. Additionally, considering R the unique positive real root of the polynomial $Q(z)$, we have $a \notin Rol[K_+]$ in each of the following cases:*
  - *there exists a root x of the polynomial $Q(z)$ such that $|x| < R$;*
  - *$x = -R$ is a simple root of the polynomial $Q(z)$ and $Q(z)$ has at least one more negative real root $y \neq -R$;*
  - *$x = -R$ is a multiple root of the polynomial $Q(z)$;*
  - *$R > 1$ and $K \cap (0,1) = \varnothing$;*

- $R=1,\ K\cap(0,1)=\varnothing$ *and there exists a root* x *of the polynomial* $Q(z)$ *such that* $|x|\neq 1$.

Proof. Let $N_Q^+=0$. According to Lemma 3.2, we have $a\in Rol[K_-^*]$ and there exists $n\in\mathbb{N}$ such that

$$H_m^{[q]}(z)=(z+1)^n Q(z)\in H[K_-^*](a).$$

Next, by applying Lemma 3.3, we obtain $a\in Rol[K_+^*]$ and

$$R(z)=(1-\alpha z)H_m^{[q]}(z)\in H[K_+^*](a),$$

$$\forall\alpha\in K,\alpha>\max\left\{-q_0,\ \max_{k=0,m-2}\frac{q_{k+1}}{q_k}\right\}.$$

Therefore, $a\in Rol[K_-^*]\cap Rol[K_+^*]$.

In the case $N_Q^+\geq 2$ we have $N_P^+\geq 2,\ \forall P(z)\vdots Q(z)$. Therefore, according to Descartes' rule of signs, $P(z)$ has at least 2 sign changes between consecutive coefficients, which implies that $a\notin Rol[K_-]\cup Rol[K_+]$ (otherwise there would exist a polynomial $P(z)\vdots Q(z)$ with 0, respectively 1 sign changes between consecutive coefficients).

Next, we will consider $N_Q^+=1$ and R the unique positive real root of the polynomial $Q(z)$. Similarly, $N_P^+\geq 1,\ \forall P(z)\vdots Q(z)$. So, according to Descartes' rule of signs, $P(z)$ has at least 1 sign change between consecutive coefficients, which implies that $a\notin Rol[K_-]$ (otherwise there would exist a polynomial $P(z)\vdots Q(z)$ with 0 sign changes between consecutive coefficients).

Assume that $a\in Rol[K_+]$ and there exists a root x of the polynomial $Q(z)$ such that $|x|<R$. Then there exists $M\in\mathbb{N}^*$ such that $a\in Rol[K_+][M]$. Considering $p=(p_k)_{k=0}^{M-1}\in G[K_+][M](a)$, we have

$$H_M^{[p]}(z)=1-\sum_{k=1}^{M}p_{k-1}z^k\in H[K_+][M],$$

where $p_k \geq 0$, $k = \overline{0, M-1}$. Since $Q(z)$ is the minimal characteristic polynomial of a, we obtain that $H_M^{[p]}(z) \vdots Q(z)$, i.e. the positive real root R of $Q(z)$ is also a root of the polynomial $H_M^{[p]}(z)$. According to Descartes' rule of signs, this is the unique positive real root of $H_M^{[p]}(z)$. Then, every root z of the polynomial $H_M^{[p]}(z)$ satisfies the condition $|z| \geq r$, where r is the unique positive real root of the polynomial

$$T(z) = \sum_{k=1}^{M} |-p_{k-1}| z^k - 1.$$

Since $T(z) = -H_M^{[p]}(z)$, we obtain $r = R$, which implies that every root z of the polynomial $H_M^{[p]}(z)$ satisfies the condition $|z| \geq R$. Finally, since $H_M^{[p]}(z) \vdots Q(z)$, we obtain that every root z of the polynomial $Q(z)$ satisfies the condition $|z| \geq R$, which contradicts our assumption. So, if there exists a root x of the polynomial $Q(z)$ such that $|x| < R$, then $a \notin Rol[K_+]$.

Next, when $x = -R$ is a root of the polynomial $Q(z)$, taking into account that $H_M^{[p]}(z) \vdots Q(z)$, we have $H_M^{[p]}(-R) = H_M^{[p]}(R) = 0$, i.e.

$$\sum_{k=1}^{M} p_{k-1} R^k = \sum_{k=1}^{M} p_{k-1} (-R)^k.$$

As a result, we obtain

$$\sum_{k-odd,\, 1 \leq k \leq M} p_{k-1} R^k = 0,$$

which implies that $p_k = 0$, for every even number k, $0 \leq k \leq M-1$. This means that the polynomial $H_M^{[p]}(z)$ is an even function, i.e.

$$H_M^{[p]}(z) = H_M^{[p]}(-z), \ \forall z \in \mathbb{R}.$$

The existence of an additional negative real root $y \neq -R$ would mean that the number $z = -y \neq R$ is also a positive real root, which contradicts the uniqueness of the positive real root R. Also, if $x = -R$ was a multiple root of the polynomial $Q(z)$, it would be a multiple root of the polynomial $H_M^{[p]}(z)$, which would imply that

$$\frac{\partial}{\partial z} H_M^{[p]}(z)\Big|_{z=-R} = 0.$$

But this would mean that

$$\sum_{k=0}^{M-1}(k+1)p_k(-R)^k = 0$$

and, taking into account that $p_k = 0$ for every even number k, $0 \le k \le M-1$, this would imply the equality

$$\sum_{k-odd,0\le k\le M-1}(k+1)p_k R^k = 0,$$

i.e. $p_k = 0$, for every odd number k. As a result, it would be $p_k = 0$, $k = \overline{0, M-1}$, which would be impossible. Therefore, $a \notin Rol[K_+]$.

Now, we consider the last case, when $K \cap (0,1) = \varnothing$ and at least one of the following conditions is true: $R > 1$ or ($R = 1$ and there exists a root x of the polynomial $Q(z)$ such that $|x| \neq 1$). If $R \ge 1$, then

$$0 = H_M^{[p]}(R) = 1 - \sum_{k=1}^{M} p_{k-1}R^k \le 1 - p_{M-1}R^M \le 1 - R^M \le 1 - 1 = 0,$$

which may happen only when $R = 1$ and $H_M^{[p]}(z) = 1 - z^M$. However, this contradicts our assumption. Therefore, $a \notin Rol[K_+]$. □

As mentioned above, each homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{R}$, and, if its minimal characteristic polynomial over $\mathbb{R}$ does not have a transcendental root over $\mathbb{Q}$, then it is also a homogeneous linear recurrence over $\mathbb{Q}$. In this case, if $Q(z) \in \mathbb{Z}[z]$ is its minimal characteristic polynomial over $\mathbb{Q}$, then it is also a homogeneous linear recurrence over $\mathbb{Z}$.

Additionally, according to Theorem 3.8, if $Q(z)$ does not have positive real roots, then it is also a homogeneous linear recurrence over $\mathbb{N}^*$. Otherwise, if $Q(z)$ has at least two distinct positive real roots or one multiple positive real root, then it is not a homogeneous linear recurrence over $\mathbb{N}^*$.

The trickiest case is the case when $Q(z)$ has exactly one simple positive real root R. Theorem 3.8 analyzes several categories. According to it, the given homogeneous linear recurrence over $\mathbb{Z}$ is not a homogeneous linear recurrence over $\mathbb{N}^*$ at least in the following cases:

- $R > 1$;
- $R = 1$ and $Q(z)$ has at least one root $|x| \neq 1$;
- $Q(z)$ has at least one root $|x| < R$;
- $x = -R$ is a simple root of the polynomial $Q(z)$ and $Q(z)$ has at least one more negative real root $y \neq -R$;
- $x = -R$ is a multiple root of the polynomial $Q(z)$.

Theorem 3.8 does not provide necessary and sufficient conditions in this case.

3.5 Littlewood, Newman and Borwein homogeneous linear recurrences

Littlewood, Newman and Borwein polynomials were defined in [2] in the following way. A Newman polynomial represents a polynomial with all coefficients in $\{0,1\}$ and free term 1. Similarly, a Littlewood polynomial is a polynomial whose coefficients all belong to $\{-1,1\}$. Finally, the polynomials with non-zero free term and all coefficients from the set $\{-1,0,1\}$ are called Borwein polynomials.

A Littlewood or Borwein polynomial is called alternating if the signs of every two consecutive non-zero coefficients are different. It is called even-alternating when the number of non-zero coefficients is odd (there is an even number of sign changes) and odd-alternating otherwise.

Next, we introduce similar definitions for Littlewood, Newman and Borwein homogeneous linear recurrences. A homogeneous linear $m-$recurrence is called Littlewood, Newman, respectively Borwein homogeneous linear $m-$recurrence, if it has at least one Littlewood, Newman, respectively Borwein characteristic polynomial of degree m. Similarly, a homogeneous linear recurrence is Littlewood, Newman, re-

spectively Borwein, if it is a Littlewood, Newman, respectively Borwein homogeneous linear $m-$recurrence for an arbitrary m.

Additionally, a Littlewood or Borwein homogeneous linear recurrence is called alternating, if it has at least one alternating Littlewood, respectively Borwein characteristic polynomial. It is called even-alternating when the corresponding characteristic polynomial is even-alternating and odd-alternating otherwise.

Taking into account these definitions, we can conclude that $Rol[\{-1,0\}][m]$ is the set of all Newman homogeneous linear m-recurrences, $Rol[\{-1,1\}][m]$ is the set of all Littlewood homogeneous linear m-recurrences and $Rol[\{-1,0,1\}][m]$ is the set of all Borwein homogeneous linear m-recurrences. Similarly, $Rol[\{-1,0\}]$ represents the set of all Newman homogeneous linear recurrences, $Rol[\{-1,1\}]$ is the set of all Littlewood homogeneous linear recurrences and $Rol[\{-1,0,1\}]$ represents the set of all Borwein homogeneous linear recurrences.

Newman polynomials were studied in [40]. It is known that all real roots of Newman polynomials must lie in the interval $\left(-\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right)$. Additionally, in [127] several bounds for the location of complex roots of Newman polynomials were obtained. These results can be considered as necessary conditions for a homogeneous linear recurrence over K to be Newman, where $K \supseteq \{-1,0\}$. For instance, if the minimal characteristic polynomial over K has one real root outside the interval $\left(-\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right)$, then the recurrence is not Newman (since the roots of minimal characteristic polynomial are also roots of any characteristic polynomial, which cannot be Newman in this case).

Additionally, the even-alternating Borwein polynomials were mentioned in [40]. Given any positive integers $d_1 < d_2 < \ldots < d_t$ with t even, the Borwein polynomial defined as

$$Q(z) = 1 - z^{d_1} + z^{d_2} - \ldots + z^{d_t}$$

has the Newman polynomial

$$Q(z)(1+z+z^2+\ldots+z^{d_t}).$$

as a multiple. Consequently, every even-alternating Borwein homogeneous linear m-recurrence is a Newman homogeneous linear $2m$-recurrence.

The main result from [40] concerns cyclotomic polynomials that do not vanish at 1. It claims that every such polynomial is a divisor of a Newman polynomial. Also, a constructive way for determining it is given. As a consequence, every homogeneous linear recurrence, with cyclotomic polynomial not vanishing at 1 as characteristic polynomial, is a Newman homogeneous linear recurrence.

According to [41], every Newman polynomial of degree at most 8 divides some Littlewood polynomials. This means that each Newman homogeneous linear m-recurrence, with $m \le 8$, is also a Littlewood homogeneous linear recurrence.

The following theorem summarises all the above results regarding Littlewood, Newman and Borwein homogeneous linear recurrences.

Theorem 3.9 *The following statements are true:*

- *If a is a homogeneous linear recurrence over K, where $K \supseteq \{-1,0\}$, and the minimal characteristic polynomial of a over K has one real root outside the interval $\left(-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$, then the recurrence is not Newman;*
- *Every even-alternating Borwein homogeneous linear m-recurrence with characteristic polynomial of the form $Q(z)=1-z^{d_1}+z^{d_2}-\ldots+z^{d_t}$ is a Newman homogeneous linear $2m$-recurrence with $Q(z)(1+z+z^2+\ldots+z^{d_t})$ as characteristic polynomial;*
- *Each homogeneous linear recurrence, with cyclotomic polynomial not vanishing at 1 as characteristic polynomial, is a Newman homogeneous linear recurrence;*
- *Every Newman homogeneous linear m-recurrence, with $m \le 8$, is also a Littlewood homogeneous linear recurrence.*

The following theorem is a direct consequence of the corresponding theorem from [41]. It describes a necessary condition for a homogeneous linear recurrence over $\mathbb{Z}$ not to be a Littlewood homogeneous linear recurrence.

Theorem 3.10 *Consider the sequence* $a \in Rol[\mathbb{Z}]$ *with minimal characteristic polynomial* $Q(z) \in H[\mathbb{Z}](a)$. *Let* $Q^*(z)$ *be the reciprocal polynomial of* $Q(z)$, *with roots of modulus strictly greater than* 1 *labelled as* $\alpha_1, \ldots, \alpha_k$, *where* $k \geq 1$. *Suppose that there exist a positive integer* N *and a real number* $\delta \geq 0$ *with the property that, for each of the* 2^N *vectors* $b = (b_1, \ldots, b_N)$, *where* $b_1, \ldots, b_N \in \{-1, 1\}$, *there are two positive integers* $n = n(b) \leq N$ *and* $i = i(b) \leq k$ *such that*

$$(|\alpha_i| - 1)\,|\alpha_i^n + b_1\alpha_i^{n-1} + \ldots + b_n| \geq 1 + \delta.$$

Then a *is not a Littlewood homogeneous linear recurrence.*

3.6 Stability and perturbations of homogeneous linear recurrences

Let $a \in Rol[\mathbb{C}][m]$ with minimal characteristic polynomial

$$P(z) = H_m^{[p]}(z) \in H[\mathbb{C}][m](a).$$

The Jury Stability Criterion, described in [60], [128] and [82], can be used to study the locations of the roots of reciprocal polynomial $P^*(z)$ of $P(z)$ relative to the unit circle, without finding the roots. The calculations are organized as a table, where

- *the columns correspond to monomials of* $P^*(z)$, *ordered in descending order by exponent;*
- *the first row contains the coefficients of* $P^*(z)$;
- *each further even row* $2k+2$ *contains the numbers from the previous row in reverse order;*
- *each further odd row* $2k+3$ *is obtained by subtracting* α *times the previous even row from the previous odd row, where* $\alpha = \beta_{2k+2} / \beta_{2k+1}$, β_{2k+2} *is the first*

element from the previous even row $2k+2$ *and* β_{2k+1} *is the first element from the previous odd row* $2k+1$*;*

- *the table is expanded until the last row of the table contains only one non-zero element.*

Since $\beta_1 = 1 > 0$, for each negative term in the sequence $\beta_1, \beta_3, \beta_5, \ldots$ the polynomial $P^*(z)$ has one root outside the unit disc, i.e. the polynomial $P(z)$ has one root inside the unit disc. So, for stability, all these values $\beta_1, \beta_3, \beta_5, \ldots$ need to be non-negative.

An additional result, coming from [60], is the fact that we need to have at least

$$P(1) > 0,\ P(-1) > 0 \text{ and } |\, p_{m-1} \,| < 1$$

in order for all the roots of $P(z)$ to lie outside the unit disc. For instance, based on [80], this does not happen when $P(z) \in \mathbb{Z}[z]$. Instead, the homogeneous linear recurrence distributions satisfy this property.

Perturbations generated by deviations in the initial state.

Next, we consider the homogeneous linear recurrence $a \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[a]} = (a_n)_{n=0}^{m-1}$, generating vector $q \in G[\mathbb{R}][m](a)$ and the corresponding characteristic polynomial $H_m^{[q]}(z) \in H[\mathbb{R}][m](a)$. Perturbations are defined as deviations in the evolution of a, caused by small deviations in the parameters, i.e. deviations of initial state and deviations of generating vector.

Initially, we consider only deviations in initial state $I_m^{[a]}$ of the homogeneous linear recurrence a, without any change in generating vector q. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = (b_n)_{n=0}^{m-1}$ and the same generating vector $q \in G[\mathbb{R}][m](b)$, where

$$b_n = a_n + \Delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where

$$\varepsilon_n = b_n - a_n,\ n = \overline{0,\infty}.$$

We have

$$\varepsilon_n = \Delta_n,\ n = \overline{0,m-1}.$$

Also, applying Theorem 2.1, we obtain $\varepsilon \in Rol[\mathbb{R}][m]$ and $q \in G[\mathbb{R}][m](\varepsilon)$. Thus,

$$\varepsilon \in Rol[\mathbb{R}][m], q \in G[\mathbb{R}][m](\varepsilon), I_m^{[\varepsilon]} = (\Delta_n)_{n=0}^{m-1}.$$

The perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim\limits_{n\to\infty} \varepsilon_n = 0$. The convergence of ε can be studied in accordance with Section 2.6.

As a remark, the asymptotic stability of perturbation ε does not depend on deviation in initial state. Since the components of generating vector q are not changed, the characteristic roots are also unchanged. This means that the asymptotic behavior of the perturbed recurrence is exactly the same as the asymptotic behavior of the original recurrence.

The maximal impact of the perturbation ε is represented by the positive value $\varepsilon^* = \max\limits_{n=0,\infty} |\varepsilon_n|$. Even if ε is asymptotically stable, it might have a big enough maximal perturbation impact.

In order to study the maximal impact of the asymptotically stable perturbation ε, we can consider the sequence $\varepsilon^2 = (\varepsilon_n^2)_{n=0}^{\infty}$. Since $\varepsilon^2 = \varepsilon \cdot \varepsilon$ and $\varepsilon \in Rol[\mathbb{R}][m]$, we have

$$\varepsilon^2 \in Rol[\mathbb{R}], dim[\mathbb{R}][\varepsilon^2] \le (dim[\mathbb{R}][\varepsilon])^2 \le m^2.$$

Consequently, $\varepsilon^2 \in Rol[\mathbb{R}][m^2]$ and its minimal generating vector can be obtained using the rank-based order minimization method.

Next, the value

$$s = \sum_{n=0}^{\infty} \varepsilon_n^2 = G^{[\varepsilon^2]}(1)$$

can be calculated. If ξ is a random variable with distribution $p = \varepsilon^2 / s$, then its mode μ and its probability p_μ can be found using the successive search algorithm,

described in Section 4. Finally, we obtain the maximal perturbation impact

$$\varepsilon^* = \sqrt{sp_\mu}\ .$$

Perturbations generated by deviations in the generating vector.

Now, we consider only deviations in generating vector q of the homogeneous linear recurrence a, without any change in initial state $I_m^{[a]}$. In this case, the perturbed recurrence represents a new homogeneous linear recurrence $b \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[b]} = I_m^{[a]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$, where

$$r_n = q_n + \delta_n,\ n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where

$$\varepsilon_n = b_n - a_n,\ n = \overline{0,\infty}\ .$$

We obtain

$$\varepsilon \in Rol[\mathbb{R}],\ dim[\mathbb{R}](\varepsilon) \le dim[\mathbb{R}](a) + dim[\mathbb{R}](b) \le m + m = 2m.$$

Consequently, $\varepsilon \in Rol[\mathbb{R}][2m]$ and its minimal generating vector can be obtained using the rank-based order minimization method.

The perturbation $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ is considered asymptotically stable if and only if $\lim\limits_{n\to\infty} \varepsilon_n = 0$. The convergence of ε can be studied in accordance with Section 2.6.

As a remark, the perturbation ε can be asymptotically stable even if the initial recurrence a is not convergent. This happens when

$$dim[\mathbb{R}](\varepsilon) < dim[\mathbb{R}](a)$$

and all roots of the minimal characteristic polynomial of a over $\mathbb{R}$, which are not greater than 1 in absolute value, disappear from the list of all roots of the minimal characteristic polynomial of ε over $\mathbb{R}$.

Similarly to the case of deviations in the initial state, in order to study the maximal impact $\varepsilon^* = \max\limits_{n=\overline{0,\infty}} |\varepsilon_n|$ of the asymptotically stable perturbation ε, we can consider the sequence ε^2, obtaining $\varepsilon^2 \in Rol[\mathbb{R}][4m^2]$. Its minimal generating vector

can be obtained using the rank-based order minimization method too.

Next, the value

$$s = \sum_{n=0}^{\infty} \varepsilon_n^2 = G^{[\varepsilon^2]}(1)$$

can be calculated. If ξ is a random variable with distribution $p = \varepsilon^2 / s$, then its mode μ and its probability p_μ can be found using the successive search algorithm, described in [102]. Finally, we obtain the maximal perturbation impact

$$\varepsilon^* = \sqrt{sp_\mu}\ .$$

Mixed perturbations of the homogeneous linear recurrences.

Mixed perturbations are generated by both types of deviations: deviations in initial state $I_m^{[a]}$ and deviations in generating vector q of the homogeneous linear recurrence a. The perturbed recurrence represents a new homogeneous linear recurrence $c \in Rol[\mathbb{R}][m]$ with initial state $I_m^{[c]}$ and the generating vector $r = (r_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](c)$, where

$$c_n = a_n + \Delta_n, r_n = q_n + \delta_n, n = \overline{0, m-1}.$$

The perturbation is given by the sequence $\eta = (\eta_n)_{n=0}^{\infty}$, where

$$\eta_n = c_n - a_n, n = \overline{0, \infty}.$$

We can study the mixed perturbations using the previous results, obtained for individual deviations in the initial state and in the generating vector of the recurrence. The deviation in initial state and the deviation in generating vector can be performed consecutively, one by one, in the following way.

Let $b \in Rol[\mathbb{R}][m]$ be the perturbed recurrence, generated by a deviation in the initial state, i.e.

$$q = (q_n)_{n=0}^{m-1} \in G[\mathbb{R}][m](b),$$

$$b_n = a_n + \Delta_n, n = \overline{0, m-1}.$$

Its perturbation is represented by the sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$, where

$$\varepsilon_n = b_n - a_n, n = \overline{0,\infty}.$$

Next, the perturbed recurrence $c \in Rol[\mathbb{R}][m]$ is obtained from $b \in Rol[\mathbb{R}][m]$ by applying the given deviation in generating vector $q = (q_n)_{n=0}^{\infty} \in G[\mathbb{R}][m](b)$:

$$c_n = b_n, r_n = q_n + \delta_n, n = \overline{0,m-1}.$$

The corresponding perturbation is represented by the sequence $\zeta = (\zeta_n)_{n=0}^{\infty}$, where

$$\zeta_n = c_n - b_n, n = \overline{0,\infty}.$$

The mixed perturbation $\eta = (\eta_n)_{n=0}^{\infty}$ is the sum of the two perturbations in this decomposition:

$$\eta_n = c_n - a_n = (c_n - b_n) + (b_n - a_n) = \zeta_n + \varepsilon_n, n = \overline{0,\infty}.$$

Therefore, this mixed perturbation is also a homogeneous linear recurrence. Consequently, the asymptotic behavior and the maximal impact of the perturbation can be similarly studied.

4 HOMOGENEOUS LINEAR RECURRENCE DISTRIBUTIONS

In this chapter, random variables with homogeneous linear recurrence distributions are studied. Their main probabilistic characteristics are analyzed, including the distribution law, expectation, variance, standard deviation, moments, mode, and median. Numerical methods for determining these characteristics are developed based on the properties of homogeneous linear recurrences established in the previous chapters. In addition, several techniques for improving the efficiency of matrix computations are presented. Finally, a combinatorial problem concerning the cardinality of the set of numbers with a fixed number of digits and a fixed digit sum is investigated and solved.

4.1 Techniques for improving the efficiency of matrix computations

In this section, several results are presented concerning optimal methods for solving the problems of matrix multiplication, determination of the characteristic polynomial, and evaluation of matrix polynomials, which can be used in the development and analysis of the computational complexity of new algorithms for determining the limit matrix and differential matrices.

Fast matrix multiplication

The classical method for multiplying two $n \times n$ matrices (the naive algorithm) requires $O(n^3)$ elementary operations. More efficient algorithms can also be used for matrix multiplication, especially when n is sufficiently large. The first matrix multiplication algorithm of complexity $O(n^{\omega})$ with $\omega < 3$ was proposed by Strassen [149] in 1969. His algorithm had $\omega \approx 2.802$. Later, this algorithm was improved by other

researchers ([158]), leading to a series of algorithms with $\omega \approx 2,796$ (Pan 1978), $\omega \approx 2,2522$ (Bini 1981), $\omega \approx 2.479$ (Strassen 1986). The most efficient algorithms to date were developed by Coppersmith–Winograd ([30]) with $\omega < 2.376$ in 1990 and by Williams ([158]) with $\omega < 2.372873$ in 2014. As of 2014, the fastest known algorithm was due to Le Gall ([103]), developed as an improvement of the Coppersmith–Winograd algorithm, who showed that $\omega \approx 2.3728639$. Even if the asymptotic complexity continues to be slightly improved, currently the best bound on the asymptotic complexity of a matrix multiplication algorithm remains about the same ($\omega \approx 2.371339$, by Williams, Alman, Duan, Xu and Zhou in [2]).

The fastest known matrix multiplication algorithms (Coppersmith–Winograd, Williams, Le Gall) are frequently used as building blocks in other algorithms to evaluate theoretical bounds on execution time. However, with the exception of Strassen's algorithm, they are considered galactic algorithms and are not used in practice due to their advantage appearing only for very large matrices, which cannot be efficiently processed by modern hardware.

On the other hand, despite these drawbacks, there exist parallel algorithms for computing the product of two $n \times n$ matrices. D'Alberto and Nicolau studied an adapted version of the Winograd matrix multiplication algorithm in [32] while Ballard, Demmel, Holtz, Lipshitz, and Schwartz described a communication-optimal parallel algorithm for Strassen matrix multiplication in [9].

Matrix inversion

Similarly to matrix multiplication, matrix inversion is one of the fundamental operations used in mathematics and computer science. There exist several algorithms for determining the inverse of an invertible matrix: Gauss–Jordan elimination, LU decomposition, the Newton iterative method, the Cayley–Hamilton method, eigenvalue decomposition, Cholesky decomposition, the method of reciprocal basis vectors, block inversion, and others.

One of the simplest methods for inverting an invertible matrix is Gauss–Jordan elimination. According to this method, the identity matrix is appended to the right of the given matrix and, subsequently, by applying a sequence of elementary row operations, a row-echelon form is obtained. The resulting left block becomes the identity matrix, while the right block is the inverse of the given invertible matrix. If the algorithm fails to reduce the left block to the identity matrix, then the initial matrix is not invertible.

The disadvantage of this method is its complexity $O(n^3)$, which is equal to the complexity of the standard matrix multiplication algorithm. However, due to its simplicity, it can be regarded as a component of more complex algorithms, at least in cases where other parts of the algorithm have complexity greater than $O(n^3)$, since the overall complexity is not significantly affected by this method.

In contrast to Gauss–Jordan elimination, the block inversion method ([31], [18]) is a divide-and-conquer type algorithm, which allows a reduction in computational complexity due to its relation to the complexity of matrix multiplication. It was shown in [31] that the block inversion method has the same complexity as the matrix multiplication algorithm used internally. Since the complexity of the fastest known matrix multiplication algorithm is $O(n^{\omega})$, it follows that the complexity of the fastest known matrix inversion algorithm is also $O(n^{\omega})$.

Determination of the characteristic polynomial

The characteristic polynomial of a matrix can be determined using the algorithms presented in [131], [65]. In [131] , several ideas were combined to obtain a new randomized Las Vegas algorithm for computing the characteristic polynomial. The complexity of this algorithm is $O(n^{\omega})$. The deterministic algorithm of Keller–Gehrig [65] works for all inputs, and its worst-case complexity is $O(n^{\omega} lg(n))$.

Evaluation of matrix polynomials

The problem of evaluating matrix polynomials is as follows: given an $n \times n$ matrix P and a polynomial $T(z) = \sum_{k=0}^{m} a_k z^k$ with numerical coefficients, determine the matrix $T(P) = \sum_{k=0}^{m} a_k P^k$.

This problem can be solved using the standard algorithm, which recursively computes the matrices

$$P^0 = I_n,\ P^1 = P^0 P,\ P^2 = P^1 P,\ \ldots,\ P^m = P^{m-1} P,$$

performing m matrix multiplications, multiplying each matrix by the corresponding coefficient a_k, $k \in \{1, 2, \ldots, m\}$ and summing over the index k. The complexity of this algorithm is $O(mn^{\omega})$.

The $r \times r$ schema, presented in [104], shows us that only $O(r)$ matrix multiplications are required, where $r = [\sqrt{m+1}]$. This means that we store the matrices P^0, P^1, …, P^r and reuse them, together with decomposing the polynomial into a set of subpolynomials of degree less than or equal to r. Therefore, the complexity of the algorithm is $O(\max\{mn^2, rn^{\omega}\})$. Under the conditions $m \approx n$ and $\omega <= 2.5$, the execution time of this algorithm is $O(n^3)$.

For the evaluation of matrix polynomials, parallel algorithms also exist. For example, Alonso, Boratto, Peinado, Ibáñez, and Sastre evaluated matrix polynomials in [3] using GPGPU algorithms.

4.2 Probabilistic characteristics of random variables

In order to characterize discrete stochastic systems with finite sequence of states, certain random variables will be defined. For these variables, polynomial-time algorithms will be developed for determining the main probabilistic characteristics. These algorithms will be based on the generating function method and the character-

istic function method. Subsequently, these methods will be adapted to the specific features of the distributions of the given random variables.

Let ξ be a random variable. We present some fundamental notions and properties.

Definition 4.1 *The function* $F_\xi : \mathbb{R} \to [0,1]$, $F_\xi(x) = \mathbb{P}(\xi < x)$, *is called the distribution function of the arbitrary random variable* ξ.

The random variable ξ is fully described by its distribution function $F_\xi(x)$. The expectation $\mathbb{E}(g(\xi))$ of the function $g(\xi)$ is computed using a Lebesgue-Stieltjes integral

$$\mathbb{E}(g(\xi)) = \int_{-\infty}^{+\infty} g(x) dF_\xi(x).$$

It is well known that the expectation is a linear function.

Definition 4.2 *The value* $\nu_n(\xi) = \mathbb{E}(\xi^n)$ *is called the moment of order* n *of the random variable* ξ.

The expectation $\mathbb{E}(\xi)$, the variance $\mathbb{V}(\xi)$ and the standard deviation $\sigma(\xi)$ are expressed in terms of moments of order 1 and 2:

$$\mathbb{E}(\xi) = \nu_1(\xi);\ \mathbb{V}(\xi) = \nu_2(\xi) - \nu_1^2(\xi);\ \sigma(\xi) = \sqrt{\mathbb{V}(\xi)}. \tag{4.1}$$

Definition 4.3 *The function* $\Phi_\xi(t) = \mathbb{E}(e^{it\xi})$ *is called the characteristic function of* ξ.

The main properties of the characteristic function were presented in [145]. These properties are:

- *The characteristic function exists for every random variable* ξ *and*

$$\left|\Phi_\xi(t)\right| \le 1;$$

- *If the random variables* $\xi_1, \xi_2, \ldots, \xi_n$ *are independent, then*

$$\Phi_{\sum_{j=1}^{n} \xi_j}(t) = \prod_{j=1}^{n} \Phi_{\xi_j}(t);$$

- *If* $\exists \mathbb{E}(|\xi|^n) < \infty$, *then* $\Phi_\xi^{(k)}(0) = i^k \mathbb{E}(\xi^k)$, $k = \overline{0,\infty}$;
- $\overline{\Phi_\xi(t)} = \Phi_\xi(-t) = \Phi_{-\xi}(t)$.

Also, the following inversion theorem holds:

Theorem 4.1 *For all points of continuity* x_1 *and* x_2 *of the function* $F_\xi(x)$, *the following formula holds:*

$$F_\xi(x_2) - F_\xi(x_1) = \frac{1}{2\pi} \lim_{y \to 0} \int_{-\infty}^{+\infty} \frac{e^{-itx_1} - e^{-itx_2}}{it} \Phi_\xi(t) e^{-t^2 y^2} dt .$$

As can be observed, the inversion theorem is an important property of the random variables. It allows us to theoretically determine the distribution function based on the characteristic function of the random variable ξ.

If the random variable ξ is discrete and it has the possible values $x_1, x_2, \ldots, x_\omega$, where ω is finite or infinite and

$$p_j = \mathbb{P}(\xi = x_j), \quad j = \overline{1,\omega},$$

is the probability that ξ takes the respective value x_j, then the variable ξ is fully described by its distribution

$$((x_1, p_1), (x_2, p_2), \ldots, (x_\omega, p_\omega)) .$$

The formula for computing the expectation $\mathbb{E}(g(\xi))$ of the function $g(\xi)$ becomes

$$\mathbb{E}(g(\xi)) = \sum_{j=1}^{\omega} g(x_j) p_j .$$

Consider the case where ξ is a non-negative integer random variable. The variable ξ is a discrete random variable with distribution $((x_j, q_j))_{j=1}^{\omega}$, where $x_j \in \mathbb{N}$, $j = \overline{1,\omega}$. Moreover, ξ can be regarded as a random variable with the set of possible values $\mathbb{N}$, and the probability that ξ takes the value $n \in \mathbb{N}$ is p_n, which is computed as follows: if $\exists x_{j_0} \in \{x_1, x_2, \ldots, x_\omega\}$ such that $x_{j_0} = n$, then $p_n = q_{j_0}$, otherwise $p_n = 0$. Therefore, the variable ξ has the distribution

$$((0, p_0), (1, p_1), (2, p_2), \ldots, (n, p_n), \ldots).$$

The formula for the expectation becomes

$$\mathbb{E}(g(\xi)) = \sum_{n=0}^{\infty} g(n) p_n .$$

Consider the sequence $p = (p_n)_{n=0}^{\infty}$ of the probabilities of the distribution of the non-negative integer random variable ξ. Denote $p = rep(\xi)$. The generating function of the random variable ξ is $G_\xi(z) = G^{[p]}(z)$. The generating function has the following properties ([145]):

- *The function* $G_\xi(z)$ *is absolutely convergent for* $|z| \le 1$;
- *If* $|z| \le 1$, *then* $|G_\xi(z)| \le 1$;
- $G_\xi(z) = \mathbb{E}(z^\xi)$; $G_\xi(1) = 1$;
- $G_\xi^{(k)}(1) = \mathbb{E}([\xi]_k)$, $\forall k \ge 1$, *where* $[\xi]_k = \prod_{j=0}^{k-1} (\xi - j)$;
- $p_n = \frac{1}{n!} G_\xi^{(n)}(0)$, $n = \overline{0, \infty}$;
- $\Phi_\xi(t) = G_\xi(e^{it})$;
- *If the random variables* $\xi_1, \xi_2, \ldots, \xi_n$ *are independent, then*

$$G_{\sum_{j=1}^{n} \xi_j}(z) = \prod_{j=1}^{n} G_{\xi_j}(z).$$

The main probabilistic characteristics of random variables are the distribution law, the expectation, the variance, the standard deviation, and the moments. For the non-negative integer random variables, the generating function allows us to determine all these characteristics.

We have

$$[\xi]_k = \sum_{j=1}^{k} s(k, j) \xi^j \; ; \; \xi^k = \sum_{j=1}^{k} S(k, j) [\xi]_j \; , \; \forall k \ge 1 ,$$

where $s(k,j)$ are the Stirling numbers of the first kind, determined by recurrences

$$s(n,0)=0\,;\ s(n,n)=1\,;\ s(n+1,k)=s(n,k-1)-ns(n,k)\,,\ k=\overline{1,n-1}\,, \tag{4.2}$$

and $S(k,j)$ are the Stirling numbers of the second kind, computed as follows:

$$S(n,1)=S(n,n)=1\,;\ S(n+1,k)=S(n,k-1)+kS(n,k)\,,\ k=\overline{2,n-1}\,. \tag{4.3}$$

Using the above properties of the generating function, we obtain the formulas:

$$\nu_k(\xi)=G_\xi^{(k)}(1)-\sum_{j=1}^{k-1}s(k,j)\nu_j(\xi)\,,\ \forall k\geq 1\,, \tag{4.4}$$

$$\nu_k(\xi)=\sum_{j=1}^{k}S(k,j)G_\xi^{(j)}(1)\,,\ \forall k\geq 1\,, \tag{4.5}$$

$$\mathbb{E}(\xi)=G'_\xi(1)\,;\ \mathbb{V}(\xi)=G''_\xi(1)+G'_\xi(1)-(G'_\xi(1))^2\,;\ \sigma(\xi)=\sqrt{\mathbb{V}(\xi)}\,. \tag{4.6}$$

Let $T(\xi)=\sum_{k=0}^{m}a_k\xi^k\in\mathbb{R}[\xi]$. Having the moments $\nu_k(\xi)$, $k=\overline{1,m}$, we can determine the expectation $\mathbb{E}(T(\xi))$ using the formula

$$\mathbb{E}(T(\xi))=a_0+\sum_{k=1}^{m}a_k\nu_k(\xi)\,.$$

Two important characteristics are the dominant value (the most probable value) and the median. The following definition recalls these notions.

Definition 4.4 *Let ξ be a non-negative integer random variable with distribution $a=rep(\xi)$. Each number μ, for which $a_\mu=\max_{n\in\mathbb{N}}a_n$, is called a dominant value and the non-negative integer number m_0, which satisfies the inequalities*

$$\mathbb{P}(\xi<m_0)<\frac{1}{2}\leq\mathbb{P}(\xi\leq m_0)\,,$$

equivalent to

$$\sum_{k=0}^{m_0-1}a_k<\frac{1}{2}\leq\sum_{k=0}^{m_0}a_k\,,$$

is called the median.

Theorem 4.2 *The median m_0 of the non-negative integer random variable ξ can be determined using the successive search algorithm.*

Proof. If $\sigma(\xi)=0$, then the variable ξ is a constant, hence the dominant value and the median are equal to the given constant. Next, we consider the case $\sigma(\xi)\neq 0$.

For determining the median, the successive search algorithm can be applied. This algorithm can be described in the following way: the initial value $\psi_0 = a_0$ is fixed and, at each step $n=1,2,3,\ldots$, the value $\psi_n = \psi_{n-1} + a_n$ is determined, until the condition $\psi_n \geq \frac{1}{2}$ is satisfied for the first time. The number $m_0 = n$, corresponding to the final step, when execution is complete, represents the median of the given random variable. The correctness of the algorithm follows from the relation

$$\psi_n = \mathbb{P}(\xi \leq n) = \sum_{k=0}^{n} a_k \ , \ \forall n \in \mathbb{N}$$

and from the fact that the sequence $(\psi_n)_{n=0}^{\infty}$ is

monotonically increasing. □

Theorem 4.3 *The number of iterations of the successive search algorithm, which determines the median of the non-negative integer random variable ξ, is bounded by*

$$N(\xi) = [\sigma(\xi)\sqrt{2} + \mathbb{E}(\xi)].$$

Proof. The truth of this theorem follows from Chebyshev's inequality. Indeed,

$$\mathbb{P}(\xi \geq \varepsilon + \mathbb{E}(\xi)) = \mathbb{P}(\xi - \mathbb{E}(\xi) \geq \varepsilon) \leq \mathbb{P}(\left|\xi - \mathbb{E}(\xi)\right| \geq \varepsilon) \leq \frac{\sigma^2(\xi)}{\varepsilon^2}, \ \forall \varepsilon > 0 \ \Leftrightarrow$$

$$\mathbb{P}(\xi \leq [\varepsilon + \mathbb{E}(\xi)]) \geq \mathbb{P}(\xi < \varepsilon + \mathbb{E}(\xi)) = 1 - \mathbb{P}(\xi \geq \varepsilon + \mathbb{E}(\xi)) \geq 1 - \frac{\sigma^2(\xi)}{\varepsilon^2}.$$

Let $\frac{\sigma^2(\xi)}{\varepsilon^2} = \frac{1}{2}$, i.e. $\varepsilon = \sigma(\xi)\sqrt{2} > 0$. The above inequality takes the form

$$\mathbb{P}(\xi \leq N(\xi)) \geq \frac{1}{2},$$

where $N(\xi) = [\sigma(\xi)\sqrt{2} + \mathbb{E}(\xi)]$. Therefore, we obtain the inequality $m_0 \leq N(\xi)$. □

Theorem 4.4 *The dominant value* μ *of the non-negative integer random variable* ξ *can be determined using the successive search algorithm. The number of iterations is bounded by*

$$n_s(\xi) = \left[\mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}} \right],$$

where s *is the minimal non-negative integer number with* $a_s > 0$.

Proof. For determining the dominant value, the successive search algorithm can be also applied. Consider the number s as the minimal non-negative integer value that satisfies the inequality $a_s > 0$. We have already obtained above the formula

$$\mathbb{P}(\xi \geq \varepsilon + \mathbb{E}(\xi)) \leq \frac{\sigma^2(\xi)}{\varepsilon^2}, \ \forall \varepsilon > 0.$$

Let $\varepsilon = \dfrac{\sigma(\xi)}{\sqrt{a_s}} > 0$. We obtain

$$\mathbb{P}\left(\xi \geq \left[\mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}} \right] + 1 \right) \leq \mathbb{P}\left(\xi \geq \mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}} \right) \leq a_s.$$

Therefore,

$$\mathbb{P}(\xi \geq n_s(\xi) + 1) \leq a_s,$$

where

$$n_s(\xi) = \left[\mathbb{E}(\xi) + \frac{\sigma(\xi)}{\sqrt{a_s}} \right],$$

which implies $a_k \leq a_s$, $\forall k > n_s(\xi)$, i.e. the following formula holds:

$$a_\mu = \max_{n \in \mathbb{N}} a_n = \max_{s \leq n \leq n_s(\xi)} a_n.$$

In this way, we theoretically justified the successive search algorithm for determining the dominant value. Therefore, firstly the non-negative integer number s is searched, followed by the computation of the value $n_s(\xi)$. Finally, the search for the dominant value μ continues based on the above formula. □

Example 4.1 *Consider a non-negative integer random variable* ξ *with geometric distribution of parameter* $p_0 = \mathbb{P}(\xi = 0) = \frac{1}{3}$. *Let* $T(\xi) = \sum_{n=0}^{5} (\mathrm{n}+1)\xi^n$. *Determine the expectation* $\mathbb{E}(\mathrm{T}(\xi))$ *and find the median and the dominant value of the random variable* ξ.

Solution. From the conditions of the problem, we have

$$p_n = \mathbb{P}(\xi = n) = \frac{1}{3}\left(\frac{2}{3}\right)^n, \ n = \overline{0,\infty}.$$

Next, we obtain the generating function

$$G_\xi(z) = \sum_{n=0}^{\infty} p_n z^n = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n z^n = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}z\right)^n = \frac{1}{3}\left(1-\frac{2}{3}z\right)^{-1} = (3-2\,\mathrm{z})^{-1}.$$

It allows us to compute $\nu_k(\xi)$, $k = \overline{1,5}$, obtaining the results presented in Table 4.1.

k	$G_\xi^{(k)}(z)$	$G_\xi^{(k)}(1)$	$(S(k,j), j = \overline{1,k})$	$\nu_k(\xi)$
1.	$2(3-2z)^{-2}$	2	(1)	2
2.	$8(3-2z)^{-3}$	8	(1,1)	10
3.	$48(3-2z)^{-4}$	48	(1,3,1)	74
4.	$384(3-2z)^{-5}$	384	(1,7,6,1)	730
5.	$3840(3-2z)^{-6}$	3840	(1,15,25,10,1)	9002

Table 4.1 Determination of the moments of the random variable ξ

Therefore, we have the following probabilistic characteristics:

$$\mathbb{E}(T(\xi)) = 1 + \sum_{n=1}^{5} (\mathrm{n}+1)\nu_n(\xi) = 57993.$$

$$\mathbb{E}(\xi) = \nu_1(\xi) = 2,$$

$$\mathbb{V}(\xi) = \nu_2(\xi) - (\nu_1(\xi))^2 = 10 - 4 = 6,$$

$$\sigma(\xi)=\sqrt{\mathbb{V}(\xi)}=\sqrt{6}\,.$$

Since $p_0>0$, we obtain

$$N(\xi)=[\sigma(\xi)\sqrt{2}+\mathbb{E}(\xi)]=[\sqrt{6}\sqrt{2}+2]=[2\sqrt{3}+2]=5\,,$$

$$n_0(\xi)=\left[\mathbb{E}(\xi)+\frac{\sigma(\xi)}{\sqrt{p_0}}\right]=\left[2+\frac{\sqrt{6}}{\sqrt{\frac{1}{3}}}\right]=[2+3\sqrt{2}]=6\,.$$

Next, according to the successive search algorithm, since

$$\psi_0=p_0=\frac{1}{3}<\frac{1}{2},$$

$$\psi_1=p_0+p_1=\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}=\frac{5}{9}\geq\frac{1}{2},$$

we obtain the median $m_0=1\leq 5$ and the dominant value $p_\mu=\max\limits_{0\leq n\leq 6}p_n=p_0=\frac{1}{3}$. □

4.3 Numerical differentiation of the rational fractions and composite functions

Consider $\forall A_0(z),\ H_0(z)\in\mathbb{C}[z]$. Let $G(z)=\dfrac{A_0(z)}{H_0(z)}$ be a rational fraction. By mathematical induction, we have the formula $G^{(k)}(z)=\dfrac{A_k(z)}{H_k(z)}$, $\forall k\in\mathbb{N}$, where the polynomials $A_k(z)$ and $H_k(z)$ are determined using the relations

$$\begin{aligned}&A_{j+1}(z)=A_j'(z)H_0(z)-(j+1)A_j(z)H_0'(z)\,;\\&H_{j+1}(z)=H_0(z)H_j(z)\,,\ j=0,1,2,\ldots.\end{aligned}\tag{4.7}$$

Consider $\forall x=(x_k)_{k=0}^{s}\in\mathbb{C}^{s+1}$, $\forall y=(y_j)_{j=0}^{t}\in\mathbb{C}^{t+1}$. Also, we consider the polynomials with corresponding coefficients:

$$X(z)=\sum_{k=0}^{s}x_kz^k\in\mathbb{C}[z]\,;\ Y(z)=\sum_{j=0}^{t}y_jz^j\in\mathbb{C}[z].$$

Let us introduce the following operations:

- $X(z)|z=x$; $x|\alpha = X(\alpha)$, $\forall \alpha \in \mathbb{C}$;
- $\Delta_0(x) = \sum_{k=0}^{s} x_k$; $\Delta_j(x) = \sum_{k=1}^{s} k^j x_k$, $\forall j \in \mathbb{N}^*$;
- $[x] = (x_1, 2x_2, \ldots, sx_s)$;
- $x^{[t]} = (x, 0, 0, \ldots, 0) \in \mathbb{C}^{t+1}$; $x_{[t]} = (0, 0, \ldots, 0, x) \in \mathbb{C}^{t+1}$, $\forall t \in \mathbb{N}$.

  If $t < s$, then the last components of the vector x are eliminated in $x^{[t]}$ and the first components are eliminated in $x_{[t]}$;
- $x \oplus y = x^{[r]} + y^{[r]}$; $x \Theta y = x^{[r]} - y^{[r]}$, where $r = \max\{s, t\}$;
- $x \otimes y = (z_i)_{i=0}^{s+t}$, where $z_i = \sum_{k+j=i;\ 0 \le k \le s;\ 0 \le j \le t} x_k y_j$, $i = \overline{0, s+t}$.

It is easy to observe the following properties:

- $\Delta_0(x) = x|1$;
- $X'(z)|z = [X(z)|z]$; $(\alpha X(z))|z = \alpha (X(z)|z)$, $\forall \alpha \in \mathbb{C}$;
- $(X(z)|z) \oplus (Y(z)|z) = (X(z) + Y(z))|z$;
- $(X(z)|z)\, \Theta\, (Y(z)|z) = (X(z) - Y(z))|z$;
- $(X(z)|z) \otimes (Y(z)|z) = (X(z)Y(z))|z$.

  Denote

$$\alpha_j = A_j(z)|z\ ,\ \chi_j = H_j(z)|z\ ,\ j = 0, 1, 2, \ldots .$$

The recurrence relations (4.7) become:

$$\begin{aligned} \alpha_{j+1} &= ([\alpha_j] \otimes \chi_0)\, \Theta\, (j+1)(\alpha_j \otimes [\chi_0]) ; \\ \chi_{j+1} &= \chi_0 \otimes \chi_j\ ,\ j = 0, 1, 2, \ldots , \end{aligned} \tag{4.8}$$

and the formula for the kth derivative of the function $G(z)$ takes the form

$$G^{(k)}(z) = \frac{\alpha_k | z}{\chi_k | z} . \tag{4.9}$$

In this way, we have provided a theoretical foundation for the following algorithm for numerical differentiation of the rational fractions:

Algorithm 4.1

Input: the numerical value $z_0 \in \mathbb{C}$, *the coefficients* α_0 *of the numerator and the coefficients* χ_0 *of the denominator of the rational fraction* $G(z)$;

Output: the values $G^{(k)}(z_0)$, $k = \overline{1,n}$;

At each step k, $k = \overline{1,n}$, *the following two steps are executed:*

1) *Determine the vectors* α_k *and* χ_k *using the formula (4.8);*

2) *Determine the value of* $G^{(k)}(z_0)$ *using the formula (4.9).*

For the derivatives $G^{(k)}(z_0)$, $k = 1,2,\ldots$, to exist, it is necessary and sufficient that the denominator $H_0(z)$ does not vanish at the point $z = z_0$, i.e. $\chi_0 | z_0 \neq 0$.

Let us consider the function $h(x) = f(g(x))$. We define the functions

$$B_\alpha(g'(x), g''(x), \ldots, g^{(n)}(x)) = \alpha_0 \prod_{j=1}^{n} (g^{(j)}(x))^{\alpha_j},$$

$$\forall \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}.$$

By mathematical induction, it is proved that the function $h^{(n)}(x)$ can be represented in the form

$$h^{(n)}(x) = \sum_{k=1}^{n} R_{\Lambda_{n,k}}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)) f^{(k)}(g(x)), \qquad (4.10)$$

where

$$R_{\Lambda_{n,k}}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)) =$$

$$= \sum_{\alpha \in \Lambda_{n,k}} B_\alpha(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)). \qquad (4.11)$$

The sets $\Lambda_{n,k}$, that occur in these relations, will be constructed below.

By differentiating the formula (4.10), we obtain

$$h^{(n+1)}(x)=\sum_{k=1}^{n+1} f^{(k)}(g(x))\left(\sum_{j=1}^{n+1-k} U_{n,\,k,\,j}(x)+V_{n,\,k-1}(x)\right), \tag{4.12}$$

where

$$U_{n,k,j}(x)=g^{(j+1)}(x)\frac{\partial R_{\Lambda_{n,\,k}}}{\partial g^{(j)}(x)}=$$

$$=\sum_{\alpha\in\Lambda_{n,\,k}} \alpha_0\alpha_j (g^{(j)}(x))^{\alpha_j-1}(g^{(j+1)}(x))^{\alpha_{j+1}+1}\cdot \prod_{r=\overline{1,\,n+1-k};\; r\notin\{j,\;j+1\}} (g^{(r)}(x))^{\alpha_r}=$$

$$=R_{\Lambda^{(j)}_{n+1,\,k}}(g'(x),g''(x),\ldots,g^{(n+2-k)}(x)), \tag{4.13}$$

$$V_{n,0}(x)=0;$$

$$V_{n,k}(x)=g'(x)R_{\Lambda_{n,k}}=R_{\Lambda^{*}_{n+1,k}}(g'(x),g''(x),\ldots,g^{(n+1-k)}(x)),\ k=\overline{1,n}, \tag{4.14}$$

and the sets $\Lambda^{(j)}_{n+1,\,k}$ and $\Lambda^{*}_{n+1,\,k}$, $k=\overline{1,n}$, $j=\overline{1,n+1-k}$, are described as follows:

$$\Lambda^{(j)}_{n+1,\,k}=\Big\{\beta=(\beta_r)_{r=0}^{n+2-k}\in\mathbb{N}^{n+3-k}\ \Big|$$

$$\alpha\in\Lambda_{n,\,k},\ \alpha_j\neq 0,\ \alpha_{n+2-k}=0;\ \beta_0=\alpha_0\alpha_j,\ \beta_j=\alpha_j-1,\ \beta_{j+1}=\alpha_{j+1}+1;$$

$$\beta_r=\alpha_r,\ r=\overline{1,\ n+2-k},\ r\notin\{j,\ j+1\}\Big\}, \tag{4.15}$$

$$\Lambda^{*}_{n+1,\,k}=\{\beta=(\alpha_0,\alpha_1+1,\alpha_2,\alpha_3,\ldots,\alpha_{n+1-k})\ |\ \alpha=(\alpha_r)_{r=0}^{n+1-k}\in\Lambda_{n,\,k}\}. \tag{4.16}$$

Next, we define the addition of subsets of the set $\mathbb{N}^{n+1}$, $\forall n\in\mathbb{N}$:

$$A+B=\Big\{\beta=(\beta_r)_{r=0}^{n}\in\mathbb{N}^{n+1}\ |$$

$$\exists\alpha_0:\gamma_{\alpha_0,\beta}=(\alpha_0,\beta_1,\beta_2,\ldots,\beta_n)\in A\cup B,\beta_0=\sum_{\gamma_{\alpha_0,\,\beta}\in A\,\cup B}\alpha_0\Big\}.$$

The following property holds:

$$R_{A+B}(g'(x),g''(x),\ldots,g^{(n)}(x))=(R_A+R_B)(g'(x),g''(x),\ldots,g^{(n)}(x)). \tag{4.17}$$

By substituting the relations (4.13) and (4.14) in (4.12) and using the property (4.17), we obtain

$$h^{(n+1)}(x)=\sum_{k=1}^{n+1}R_{\Lambda_{n+1,k}}(g'(x),g(x),\ldots,g^{(n+2-k)}(x))f^{(k)}(g(x)),$$

where

$$\Lambda^{*}_{n+1,\,0}=\emptyset;\ \Lambda_{n+1,\,k}=\sum_{j=1}^{n+1-k}\Lambda^{(j)}_{n+1,\,k}+\Lambda^{*}_{n+1,\,k-1},\ k=\overline{1,\ n+1}. \tag{4.18}$$

Since $h'(x)=g'(x)f'(g(x))$, we have $\Lambda_{1,1}=\{(1,1)\}$.

The algorithm for numerical differentiation of the composite function is described in the following way:

Algorithm 4.2

Input: the numerical values $x_0\in\mathbb{C}$, $g^{(n)}(x_0)$, $f^{(n)}(g(x_0))$, $n=\overline{1,N}$;

Output: the values $h^{(n)}(x_0)$, $n=\overline{1,N}$, *where* $h(x)=f(g(x))$;

1) *Fix the set* $\Lambda_{1,\,1}=\{(1,1)\}$;
2) *For each* $n=\overline{1,N-1}$, $k=\overline{1,n}$, *perform the following steps:*
   a) *Determine the sets* $\Lambda^{(j)}_{n+1,\,k}$, $j=\overline{1,n+1-k}$, *using the formula (4.15);*
   b) *Determine the set* $\Lambda^{*}_{n+1,\,k}$ *by applying the relation (4.16);*
   c) *Obtain the set* $\Lambda_{n+1,\,k}$ *according to the formula (4.18);*
3) *By using (4.11), determine the values*

$$R_{\Lambda_{n,k}}(g'(x_0),g''(x_0),\ldots,g^{(n+1-k)}(x_0)),\ n=\overline{1,N},\ k=\overline{1,n},$$

4) *Determine the value* $h^{(n)}(x_0)$ *according to the formula (4.10).*

Example 4.2 *Let* $f:\mathbb{R}\to\mathbb{R}$, $f(z)=\dfrac{z-1}{z^2+1}$. *Find* $f^{(k)}(1)$, $k=\overline{1,3}$.

Solution. By applying Algorithm 4.1, we obtain:

k	α_k	χ_k	$f^{(k)}(1)$
0	(-1, 1)	(1, 0, 1)	0
1	(1, 2, -1)	(1, 0, 2, 0, 1)	$\frac{1}{2}$
2	(2, -6, -6, 2)	(1, 0, 3, 0, 3, 0, 1)	-1
3	(-6, -24, 36, 24, -6)	(1, 0, 4, 0, 6, 0, 4, 0, 1)	$\frac{3}{2}$

Table 4.2 Numerical differentiation of rational fraction

Example 4.3 *Let* $f(x)=e^x$, $g(x)=x^3$ *and* $h(x)=f(g(x))$. *Find* $h^{(n)}(1)$, $n=\overline{1,3}$.

Solution. By applying Algorithm 4.2, we have:

k / n	0	1	2	3
1	$\Lambda_{10}^{*}=\varnothing$	$\Lambda_{11}=\{(1,1)\}$		
2	$\Lambda_{20}^{*}=\varnothing$	$\Lambda_{21}^{(1)}=\{(1,0,1)\}$, $\Lambda_{21}^{*}=\{(1,2)\}$, $\Lambda_{21}=\{(1,0,1)\}$	$\Lambda_{22}=\{(1,2)\}$	
3	$\Lambda_{30}^{*}=\varnothing$	$\Lambda_{31}^{(1)}=\varnothing$, $\Lambda_{31}^{(2)}=\{(1,0,0,1)\}$, $\Lambda_{31}^{*}=\{(1,1,1)\}$, $\Lambda_{31}=\{(1,0,0,1)\}$	$\Lambda_{32}^{(1)}=\{(2,1,1)\}$, $\Lambda_{32}^{*}=\{(1,3)\}$, $\Lambda_{32}=\{(3,1,1)\}$	$\Lambda_{33}=\{(1,3)\}$

Table 4.3 Numerical differentiation of the composite function

At the end, for the current problem, we obtain

n	$g^{(n)}(1)$	$f^{(n)}(g(1))$	$h^{(n)}(1)$
0	1	e	$f(g(1))=e$
1	3	e	$g'(1)\,\mathrm{f}'(g(1))=3e$
2	6	e	$g''(1)f'(g(1))+(\mathrm{g}'(1))^2 f''(g(1))=15e$
3	6	e	$g^{(3)}(1)f'(g(1))+3g'(1)\,g''(1)\,f''(g(1))+$ $+(g'(1))^3 f^{(3)}(g(1))=87e$

Table 4.4 Final results for numerical differentiation of composite function

4.4 Homogeneous linear recurrence distributions

In this section, we study the homogeneous linear recurrence distributions. The following theorem holds:

Theorem 4.5 *If* ξ *is a random variable with distribution* $a=rep(\xi)\in Rol[\mathbb{C}]$, *then* $a\in Rol[\mathbb{R}]$ *and* $\dim[\mathbb{R}](a)=\dim[\mathbb{C}](a)$.

Proof. Let ξ be a discrete random variable with homogeneous linear recurrence distribution $a=rep(\xi)\in Rol[\mathbb{C}]$ of dimension $\dim[\mathbb{C}](a)=m$ and generating vector $q\in G[\mathbb{C}][m](a)$. Since $a=rep(\xi)$, we have

$$a_n=\mathbb{P}(\xi=n)\in[0;1],\ \forall n\in\mathbb{N}$$

and $\sum_{n=0}^{\infty}a_n=1$, hence $a\subseteq\mathbb{R}$ and a is not identically zero. By applying Theorem 2.26, we have $q\in\mathbb{R}^m$ and, consequently, $a\in Rol[\mathbb{R}][m]$, which implies the inequality $\dim[\mathbb{R}](a)\le m$. On the other hand, we have

$$\dim[\mathbb{R}](a)\ge\dim[\mathbb{C}](a)=m.$$

Therefore, $a\in Rol[\mathbb{R}]$ and $\dim[\mathbb{R}](a)=\dim[\mathbb{C}](a)=m$. □

Theorem 4.6 *If* ξ *is a random variable with distribution* $a=rep(\xi)\in Rol[\mathbb{C}][m]$, $q\in G[\mathbb{C}][m](a)$ *and* $H_m^{[q]}(1)=0$, *then* $\dim[\mathbb{R}](a)\ne m$ *and the root* $z=1$ *is fictitious.*

Proof. Since $a = rep(\xi)$, it follows that $\sum_{n=0}^{\infty} a_n = 1$ and, therefore, we have $\lim_{n\to\infty} a_n = 0$, i.e. a is convergent and $H_m^{[q]}(1) = 0$. According to the convergence theorem for the homogeneous linear recurrences, we have $0 = \lim_{n\to\infty} a_n$ is a component of the vector $x = I_m^{[a]}((B^{[a]})^T)^{-1}$, corresponding to the root $z = 1$. If we apply the order minimization method by eliminating the characteristic zeros, we obtain that $z = 1$ is a fictitious root, i.e. $\dim[\mathbb{C}](a) \neq m$. From Theorem 4.5, we have

$$\dim[\mathbb{R}](a) = \dim[\mathbb{C}](a) \neq m$$

and $z = 1$ is a fictitious root, i.e. $a \in Rol[\mathbb{R}][m-1]$ and

$$\frac{H_m^{[q]}(z)}{z-1} \in H[\mathbb{R}][m-1](a).$$ □

Theorem 4.5 and Theorem 4.6 allow us to consider only homogeneous linear recurrence distributions over $\mathbb{R}$. Moreover, $z = 1$ is not a root of the minimal characteristic polynomial.

We consider a nonnegative integer random variable ξ with distribution $a = rep(\xi) \in Rol[\mathbb{R}][m]$. Let $q \in G[\mathbb{R}][m](a)$ be a generating vector of the sequence a. We also assume that $H_m^{[q]}(1) \neq 0$, i.e. $z = 1$ is not a root of the characteristic polynomial $H_m^{[q]}(z)$. If this condition is not satisfied, we use Horner's scheme to divide $H_m^{[q]}(z)$ by binomial $(1-z)$ as many times as possible. We thus obtain a polynomial that, according to the previously proved properties, is also a unit characteristic polynomial for the distribution of ξ and, unlike the original polynomial $H_m^{[q]}(z)$, does not vanish at $z = 1$.

Since $a = rep(\xi) \in Rol[\mathbb{R}][m]$, the generating function of the sequence a is a regular rational fraction of the form

$$G^{[a]}(z)=\frac{G_m^{[a]}(z)-z\sum\limits_{k=0}^{m-1}q_k z^k G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Therefore, the generating function of the random variable ξ is

$$G_\xi(z)=\frac{A_0(z)}{H_0(z)},$$

where

$$\begin{gathered}A_0(z)=G_m^{[a]}(z)-z\sum_{k=0}^{m-1}q_k z^k G_{m-1-k}^{[a]}(z)\in\mathbb{R}[z];\\ H_0(z)=H_m^{[q]}(z)\in\mathbb{R}[z];\ q\in G[\mathbb{R}][m](a)\end{gathered}\tag{4.19}$$

and the conditions $H_0(1)\neq 0$ and $\deg(A_0(z))<\deg(H_0(z))=m$ are satisfied.

Let $\alpha_0=A_0(z)\,|\,z$ and $\chi_0=H_0(z)\,|\,z$. By using the relations (4.19), we obtain

$$\chi_0=(1,-q_0,-q_1,\ldots,-q_{m-1}),\tag{4.20}$$

$$\alpha_0=I_m^{[a]}\Theta\sum_{k=0}^{m-2}q_k(\delta_{k+1}\otimes I_{m-1-k}^{[a]}),\tag{4.21}$$

where $\delta_s=(0,0,\ldots,0,1)\in\mathbb{C}^{s+1}$, $\forall s\in\mathbb{N}$. Therefore, we can determine the values $G_\xi^{(k)}(1)$, $k=1,2,\ldots$, using Algorithm 4.1 for $z_0=1$. In this way, the probabilistic characteristics of the random variable with homogeneous linear recurrence distribution are obtained by applying the following algorithm:

Algorithm 4.3

Input: $q\in G[\mathbb{R}][m](a)$, $I_m^{[a]}\in\mathbb{R}^m$, *where* $a=rep(\xi)\in Rol[\mathbb{R}][m]$;

Output: $\mathbb{E}(\xi)$, $\mathbb{V}(\xi)$, $\sigma(\xi)$, $\nu_k(\xi)$, $k=\overline{1,n}$, $n\geq 2$;

1) Determine χ_0 *according to the formula (4.20);*

2) Obtain the vector $\mu=(\mu_k)_{k=0}^{m-1}$ *according to the recurrence formula*

$$\mu_{m-1}=-(\chi_0)_m,\ \mu_k=\mu_{k+1}-(\chi_0)_{k+1},\ k=m-2,m-3,\ldots,0;$$

3) *Compute* $\Delta_0(\chi_0)=1-\mu_0$. *If* $\Delta_0(\chi_0)=0$, *then* $\chi_0:=\mu$, $m:=m-1$ *and go to the step 2), otherwise* $q:=-(\chi_0)_{[m-1]}$ *and go to the step 4);*

4) *Determine* α_0 *according to the formula (4.21);*

5) *Find the values* $G_\xi^{(k)}(1)$, $k=\overline{1,n}$, *by using Algorithm 4.1;*

6) *Determine* $\mathbb{E}(\xi)$, $\mathbb{V}(\xi)$ *and* $\sigma(\xi)$ *by using the relation (4.6);*

7) *Obtain the Stirling numbers of the second kind (or the Stirling numbers of the first kind)* $S(k,j)$, $k=\overline{1,n}$, $j=\overline{1,k}$, *according to relations (4.2) and (4.3);*

8) *Determine the moments* $\nu_k(\xi)$, $k=\overline{1,n}$, *according to (4.4) and (4.5).*

Remark 4.1 *The second step of Algorithm 4.3 is Horner's scheme for dividing the polynomial* $\chi_0 \mid z$ *by the binomial* $1-z$, *the quotient being* $\mu \mid z$, *required to ensure* $\Delta_0(\chi_0)\neq 0$.

Remark 4.2 *The complexity of Algorithm 4.3 is* $O(m^2n^2)$. *If the moments are not required to be computed, then Algorithm 4.3 is reduced to the first six steps only, applied with* $n=2$, *the complexity being* $O(m^2)$.

Next, we provide an alternative to Algorithm 4.3, which does not require the numerical differentiation using Algorithm 4.1. The following theorem allows us to compute the moment $\nu_k(\xi)$ of order $k\geq 1$ of the non-negative integer random variable ξ with distribution $a=rep(\xi)\in Rol[\mathbb{R}][m]$.

Theorem 4.7 *Consider the non-negative integer random variable* ξ *with distribution* $a=rep(\xi)\in Rol[\mathbb{R}][m]$ *and generating vector* $q\in G[\mathbb{R}][m](a)$. *Then*

$$c^{(k)}=(n^k a_n)_{n=0}^{\infty}\in Rol[\mathbb{R}][M_k],$$

$$q^{(k)}\in G[\mathbb{R}][M_k](c^{(k)})$$

and

$$\nu_k(\xi)=G^{[c^{(k)}]}(1),\ \forall k\geq 1, \tag{4.22}$$

where $M_k=m(k+1)$ *and*

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in H[\mathbb{R}][M_k](c^{(k)}). \tag{4.23}$$

Proof. Let $\forall k \geq 1$. Consider the sequence $d^{(k)} = (n^k)_{n=0}^{\infty}$. We have $c^{(k)} = d^{(k)}a$. From Theorem 1.23, we obtain $d^{(k)} \in Rol[\mathbb{R}][k+1]$ and

$$(1-z)^{k+1} \in H[\mathbb{R}][k+1](d^{(k)}).$$

By applying Theorem 1.25, we have $c^{(k)} \in Rol[\mathbb{C}]$ and

$$P_k(z) = LCM(\{(z-z_t)^{s_t+k} \mid t = \overline{0, p-1}\}) \in H[\mathbb{C}](c^{(k)}),$$

where $z_0, z_1, \ldots, z_{p-1}$ are all the distinct roots, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-1}$, of the polynomial $H_m^{[q]}(z)$. From the relations

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \vdots H_m^{[q]}(z)\left(\prod_{t=0}^{p-1}(z-z_t)\right)^k = \prod_{t=0}^{p-1}(z-z_t)^{s_t+k} \vdots P_k(z)$$

and $H_{M_k}^{[q^{(k)}]}(z) \in \mathbb{R}[z]$, by using Corollary 1.2, it follows that

$$c^{(k)} \in Rol[\mathbb{R}][M_k],\ q^{(k)} \in G[\mathbb{R}][M_k](c^{(k)}) \text{ and } H_{M_k}^{[q^{(k)}]}(z) \in H[\mathbb{R}][M_k](c^{(k)}).$$

Next, we obtain the formula

$$\nu_k(\xi) = \sum_{n=0}^{\infty} n^k a_n = G^{[c^{(k)}]}(1).$$

These values can be obtained from Theorem 1.1 and the formula (4.23). □

Taking into account these results, we have obtained a new algorithm for the characterization of random variables with homogeneous linear recurrence distribution:

Algorithm 4.4

Input: $q \in G[\mathbb{R}][m](a)$, $I_m^{[a]} \in \mathbb{R}^m$, where $a = rep(\xi) \in Rol[\mathbb{R}][m]$;

Output: $\mathbb{E}(\xi)$, $\mathbb{V}(\xi)$, $\sigma(\xi)$, $\nu_k(\xi)$, $k = \overline{1,t}$, $t \geq 2$;

1) If $H_m^{[q]}(1) = 0$, go to step 2, otherwise go to step 3;

2) *By using Horner's scheme, determine* $q^* \in \mathbb{R}^{m-1}$, *which satisfies the equality* $H_{m-1}^{[q^*]}(z) = \dfrac{H_m^{[q]}(z)}{1-z}$. *Next, set* $q := q^*$ *and* $m := m-1$ *and go back to step 1;*

3) *Compute the values* $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$, $n = \overline{m, m(t+1)-1}$;

4) *For each* $k = \overline{1,t}$, *execute the following steps:*

   a. *Determine the initial state* $I_{M_k}^{[c^{(k)}]} = (n^k a_n)_{n=0}^{m(k+1)-1}$;

   b. *Obtain the generating vector* $q^{(k)}$ *using the formula (4.23);*

   c. *Compute the value* $\nu_k(\xi)$ *using the relation (4.22) and Theorem 1.1;*

5) *Compute the values* $\mathbb{E}(\xi)$, $\mathbb{V}(\xi)$ *and* $\sigma(\xi)$ *with the help of the formula (4.1).*

This algorithm can also be extended to the degenerate case. The following theorems hold:

Theorem 4.8 *If* $a = rep(\xi) \in Rol^*[\mathbb{C}]$, *then* $a \in Rol^*[\mathbb{R}]$ *and*

$$dim[\mathbb{R}](a) = dim[\mathbb{C}](a).$$

Proof. Consider the sequence $a = rep(\xi) \in Rol^*[\mathbb{C}][m]$ with generating vector $q \in G^*[\mathbb{C}][m](a)$ and dimension $m = dim[\mathbb{C}](a)$. Then,

$$b = (a_n)_{n=m-r}^{\infty} \in Rol[\mathbb{C}][r],\ Q = (q_k)_{k=0}^{r-1} \in G[\mathbb{C}][r](b),$$

where $r = deg(H_m^{[q]}(z))$. This relation implies

$$c = (c_n)_{n=0}^{\infty} = \frac{b}{1-S} \in Rol[\mathbb{C}][r],\ Q \in G[\mathbb{C}][r](c),$$

where $S = \sum_{k=0}^{m-r-1} a_k$. Since $\sum_{n=0}^{\infty} c_n = 1$, we can consider a random variable η with distribution $c = rep(\eta)$. Next, we obtain

$$c \in Rol[\mathbb{R}][r],\ Q \in G[\mathbb{R}][r](c),$$

which implies

$$b \in Rol[\mathbb{R}][r],\ Q \in G[\mathbb{R}][r](b).$$

Therefore,

$$a \in Rol^*[\mathbb{R}][m],\ q \in G^*[\mathbb{R}][m](a).$$

Since $dim[\mathbb{R}](a) \geq dim[\mathbb{C}](a) = m$ and $a \in Rol^*[\mathbb{R}][m]$, we have

$$dim[\mathbb{R}](a) = dim[\mathbb{C}](a) = m. \qquad \square$$

Theorem 4.9 *If* $a = rep(\xi) \in Rol^*[\mathbb{C}][m],\ q \in G^*[\mathbb{C}][m](a)$ *and* $H_m^{[q]}(1) = 0$, *then* $dim[\mathbb{R}](a) \neq m$ *and the root* $z = 1$ *is fictitious.*

Proof. Consider the sequence $a = rep(\xi) \in Rol^*[\mathbb{C}][m]$ with $q \in G^*[\mathbb{C}][m](a)$ and $H_m^{[q]}(1) = 0$. Then

$$b = (a_n)_{n=m-r}^{\infty} \in Rol[\mathbb{C}][r],\ Q = (q_k)_{k=0}^{r-1} \in G[\mathbb{C}][r](b),$$

where $r = deg(H_m^{[q]}(z))$. This relation implies

$$c = (c_n)_{n=0}^{\infty} = \frac{b}{1-S} \in Rol[\mathbb{C}][r],\ Q \in G[\mathbb{C}][r](c),$$

where $S = \sum_{k=0}^{m-r-1} a_k$. Since $\sum_{n=0}^{\infty} c_n = 1$, we can consider a random variable η with distribution $c = rep(\eta)$. We obtain $dim[\mathbb{R}](c) \neq r$ and the root $z = 1$ of the polynomial $H_m^{[q]}(z)$ is fictitious. Therefore, $dim[\mathbb{R}](a) \neq m$ and the root $z = 1$ of the polynomial $H_m^{[q]}(z)$ is fictitious. $\square$

Theorem 4.10 *Let* ξ *be a non-negative integer random variable with homogeneous linear recurrence distribution* $a = rep(\xi) \in Rol^*[\mathbb{R}][m]$ *and* $q \in G^*[\mathbb{R}][m](a)$. *Then*

$$c^{(k)} = (n^k a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][M_k],\ q^{(k)} \in G^*[\mathbb{R}][M_k](c^{(k)})$$

and

$$\nu_k(\xi) = G^{[c^{(k)}]}(1),\ \forall k \geq 1, \tag{4.24}$$

where $M_k = m(k+1)$ *and*

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in H^*[\mathbb{R}][M_k](c^{(k)}). \tag{4.25}$$

Proof. For $\forall k \geq 1$, we consider the sequence $d^{(k)} = (n^k)_{n=0}^{\infty}$. We have $c^{(k)} = d^{(k)} a$. From Theorem 1.23, we obtain

$$d^{(k)} \in Rol[\mathbb{R}][k+1],\ (1-z)^{k+1} \in H[\mathbb{R}][k+1](d^{(k)}).$$

By applying Theorem 1.32, we have $c^{(k)} \in Rol^*[\mathbb{C}][m+kp]$ and

$$P_k(z) = LCM(\{(z-z_t)^{s_t+k} \mid t = \overline{0, p-1}\}) =$$

$$= H_m^{[q]}(z) \prod_{t=0}^{p-1} (z-z_t)^k \in H^*[\mathbb{C}][m+kp](c^{(k)}),$$

where $z_0, z_1, \ldots, z_{p-1}$ are all the distinct roots, with corresponding multiplicities $s_0, s_1, \ldots, s_{p-1}$, of the polynomial $H_m^{[q]}(z)$. Since

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in \mathbb{R}[z]$$

is divisible by

$$P_k(z) = H_m^{[q]}(z) \left(\prod_{t=0}^{p-1} (z-z_t) \right)^k,$$

by using Theorem 1.30 and the inequality

$$deg(H_m^{[q]}(z)) \leq m,$$

we have

$$c^{(k)} \in Rol^*[\mathbb{R}][M_k],\ q^{(k)} \in G^*[\mathbb{R}][M_k](c^{(k)}),$$

$$H_{M_k}^{[q^{(k)}]}(z) \in H^*[\mathbb{R}][M_k](c^{(k)}).$$

Next, we obtain the formula

$$\nu_k(\xi) = \sum_{n=0}^{\infty} n^k a_n = G^{[c^{(k)}]}(1).$$

These values can be obtained from Theorem 1.29 and formula (4.25). □

Therefore, Algorithm 4.4 can also be used for degenerate homogeneous linear recurrence distributions and it has polynomial complexity.

Example 4.4 *Let* ξ *be a non-negative integer random variable with distribution* $a = rep(\xi) \in Rol[\mathbb{R}][2]$, $q = (1, -0.21) \in G[\mathbb{R}][2](a)$ *and* $I_2^{[a]} = (0, 0.21)$. *Determine the expectation, the variance, the standard deviation and the moments of order 1 and 2 of the random variable* ξ.

Solution. By applying Algorithm 4.4, we obtain

n	a_n	na_n	$G_n^{[c^{(1)}]}(1)$	$G_{3-n}^{[c^{(1)}]}(1)$	$q_n^{(1)}$	$q_n^{(1)} G_{3-n}^{[c^{(1)}]}(1)$
0	0	0	0	0.63	2	1.26
1	0.21	0.21	0	0.21	-1.42	-0.2982
2	0.21	0.42	0.21	0	0.42	0
3	0.1659	0.4977	0.63	0	-0.0441	0
4	--	--	1.1277	--	--	--
Total	--	--	--	--	0.9559	0.9618

Table 4.5 Homogeneous linear recurrence distribution – step 1

n	a_n	$n^2 a_n$	$G_n^{[c^{(2)}]}(1)$	$G_{5-n}^{[c^{(2)}]}(1)$	$q_n^{(2)}$	$q_n^{(2)} G_{5-n}^{[c^{(2)}]}(1)$
0	0	0	0	4.4919	3	13.4757
1	0.21	0.21	0	2.5431	-3.63	-9.231453
2	0.21	0.84	0.21	1.05	2.26	2.373
3	0.1659	1.4931	1.05	0.21	-0.7623	-0.160083
4	0.1218	1.9488	2.5431	0	0.1323	0
5	0.086961	2.174025	4.4919	0	-0.00926	0
6	--	--	6.665925	--	--	--
Total	--	--	--	--	0.990739	6.457164

Table 4.6 Homogeneous linear recurrence distribution – step 2

k	M_k	$H_{M_k}^{[q^{(k)}]}(1)$	$G_{M_k}^{[c^{(k)}]}(1) - \sum_{j=0}^{M_k-1} q_k G_{M_k-1-j}^{[c^{(k)}]}(1)$	$G^{[c^{(k)}]}(1)$
1	4	0.0441	0.1659	3.761905
2	6	0.009261	0.208761	22.54195

Table 4.7 Homogeneous linear recurrence distribution – step 3

$\nu_1(\xi)$	$\nu_2(\xi)$	$\mathbb{E}(\xi)$	$\mathbb{V}(\xi)$	$\sigma(\xi)$
3.761905	22.54195	3.761905	8.39002	2.89655

Table 4.8 Homogeneous linear recurrence distribution – final step

4.5 Parallelization aspects for homogeneous linear recurrence distributions

Let ξ be a non-negative integer random variable with distribution $a = rep(\xi)$. According to Section 4.4, for finding the probabilistic characteristics of homogeneous linear recurrence distributions we have to go through the following stages:

- Establish the order m of the recurrence and its double initial state $(a_n)_{n=0}^{2m-1}$;
- Determine the minimal generating vector;
- Compute the moment $\nu_k(T)$, $k = \overline{1,r}$.

Usually, the first stage follows from the theoretical foundation of the specific applied problem. The computations can be organized in layers, with each component of the double initial state being computed through a sequence of elementary operations applied to the values obtained at the preceding layers. These operations can be distributed across multiple processors, with the resulting values being accumulated and aggregated.

According to the rank-based order minimization method, given by Theorem 2.15, the second stage is reduced to solving a system of linear equations. A number of parallel algorithms for solving such systems are known. For example, [35] presents

parallel implementations of Gaussian elimination, Gauss–Jordan elimination, Huard's variant, and Enright's algorithm.

At the third stage, the method of determining the moment of order k, $k=\overline{1,r}$, is based on Theorem 4.7 and Theorem 1.1. According to Theorem 4.7, we can parallelize the operations performed at step k, $k=\overline{1,r}$, based on the computations performed at the preceding step. In this way, considering $H_{M_0}^{[q^{(0)}]}(z)=H_m^{[q]}(z)$, we can compute the polynomial $H_{M_k}^{[q^{(k)}]}(z)$:

$$H_{M_k}^{[q^{(k)}]}(z)=H_{M_{k-1}}^{[q^{(k-1)}]}(z)\cdot H_m^{[q]}(z),\ k=\overline{1,r}.$$

Here, the polynomial multiplications can be parallelized by applying the algorithm from [135].

Subsequently, still at step k, when Theorem 1.1 is applied for $z=1$, $c^{(k)}=(n^k a_n)_{n=0}^{\infty}$ and $q=q^{(k)}$, the computations can also be parallelized. The denominator can be determined using a parallel summation algorithm based on the divide-and-conquer principle. To compute the numerator, the processors must be assigned the task of computing products of the form $q_j G_{m-1-j}^{[c^{(k)}]}$, after which the results are summed using the same parallel summation algorithm as above. The values $G_j^{[c^{(k)}]}(1)$, $j=\overline{1,m}$, are computed sequentially in advance on one of the processors.

4.6 Counting numbers with fixed digit count and digit sum

Let $B_b(s,t)$ be the number of $t-$digit numbers, written in base b, with the sum of digits equal to s. We define a number in base b as any sequence of $t\geq 1$ digits from the set $\{0,1,\ldots,b-1\}$.

Theorem 4.11 *The following recurrence formula holds:*

$$B_b(s,t)=\sum_{k=0}^{b-1}B_b(s-k,t-1),\ \forall s\geq 0,\ t\geq 2,$$

where

$$B_b(s,1)=\begin{cases}1, & 0\le s<b\\ 0, & s\ge b\end{cases};$$

$$B_b(0,t)=1,\ \forall t\ge 1,$$

$$B_b(s,t)=0,\ if\ s<0\ \ or\ \ t\le 0.$$

Proof. These relations are obvious for $s\le 0$ and $t\le 1$. Let $s\ge 0$ and $t\ge 2$. Each $t-$ digit number, with the sum of digits equal to s and the last digit equal to k, is obtained from a $(t-1)-$ digit number with the sum of digits equal to $(s-k)$ by concatenating the digit k at the end. Thus, the recurrence formula above holds. □

Theorem 4.12 *The following formula holds:*

$$B_b(s,t)=B_b(s,t-1)+B_b(s-1,t)-B_b(s-b,t-1),\ \forall t\ge 2.$$

Proof. Consider all $t-$digit numbers, written in base b, which have the sum of digits equal to s. They can be distributed in the following categories:

- The numbers that have the last digit equal to zero are obtained from $(t-1)-$ digit numbers with the sum of digits s, by concatenating a zero at the end. Therefore, there are $B_b(s,t-1)$ numbers in this category;
- The numbers, whose last digit is not zero, with the sum of digits equal to s, are obtained from $t-$digit numbers, that have the last digit not equal to $(b-1)$ and the sum of digits equal to $(s-1)$, by increasing the last digit with one unit. In this category, there are $(B_b(s-1,t)-N)$ numbers, where N is the number of numbers from the following category;
- The numbers with last digit $(b-1)$ and the sum of digits equal to $(s-1)$ are obtained from $(t-1)-$ digit numbers, with the sum of digits equal to $(s-b)$, by concatenating the digit $(b-1)$ at the end. Therefore, in this category there are $N=B_b(s-b,t-1)$ numbers.

In conclusion, according to these categories, we obtain the formula from the statement of the theorem. □

The following properties are true:

- The number of $t-$digit numbers, written in base b, with the sum of digits equal to s and the first digit not equal to zero is
$$n = B_b(s,t) - B_b(s,t-1);$$
- The relation
$$B_2(s,t) = C_t^s,\ \forall s,t:\ 0 \le s \le t,$$
holds. This follows from the fact that, in base 2, a $t-$digit number has the sum of digits equal to s if and only if the number contains s ones and $t-s$ zeros. Therefore, $B_2(s,t)$ represents the number of ways to place s ones in t positions, i.e. C_t^s;
- The number of $2t-$digit numbers, written in base b, with the sum of the first t digits equal to the sum of the last t digits, is
$$N_b(t) = \sum_{s=0}^{(b-1)t} (B_b(s,t))^2 .$$
This formula follows from the fact that every number with the given property is obtained by concatenating two $t-$digit numbers (in base b) with the same sum s of digits, $0 \le s \le (b-1)t$.

The generating functions, with respect to the first argument, are given by the following theorem:

Theorem 4.13 *The generating functions of the sequences*
$$a^{(t)} = (B_b(s,t))_{s=0}^{\infty},\ t \ge 1,$$
are
$$G^{[a^{(t)}]}(z) = \left(\frac{z^b - 1}{z-1}\right)^t,\ \forall z \ne 1,$$
$$G^{[a^{(t)}]}(1) = b^t.$$

Proof. Each digit of any $t-$digit number, written in base b, does not exceed $(b-1)$. Therefore, $s \le (b-1)t$, i.e.

$$G^{[a^{(t)}]}(z) = \sum_{s=0}^{(b-1)t} B_b(s,t) z^s .$$

We have

$$G^{[a^{(t)}]}(1) = \sum_{s=0}^{(b-1)t} B_b(s,t) = N(b,t) = b^t ,$$

where $N(b,t)$ is the number of $t-$digit numbers, written in base b.

Let $a^{(0)} = (B_1(s,1))_{s=0}^{\infty}$. We observe that $G^{[a^{(0)}]}(z) = 1$. For $t = 1$, we have

$$G^{[a^{(1)}]}(z) = \sum_{s=0}^{\infty} B_b(s,1) z^s = \sum_{s=0}^{b-1} z^s = \frac{z^b - 1}{z-1} = \frac{z^b - 1}{z-1} G^{[a^{(0)}]}(z) .$$

Now, we consider $t \geq 2$. From Theorem 4.11, we have

$$B_b(s,t) = \sum_{k=0}^{b-1} B_b(s-k, t-1), \ \forall s \geq 0, \ t \geq 2 .$$

The generating function is

$$G^{[a^{(t)}]}(z) = \sum_{s=0}^{\infty} B_b(s,t) z^s = \sum_{s=0}^{\infty} \sum_{k=0}^{b-1} B_b(s-k, t-1) z^s =$$

$$= \sum_{k=0}^{b-1} \sum_{s=0}^{\infty} B_b(s-k, t-1) z^s = \sum_{k=0}^{b-1} \sum_{s=-k}^{\infty} B_b(s, t-1) z^{s+k} =$$

$$= \sum_{k=0}^{b-1} \sum_{s=0}^{\infty} B_b(s, t-1) z^{s+k} = \sum_{k=0}^{b-1} z^k \sum_{s=0}^{\infty} B_b(s, t-1) z^s = \sum_{k=0}^{b-1} z^k G^{[a^{(t-1)}]}(z) =$$

$$= G^{[a^{(t-1)}]}(z) \sum_{k=0}^{b-1} z^k = \frac{z^b - 1}{z-1} G^{[a^{(t-1)}]}(z), \ \forall t \geq 1 .$$

It follows that

$$G^{[a^{(t)}]}(z) = \left(\frac{z^b - 1}{z-1} \right)^t, \ t \in \mathbb{N}, \ \forall z \neq 1 . \qquad \square$$

The generating functions, with respect to the second argument, are given by the following theorem:

Theorem 4.14 *The generating functions of the sequences*

$$d^{(s)} = (B_b(s,t))_{t=0}^{\infty}, \ \forall s \in \mathbb{Z},$$

are

$$G^{[d^{(s)}]}(z)=\frac{z}{1-z}\sum_{k=0}^{b-2}G^{[d^{(s-1-k)}]}(z),\ \forall s\geq b,$$

$$G^{[d^{(s)}]}(z)=\frac{z}{(1-z)^{s+1}},\ s=\overline{0,b-1},\ \forall z\in D_s,$$

where D_s *is the domain of convergence of the series* $G^{[d^{(s)}]}(z),\ s\in\mathbb{Z}$.

Proof. From Theorem 4.11, we have

$$B_b(s,t)=\sum_{k=0}^{b-1}B_b(s-k,t-1),\ \forall s\geq 0,\ t\geq 2.$$

Consider $\forall z\in D_s$, where D_s is the convergence domain of $G^{[d^{(s)}]}(z),\ s\in\mathbb{N}$.

We have

$$G^{[d^{(s)}]}(z)=\sum_{t=0}^{\infty}B_b(s,t)z^t=zB_b(s,1)+\sum_{t=2}^{\infty}\sum_{k=0}^{b-1}B_b(s-k,t-1)z^t=$$

$$=zB_b(s,1)+\sum_{k=0}^{b-1}\sum_{t=2}^{\infty}B_b(s-k,t-1)z^t=zB_b(s,1)+z\sum_{k=0}^{b-1}\sum_{t=0}^{\infty}B_b(s-k,t)z^t=$$

$$=zB_b(s,1)+z\sum_{k=0}^{b-1}G^{[d^{(s-k)}]}(z)=zB_b(s,1)+zG^{[d^{(s)}]}(z)+z\sum_{k=0}^{b-2}G^{[d^{(s-1-k)}]}(z).$$

It follows that

$$(1-z)G^{[d^{(s)}]}(z)=zB_b(s,1)+z\sum_{k=0}^{b-2}G^{[d^{(s-1-k)}]}(z),$$

hence

$$G^{[d^{(s)}]}(z)=\frac{z}{1-z}\left[B_b(s,1)+\sum_{k=0}^{b-2}G^{[d^{(s-1-k)}]}(z)\right].$$

For $s\geq b$, we have

$$G^{[d^{(s)}]}(z)=\frac{z}{1-z}\sum_{k=0}^{b-2}G^{[d^{(s-1-k)}]}(z).$$

Let us prove that the following formula holds:

$$G^{[d^{(s)}]}(z)=\frac{z}{(1-z)^{s+1}},\ s=\overline{0,b-1}.$$

For $s=0$, we have

$$G^{[d^{(0)}]}(z)=\frac{z}{1-z}\left[1+\sum_{k=0}^{b-2}G^{[d^{(-1-k)}]}(z)\right]=\frac{z}{1-z},$$

hence the statement is true. Assume that the statement is true for $s=\overline{0,r-1}$, $r\le b-1$. Let us prove it for $s=r$. We have

$$G^{[d^{(r)}]}(z)=\frac{z}{1-z}\left[1+\sum_{k=0}^{b-2}G^{[d^{(r-1-k)}]}(z)\right]=\frac{z}{1-z}\left[1+\sum_{k=0}^{r-1}G^{[d^{(r-1-k)}]}(z)\right]=$$

$$=\frac{z}{1-z}\left[1+\sum_{k=0}^{r-1}\frac{z}{(1-z)^{r-k}}\right]=\frac{z}{1-z}\left[1+\frac{z}{(1-z)^{r}}\sum_{k=0}^{r-1}(1-z)^{k}\right]=$$

$$=\frac{z}{1-z}\left[1+\frac{z}{(1-z)^{r}}\frac{1-(1-z)^{r}}{z}\right]=\frac{z}{(1-z)^{r+1}}.$$

Finally, by mathematical induction, we obtain

$$G^{[d^{(s)}]}(z)=\frac{z}{(1-z)^{s+1}},\ s=\overline{0,b-1}.$$ □

The direct formula is given by the following theorem:

Theorem 4.15 *The value* $B_b(s,t)$ *satisfies the explicit formula*

$$B_b(s,t)=\sum_{j=0}^{[s/b]}(-1)^j C_t^j C_{t-1+s-bj}^{t-1},\ 0\le s\le (b-1)t,$$

where $[s/b]$ *is the integer part of the real number* s/b. *Also, we have*

$$B_b(s,t)=0,\ \forall s>(b-1)t.$$

Proof. Let

$$a^{(t)}=(B_b(s,t))_{s=0}^{\infty},\ t\ge 1.$$

From Theorem 4.13, we have

$$G^{[a^{(t)}]}(z)=u(z)v(z),\ \forall z\ne 1,\ \forall t\ge 1,$$

where

$$u(z)=(z^b-1)^t,\ v(z)=(z-1)^{-t}.$$

Therefore, the following formula holds:

$$G^{[a^{(t)}](s)}(z)=\sum_{k=0}^{s}C_s^k u^{(k)}(z)v^{(s-k)}(z).$$

This implies

$$G^{[a^{(t)}](s)}(0)=\sum_{k=0}^{s}C_s^k u^{(k)}(0)v^{(s-k)}(0).$$

Let us compute the values $u^{(k)}(0)$ and $v^{(s-k)}(0)$, $k=\overline{0,s}$. By differentiating $(s-k)$ times successively the function $v(z)$, we obtain

$$v^{(s-k)}(z)=(z-1)^{-t-s+k}\prod_{r=-t+1-(s-k)}^{-t}r,$$

which implies

$$v^{(s-k)}(0)=(-1)^{s-k}\frac{(t-1+s-k)!}{(t-1)!}(-1)^{-t-s+k}=(-1)^t\frac{(t-1+s-k)!}{(t-1)!}.$$

Also, we have the following formula:

$$u(z)=(z^b-1)^t=\sum_{j=0}^{t}C_t^j z^{bj}(-1)^{t-j}=(-1)^t\sum_{j=0}^{t}C_t^j(-1)^j z^{bj}.$$

By differentiating k times successively, we obtain

$$u^{(k)}(z)=(-1)^t\sum_{j=0}^{t}C_t^j(-1)^j z^{bj-k}\prod_{r=bj+1-k}^{bj}r.$$

It follows that

$$u^{(k)}(0)=\begin{cases}(-1)^{j+t}k!C_t^j, & if\quad k=bj,\quad 0\le j\le t\\ 0, & otherwise\end{cases}.$$

For the generating function, we have

$$G^{[a^{(t)}](s)}(0)=\sum_{k=0}^{s}C_s^k u^{(k)}(0)v^{(s-k)}(0)=$$

$$=\sum_{j=0}^{[s/b]}C_s^{bj}C_t^j(-1)^{j+t}(bj)!(-1)^t\frac{(t-1+s-bj)!}{(t-1)!}=$$

$$= \sum_{j=0}^{[s/b]} \frac{s!}{(s-bj)!} C_t^j (-1)^j \frac{(t-1+s-bj)!}{(t-1)!} = s! \sum_{j=0}^{[s/b]} C_t^j (-1)^j C_{t-1+s-bj}^{t-1} .$$

In conclusion, we obtain

$$B_b(s,t) = a_s^{(t)} = \frac{1}{s!} G^{[a^{(t)}](s)}(0) = \sum_{j=0}^{[s/b]} (-1)^j C_t^j C_{t-1+s-bj}^{t-1} , \ 0 \le s \le (b-1)t . \qquad \square$$

Theorem 4.16 *The number of decompositions of the number* s *as an unordered sum of* t *positive integers is* $n = C_{s-1}^{t-1}$.

Proof. It is observed that there exists a bijection between these decompositions and the $t-$digit numbers in base $(s-t+1)$ (each decomposition consists of the digits of such a number obtained by adding one unit to each digit). Therefore,

$$n = B_{s-t+1}(s-t,t) .$$

From the explicit formula, for $b > s$, we have

$$B_b(s,t) = C_{t-1+s}^{t-1} .$$

In conclusion, we obtain the formula $n = C_{s-1}^{t-1}$. $\square$

Theorem 4.17 *The following formula holds:*

$$\sum_{s=0}^{t} \sum_{j=0}^{[s/2]} (-1)^{s+j} C_t^j C_{t-1+s-2j}^{t-1} = 0 , \ \forall t \ge 1.$$

Proof. The truth of the statement of this theorem follows from Theorem 4.15. We have the following sequence of equalities:

$$\sum_{s=0}^{t} \sum_{j=0}^{[s/2]} (-1)^{s+j} C_t^j C_{t-1+s-2j}^{t-1} = \sum_{s=0}^{t} (-1)^s B_2(s,t) =$$

$$= \sum_{s=0}^{t} (-1)^s C_t^s = (-1+1)^t = 0 , \ \forall t \ge 1. \qquad \square$$

Example 4.5 *A gambling agency offered for sale 10,000 tickets at a price of 5 euros each, numbered from 0000 to 9999, randomly marked, and concealed by a protective layer. A ticket is considered lucky if its number satisfies the following condition: the sum of the first two digits is equal to the sum of the last two digits. Knowing that the agency retains 10% of the revenue to cover expenses, while the remaining amount is divided equally among all winners, and assuming that all tickets were sold, determine the probability that a player wins and the prize received by each winner.*

Solution: We have

$$N_{10}(2) = \sum_{s=0}^{18} (B_{10}(s,2))^2 =$$

$$= \sum_{s=0}^{18} \left(\sum_{j=0}^{\left[\frac{s}{10}\right]} (-1)^j C_2^j C_{1+s-10j}^1 \right)^2 =$$

$$= \sum_{s=0}^{18} \left(\sum_{j=0}^{\left[\frac{s}{10}\right]} (-1)^j C_2^j (1+s-10j) \right)^2 =$$

$$= \sum_{s=0}^{9} (1+s)^2 + \sum_{s=10}^{18} ((1+s)+(-2(s-9)))^2 =$$

$$= \sum_{s=1}^{10} s^2 + \sum_{s=10}^{18} (19-s)^2 = \sum_{s=1}^{10} s^2 + \sum_{s=1}^{9} s^2 = 2\sum_{s=1}^{9} s^2 + 100 = 670.$$

Therefore, the probability that a player wins is

$$p = \frac{670}{10000} = 0.067 = 6.7\%.$$

In consequence, the net gain of each winner is

$$C = \frac{5 \cdot 10000 \cdot (1-0.1)}{670} - 5 \approx 62.16 \text{ euros.}$$ □

5 ASYMPTOTIC BEHAVIOR OF MARKOV PROCESSES

In this chapter, the asymptotic behavior of the transition probability matrix of discrete Markov stochastic systems is investigated. The limiting matrix and the corresponding differential components are determined. Two numerical methods are proposed for solving this problem. The first is based on the decomposition of the transition matrix with respect to its eigenvalues, while the second relies on the properties of homogeneous linear recurrences established in the preceding chapters. Furthermore, several improvements to these methods are developed to enhance their computational efficiency.

5.1 Historical development of Markov processes

The development of Markov processes is closely connected with the formalization of uncertainty and the study of systems evolving in time. Bartlett presented stochastic processes as mathematical models for time-dependent phenomena ([10]). Feller later developed a broad analytical framework for probability theory and its applications ([46]). Shiryaev connected classical probability with the analysis of stochastic processes ([145]), while Rossberg, Jesiak and Siegel developed analytical methods for probability theory and related stochastic models ([142]).

The central idea of a Markov process is that the future depends on the past only through the present state. In discrete time, this principle leads to Markov chains and transition matrices. Chung developed a systematic treatment of Markov chains with stationary transition probabilities ([28]). The theory of finite chains was subsequently presented by Kemeny and Snell ([66]) and extended to denumerable chains by Kemeny, Snell, and Knapp ([67]). Their later work provided a further treatment of

finite Markov chains and clarified the classification of states, recurrence, transience, and limiting behavior ([68]). Cinlar placed Markov chains within the broader theory of stochastic processes ([29]), while Norris offered a modern account of transition probabilities, recurrence, hitting times, and invariant distributions ([126]). Seneta emphasized the relationship between Markov chains and non-negative matrices, particularly in the study of asymptotic behavior ([143]). Karlin and Taylor also contributed to the probabilistic background of stochastic-process theory through their treatment of transition mechanisms, dependence, and limiting phenomena ([62]). Grimmett and Stirzaker provided a broader account of probability and random processes, thereby placing Markov models within the general development of modern probability ([51]). Godonoaga and Baractari considered a probabilistic method for minimax problems with general constraints, linking probabilistic models with optimization ([52]).

The increasing importance of computation led to algorithmic methods for analyzing Markov systems. Fox and Landi proposed an algorithm for identifying ergodic subchains and transient states in a stochastic matrix ([48]). Stewart developed numerical methods for calculating stationary distributions, limiting probabilities, and other quantities associated with Markov chains ([147]). He later extended this computational perspective to queues, simulation, and applications of Markov chains ([148]). The computational setting was supported by developments in numerical linear algebra and optimization. Antoniou and Lu surveyed practical optimization algorithms and engineering applications ([5]), while Dantzig and Thapa presented the foundations and extensions of linear programming ([33], [34]). The polynomial-time viewpoint was strengthened by Karmarkar's algorithm ([63]) and by Khachian's polynomial algorithms for linear programming ([69], [70], [71]). Roos, Terlaky, and Vial further developed the theory and algorithms of interior-point methods ([141]), Vanderbei presented linear programming from both foundational and algorithmic perspectives ([151]), and Wright analyzed primal-dual interior-point methods ([159]).

A major transformation occurred when Markov models were combined with decision-making. Bellman’s work on Markovian decision processes introduced one of the fundamental formulations of controlled stochastic systems ([12]). His principle of

optimality and dynamic programming provided the main recursive method for optimizing sequential decisions ([13]). Bellman subsequently connected these ideas with control processes, linear systems, and quadratic criteria ([14]). Bellman and Dreyfus demonstrated the practical use of dynamic programming in multistage optimization ([15]), while Bellman and Kalaba related dynamic programming to modern control theory ([16]). Bellman and Kalaba also considered dynamic programming for control processes governed by general functional equations ([17]). This line of research was developed further by Bertsekas in his treatment of dynamic programming and optimal control ([20]) and by Dreyfus and Law in their analysis of the theory and practice of dynamic programming ([38]).

Howard developed the theory of dynamic programming and Markov processes ([56]). He later extended it to semi-Markov decision processes, in which transition durations may vary ([55], [57]). Derman presented a systematic theory of finite-state Markov decision processes ([37]), and Mine and Osaki provided another early treatment of Markovian decision processes ([123]). Controlled Markov processes were studied by Dynkin and Yushkevich ([45]). Fleming and Rishel connected stochastic control with deterministic control theory ([47]), while Bertsekas and Shreve established a rigorous framework for discrete-time stochastic optimal control ([21]). The general theory of optimal control was also developed by Berkovitz ([19]) and Kirk ([72]). Hull presented control theory from an applications-oriented perspective ([59]), while Kalaba and Spingarn connected control, identification, and input optimization ([61]). These works helped extend Markovian decision models toward broader controlled dynamical systems.

The optimization of Markov processes was also expressed through mathematical programming. Blackwell examined discounted dynamic programming and the characterization of optimal policies ([22]). Denardo formulated Markov decision processes through linear programming ([36]). Papadimitriou and Tsitsiklis investigated their computational complexity ([130]). Puterman provided a comprehensive treatment of Markov decision processes and discrete stochastic dynamic programming ([136], [137]). Related developments were presented by White in his works on

finite dynamic programming ([155]) and Markov decision processes ([156]), as well as by White ([154]). The relationship between dynamic programming and combinatorial optimization was reinforced by Papadimitriou and Steiglitz ([129]). The limits of simplex-based optimization were illustrated by Klee and Minty ([73]), whereas Dantzig's work established the linear-programming framework in which many decision-process formulations can be expressed ([33], [34]).

Stochastic dynamic programming was progressively applied to economics, management, engineering, and complex dynamical systems. Van der Wal presented a general framework for stochastic dynamic programming ([150]). Sethi and Thompson studied optimal control in management science and economics ([144]), while Weber and Kryazhimskiy considered its economic applications ([153]). Robinett, Wilson, Eisler and Hurtado applied dynamic programming to the optimization of dynamical systems ([139]). Sniedovich provided a broad account of the principles and foundations of dynamic programming ([146]). Bäuerle and Rieder focused on financial applications of Markov decision processes ([11]), and Hu and Yue surveyed their wider applications ([58]). Additional contributions considered discrete and time-dependent control problems. Bortakovskii examined optimal and suboptimal control for sets of trajectories of deterministic continuous-discrete systems ([24]). Krabs and Pickl studied controllability, optimization, and dynamical games in time-discrete systems ([76], [77]). Kurina and Nekrasova investigated an asymptotic discrete optimal-control problem ([78]), while Lawler's work on optimal cycles in doubly weighted directed graphs is relevant to cyclic and network-based optimization ([79]).

Further developments addressed multiple objectives, network structures, and optimization. Ghosh studied Markov decision processes with multiple costs ([50]), while Durinovic, Lee, Katehakis and Filar considered multiobjective models with an average-reward criterion ([44]). Gheorghe examined decision processes in dynamic probabilistic systems ([49]). Karp’s characterization of the minimum cycle mean became important for cyclic systems and average-cost optimization ([64]). Butkovič and Cuninghame-Green developed algorithms for maximum cycle means in matrices ([27]). Romanovski studied stationary control of discrete processes ([140]). Lozovanu

investigated multiobjective control and dynamic games on networks ([106], [107]) and polynomial-time algorithms for cyclic games ([108]). Pickl and Lozovanu proposed algorithms for finite Markov processes, state-time probabilities, and limit matrices ([133], [134]). The algorithmic background was broadened by Lozovanu's work on extremal-combinatorial problems ([105]) and by his linear-parametrical approach to bilinear programming with Fonoberova ([109]). Further joint studies with Pickl addressed stochastic networks ([112]), state-time probabilities and limit matrices ([113]), variable transition times ([114]), random graphs ([115]), stochastic discrete control ([116]), stationary stochastic control on networks ([117]), multiobjective time-discrete systems ([118]), stochastic systems on complex networks ([119]), and polynomial-time algorithms for optimal strategies ([120]). Related work considered varying transition times in a discrete dynamical system ([111]). Luenberger and Ye provided a general treatment of linear and nonlinear programming ([121]), while Malozemov and Omelchenko studied a discrete optimal-control problem with an explicit solution ([122]).

Finally, the study of prescribed sequences of states is connected with algorithms from string matching and combinatorics. Aho and Corasick developed an algorithm for locating multiple words in a text ([1]). Knuth, Morris, and Pratt introduced an efficient method for matching a prescribed string ([74]), while Guibas and Odlyzko studied periods and repetitions in strings ([53]). These results provide useful algorithmic tools for identifying final sequences during stochastic evolution. The combinatorial setting also includes random-graph models, such as those studied by Bollobas ([23]), and polynomial methods related to combinatorial sequences. Drungilas, Jankauskas, and Šiurys investigated polynomial multiples of Borwein polynomials ([39]), while Zbaganu considered a combinatorial problem involving the generation of a prescribed sequence ([160]). These works are not Markov-process theories themselves, but they illustrate the wider mathematical context in which sequence recognition and discrete stochastic evolution can be studied.

Therefore, the development of Markov processes proceeded from classical probability theory to Markov chains, matrix analysis, numerical computation, sto-

chastic control, dynamic programming, and algorithmic decision-making. The incorporation of semi-Markov behavior, multiple objectives, networks, games, and sequence-recognition methods created the conceptual basis for studying stochastic systems subject to prescribed final sequences of states.

5.2 Numerical algorithm for determining the limiting probabilities

Let L be a stochastic system with a finite set of states $V = \{v_1, v_2, \ldots, v_n\}$. The transition of the system from the state v_i to the state v_j is performed with probability p_{ij}, $i, j = \overline{1,n}$, at each discrete time $t = 0, 1, 2, \ldots$. The transition time is equal to one.

The matrix $P = (p_{i,j})_{i,j=\overline{1,n}}$ represents the transition probability matrix of the system L. The following equalities hold:

$$\sum_{j=1}^{n} p_{ij} = 1, \ i = \overline{1,n}. \tag{5.1}$$

Since $p_{i,j}$, $i, j = \overline{1,n}$, are probabilities, the following inequalities are also satisfied:

$$0 \le p_{i,j} \le 1, \ i, j = \overline{1,n}. \tag{5.2}$$

Definition 5.1 *The matrix P, which satisfies the conditions (5.1) and (5.2), is called a stochastic matrix.*

Further, we denote by $p_{i,j}(t)$ the probability that the stochastic system L is in the state v_j at time t, provided that the initial state of the system is v_i, $i, j = \overline{1,n}$, $t = 1, 2, \ldots$. The matrix $P(t) = (p_{i,j}(t))_{i,\ j=\overline{1,\ n}}$ represents the $t-$step transition probability matrix. The formula $P(t) = P^t$, $\forall t > 0$, is known.

If the limit probabilities $p_j^* = \lim_{t\to\infty} p_{i,j}(t)$, $j = \overline{1,n}$, exist and do not depend on the initial state v_i of the stochastic system L, then the system is ergodic and the lim-

iting probabilities p_j^*, $j=\overline{1,n}$, are determined as the unique solution of the system of linear equations

$$\begin{cases} P^T p^* = p^* \\ \sum_{j=1}^{n} p_j^* = 1 \end{cases}, \tag{5.3}$$

where P^T represents the transpose of the matrix P and $p^* = (p_1^*,\ p_2^*,\ \ldots,\ p_n^*)^T$.

If the system is not ergodic, then the system of linear equations (5.3) may not have a unique solution. Nevertheless, there are values $p_{i,j}^*$, $i,j=\overline{1,n}$, which represent the limiting transition probabilities of the system from one state to another. In other words, the value $p_{i,j}^*$ represents the probability that, in the limit as time tends to infinity, the stochastic system L reaches the state v_j, knowing that the system started from the state v_i, $i,j=\overline{1,n}$.

In what follows, we present the definition of the limiting probability matrix using the Cesàro limit. This limit always exists.

Definition 5.2 *The matrix* $P^* = \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} P^k$ *is called the limiting probability matrix of the discrete Markov process with transition probability matrix* P.

The problem of determining the limiting probabilities ([110], [95]) is:

Problem 5.1 *Find the matrix* $P^* = (p_{i,j}^*)_{i,j=\overline{1,n}}$ *of the limiting transition probabilities.*

Problem 5.1 asks to determine the matrix of limiting probabilities

$$P^* = (p_{i,j}^*)_{i,j=\overline{1,n}}$$

of the stochastic Markov process L with a finite set of states $V = \{v_1, v_2, \ldots, v_n\}$ and the matrix of transition probabilities $P = (p_{i,j})_{i,j=\overline{1,n}}$.

Consider the function

$$A : \mathbb{C} \to M_n(\mathbb{C}),\ A(z) = I_n - zP,$$

where I_n is the identity matrix of size $n \times n$. Let $A_{i,j}(z)$, $i, j = \overline{1,n}$, be the entries of the matrix $A(z)$. We have

$$A_{i,j}(z) = \delta_{i,j} - zp_{i,j} \in \mathbb{C}[z], \ \deg(A_{i,j}(z)) \leq 1,$$

where

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \ i, j = \overline{1,n},$$

represents the Kronecker symbol.

We denote $\Delta(z) = |A(z)|$. From the definition of the determinant and the preceding facts, it follows that $\Delta(z) \in \mathbb{C}[z]$, $\deg(\Delta(z)) \leq n$.

Consider $D = \{z \in \mathbb{C} \mid \Delta(z) \neq 0\}$. We have $|\mathbb{C} \setminus D| \leq \deg(\Delta(z)) \leq n$. Also, for $\forall z \in D$ the matrix $A(z)$ is invertible. We define the function

$$H : D \to M_n(\mathbb{C}), \ H(z) = (A(z))^{-1}.$$

Let $H_{i,j}(z)$, $i, j = \overline{1,n}$, be the entries of the matrix $H(z)$. According to the method for inverting the matrices, we have the formula

$$H_{i,j}(z) = \frac{M_{j,i}(z)}{\Delta(z)}, \ i, j = \overline{1,n},$$

where

$$M_{i,j}(z) = (-1)^{i+j} R_{i,j}(z)$$

and $R_{i,j}(z)$ is the matrix obtained from $A(z)$ by eliminating the row i and the column j, $i, j = \overline{1,n}$. Therefore, $M_{j,i}(z) \in \mathbb{C}[z]$ is a polynomial of degree

$$\deg(M_{j,i}(z)) \leq n-1, \ i, j = \overline{1,n}.$$

We have

$$\Delta(1) = |A(1)| = \left| (A_{P,\,ij}(1))_{i,\ j=\overline{1,\,n}} \right| = \left| (\delta_{ij} - p_{ij})_{i,\ j=\overline{1,\,n}} \right|.$$

Since

$$\sum_{j=1}^{n}(\delta_{i,j}-p_{i,j})=\sum_{j=1}^{n}\delta_{i,j}-\sum_{j=1}^{n}p_{i,j}=\delta_{ii}-1=0,\ i=\overline{1,n},$$

it follows that $1\in\mathbb{C}\setminus D$ and $\Delta(z)\vdots(z-1)$.

Let $m(z)$ be the multiplicity of the root z of the polynomial $\Delta(z)$, $\forall z\in\mathbb{C}\setminus D$. Since $H_{i,j}(z)$ is a rational fraction with the denominator $\Delta(z)$, then $H_{i,j}(z)$ can be uniquely decomposed in the form

$$H_{i,j}(z)=B_{i,j}(z)+\sum_{y\in\mathbb{C}\setminus D}\sum_{k=1}^{m(y)}\frac{\alpha_{i,j,k}(y)}{(z-y)^k},\ i,j=\overline{1,n},\tag{5.4}$$

where

$$\alpha_{i,j,k}(y)\in\mathbb{C},\ \forall y\in\mathbb{C}\setminus D,\ k=\overline{1,m(y)},\ i,j=\overline{1,n},$$

and the polynomial $B_{i,j}(z)\in\mathbb{C}[z]$ has

$$\deg(B_{i,j}(z))=\deg(M_{j,i}(z))-\deg(\Delta(z))$$

if $\deg(M_{j,i}(z))\geq\deg(\Delta(z))$, otherwise $B_{i,j}(z)=0$.

For representing the relation (5.4) in a more convenient form, we will use the properties of the functions

$$\nu_k(z)=(1-z)^{-k},\ \forall k\in\mathbb{N}^*.$$

For $k=1$ the formula

$$\nu_1(z)=\sum_{t=0}^{\infty}z^t$$

holds. Generally, for $\forall k\geq 1$ the recurrence relation

$$\nu_{k+1}(z)=\frac{d\nu_k(z)}{k\,dz}$$

holds. Using these properties and the mathematical induction method, we obtain the decomposition

$$\nu_k(z)=\sum_{t=0}^{\infty}T_{k-1}(t)z^t,\ \forall k\geq 1,$$

where $T_{k-1}(t)$ is a polynomial of degree which does not exceed $(k-1)$.

The decomposition (5.4) is transformed in the following way:

$$H_{i,j}(z)=B_{i,j}(z)+\sum_{y\in\mathbb{C}\setminus D}\sum_{k=1}^{m(y)}\frac{(-y)^{-k}\alpha_{i,j,k}(y)}{(1-y^{-1}z)^{k}}=$$

$$=B_{i,j}(z)+\sum_{y\in\mathbb{C}\setminus D}\sum_{k=1}^{m(y)}(-y)^{-k}\alpha_{i,j,k}(y)\nu_k\left(\frac{z}{y}\right)=$$

$$=B_{i,j}(z)+\sum_{y\in\mathbb{C}\setminus D}\sum_{k=1}^{m(y)}\left(-\frac{1}{y}\right)^{k}\alpha_{i,j,k}(y)\sum_{t=0}^{\infty}T_{k-1}(t)\left(\frac{z}{y}\right)^{t}=$$

$$=B_{i,j}(z)+\sum_{t=0}^{\infty}\left(\frac{z}{y}\right)^{t}\sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\left(-\frac{1}{y}\right)^{k+1}\alpha_{i,j,k+1}(y)T_k(t),\ i,j=\overline{1,n}.$$

It is easy to observe that the expressions

$$\sum_{k=0}^{m(y)-1}\left(-\frac{1}{y}\right)^{k+1}\alpha_{i,j,k+1}(y)T_k(t),\ i,j=\overline{1,n},$$

are polynomials of degree less than or equal to $m(y)-1$ and, therefore, they can be written in the polynomial form

$$\sum_{k=0}^{m(y)-1}\beta_{i,j,k}(y)t^{k},$$

where

$$\beta_{i,j,k}(y)\in\mathbb{C},\ i,j=\overline{1,n},\ k=\overline{0,m(y)-1}.$$

In this way, the following transformations hold:

$$H_{i,j}(z)=B_{i,j}(z)+\sum_{t=0}^{\infty}z^{t}\sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\frac{t^{k}}{y^{t}}\beta_{i,j,k}(y)=$$

$$=W_{i,j}(z)+\sum_{t=1+\deg(B_{i,j}(z))}^{\infty}z^{t}\sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\frac{t^{k}}{y^{t}}\beta_{i,j,k}(y),\ i,j=\overline{1,n},\qquad(5.5)$$

where the polynomial $W_{i,j}(z)\in\mathbb{C}[z]$ has the degree

$$\deg(W_{i,j}(z))=\deg(B_{i,j}(z)),\ i,j=\overline{1,n}.$$

Since

$$\|P\| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} p_{i,j} = 1,$$

it follows that

$$\|zP\| = |z|\|P\| = |z|.$$

Let $|z| < 1$. We have

$$H(z) = (I_n - zP)^{-1} = \sum_{t=0}^{\infty} P^T z^t,$$

which implies

$$H_{i,j}(z) = \sum_{t=0}^{\infty} p_{i,j}(t) z^t, \ i,j = \overline{1,n}. \tag{5.6}$$

From the relations (5.5) and (5.6) and from the uniqueness of the generating function, we obtain

$$p_{i,j}(t) = \sum_{y \in \mathbb{C} \setminus D} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y), \ \forall t > \deg(B_{i,j}(z)), \ i,j = \overline{1,n}. \tag{5.7}$$

And, since we have the inequalities

$$0 \le p_{i,j}(t) \le 1, \ i,j = \overline{1,n}, \ \forall t \ge 0,$$

it follows that

$$|y| \ge 1, \ \forall y \in \mathbb{C} \setminus D, \ \beta_{i,j,k}(1) = 0, \ \forall k \ge 1,$$

which implies

$$\alpha_{i,j,k}(1) = 0, \ \forall k \ge 2.$$

Consider the polynomial

$$\Delta(z) = (z-1)^{m(1)} T(z).$$

The relation (5.4) takes the form

$$H_{i,j}(z) = \frac{\alpha_{i,j,1}(1)}{z-1} + B_{i,j}(z) + \sum_{y \in (\mathbb{C} \setminus D) \setminus \{1\}} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k} =$$

$$= \frac{\alpha_{i,j,1}(1)}{z-1} + \frac{Y_{i,j}(z)}{T(z)},\ i,j = \overline{1,n},$$

where $Y_{i,j}(z) \in \mathbb{C}[z]$ and

$$\deg(Y_{i,j}(z)) = \deg(B_{i,j}(z)) + \deg(T(z)) = \deg(B_{i,j}(z)) + \deg(\Delta(z)) - m(1) =$$

$$= \deg(M_{j,i}(z)) - m(1) \le n-1-m(1) \le n-2,\ i,j = \overline{1,n}.$$

Further, we denote

$$Y(z) = (Y_{i,j}(z))_{i,j=\overline{1,n}},\ \alpha_1(1) = (\alpha_{i,j,1}(1))_{i,j=\overline{1,n}}.$$

The decomposition of the matrix $H(z)$ takes the form

$$H(z) = \frac{1}{z-1}\alpha_1(1) + \frac{1}{T(z)}Y(z) \tag{5.8}$$

It was shown in [56] that the limiting probability matrix satisfies the relation

$$P^* = -\alpha_1(1), \tag{5.9}$$

i.e. the limiting probability matrix is the coefficient of $(1-z)^{-1}$ in the above decomposition of the matrix $H(z) = (I_n - zP)^{-1}$. From the decomposition (5.8) and the relation (5.9), we obtain the formula

$$P^* = \lim_{z \to 1}(1-z)(I_n - zP)^{-1}$$

for determining the limiting probability matrix P^*.

We consider the characteristic polynomial of the matrix P,

$$K(z) = \left|P - zI_n\right| = \sum_{k=0}^{n} \nu_k z^k.$$

The leading coefficient of the polynomial $K(z)$ is $\nu_n = \left|-I_n\right| = (-1)^n \neq 0$. Therefore, $\deg(K(z)) = n$ and the polynomial $K(z)$ can be written in the form

$$K(z) = (-1)^n (z^n - \alpha_1 z^{n-1} - \alpha_2 z^{n-2} - \ldots - \alpha_n).$$

If we denote $\alpha_0 = -1$, then we readily obtain the relation

$$\nu_k = (-1)^{n+1}\alpha_{n-k},\ k = \overline{0,n}.$$

The Leverrier method for determining the coefficients α_k, $k=\overline{1,n}$, was described in [54]. It consists of the following steps:

- Determine the matrices $P^k=(p_{i,j}(k))_{i,j=\overline{1,n}}$, $k=\overline{1,n}$;
- Obtain the traces of the matrices: $s_k=tr\,P^k=\sum_{j=1}^{n}p_{j,j}(k)$, $k=\overline{1,n}$;
- Determine the coefficients $\alpha_k=\frac{1}{k}\left(s_k-\sum_{j=1}^{k-1}\alpha_j s_{k-j}\right)$, $k=\overline{1,n}$.

Next, we determine the coefficients of the polynomial $\Delta(z)=\sum_{k=0}^{n}\beta_k z^k$. Let $\forall z\in\mathbb{C}\setminus\{0\}$. We have the following transformations:

$$\Delta(z)=\left|I_n-zP\right|=(-z)^n\left|P-z^{-1}I_n\right|=(-1)^n z^n K(z^{-1})=(-1)^n z^n\sum_{k=0}^{n}\nu_k z^{-k}=$$

$$=(-1)^n\sum_{k=0}^{n}\nu_k z^{n-k}=\sum_{k=0}^{n}(-1)^n\nu_{n-k}z^k=\sum_{k=0}^{n}(-1)^n(-1)^{n+1}\alpha_k z^k=\sum_{k=0}^{n}(-\alpha_k)z^k.$$

For $z=0$, we have

$$\Delta(0)=\left|I_n\right|=1=-\alpha_0=\sum_{k=0}^{n}(-\alpha_k)z^k.$$

Therefore,

$$\Delta(z)=\sum_{k=0}^{n}(-\alpha_k)z^k,\ \forall z\in\mathbb{C}.$$

We obtain $\beta_k=-\alpha_k$, $k=\overline{0,n}$. The formula for the coefficients β_k, $k=\overline{0,n}$, is

$$\beta_k=-\alpha_k=-\frac{1}{k}\left(s_k-\sum_{j=1}^{k-1}\alpha_j s_{k-j}\right)=$$

$$=-\frac{1}{k}\left(s_k+\sum_{j=1}^{k-1}\beta_j s_{k-j}\right),\ k=\overline{1,n},\ \beta_0=-\alpha_0=1.$$

Therefore, the coefficients of the polynomial $\Delta(z)$ can be determined using the following algorithm:

Algorithm 5.1

Input: $P \in M_n(\mathbb{C})$;

Output: β_k, $k = \overline{0,n}$, *where* $\sum_{k=0}^{n} \beta_k z^k = \Delta(z)$;

- *Determine the matrix* $P^k = (p_{i,j}(k))_{i,j=\overline{1,n}}$, $k = \overline{1,n}$;
- *Determine the traces of the matrices:* $s_k = tr\, P^k = \sum_{j=1}^{n} p_{j,j}(k)$, $k = \overline{1,n}$;
- *Determine the coefficients:* $\beta_0 = 1$, $\beta_k = -\frac{1}{k}\left(s_k + \sum_{j=1}^{k-1} \beta_j s_{k-j} \right)$, $k = \overline{1,n}$.

Let $\delta(z) = (z-1)T(z)$ and $N = \deg(\delta(z)) = n - (m(1)-1)$. As shown above, the matrix $H(z)$ can be written in the form

$$H(z) = \frac{1}{\delta(z)} \sum_{k=0}^{N-1} z^k R^{(k)},$$

where the matrices $R^{(k)}$ can be obtained from the relation

$$(z-1)^{m(1)-1} \sum_{k=0}^{N-1} z^k R_{i,j}^{(k)} = M_{j,i}(z),\ i,j = \overline{1,n}.$$

From the relation

$$I_n = (I_n - zP)(I_n - zP)^{-1},$$

we obtain the formula for the matrix $\delta(z) I_n$:

$$\delta(z) I_n = \sum_{k=0}^{N-1} z^k R^{(k)} - \sum_{k=0}^{N-1} z^{k+1} (PR^{(k)}) =$$

$$= R^{(0)} + \sum_{k=1}^{N-1} z^k (R^{(k)} - PR^{(k-1)}) - z^n (PR^{(N-1)}).$$

Let $\delta(z) = \sum_{k=0}^{N} \beta_k^* z^k$. We obtain the matrices $R^{(k)}$, $k = \overline{0, N-1}$:

$$R^{(0)} = \beta_0^* I_n,\ R^{(k)} = \beta_k^* I_n + PR^{(k-1)},\ k = \overline{1, N-1}. \qquad (5.10)$$

Therefore,

$$H(z)=\left(\frac{V_{i,j}(z)}{\delta(z)}\right)_{i,j=\overline{1,n}},$$

where

$$V_{i,j}(z)=\sum_{k=0}^{N-1}R_{i,j}^{(k)}z^k,\ i,j=\overline{1,n}.$$

Consider the polynomials

$$T(z)=\sum_{k=0}^{N-1}\gamma_k z^k,\ Y(z)=\sum_{k=0}^{N-2}y^{(k)}z^k,\ y^*=\alpha_1(1).$$

According to (5.8), we have

$$\frac{V_{i,j}(z)}{\delta(z)}=H_{i,j}(z)=\frac{y_{i,j}^*}{z-1}+\frac{\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k}{T(z)},\ i,j=\overline{1,n}.$$

This relation implies

$$\sum_{k=0}^{N-1}R_{i,j}^{(k)}z^k=V_{i,j}(z)=y_{i,j}^*T(z)+(z-1)\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=$$

$$=y_{i,j}^*\sum_{k=0}^{N-1}\gamma_k z^k+\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^{k+1}-\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=$$

$$=\sum_{k=0}^{N-1}\gamma_k y_{i,j}^*z^k+\sum_{k=1}^{N-1}y_{i,j}^{(k-1)}z^k-\sum_{k=0}^{N-2}y_{i,j}^{(k)}z^k=$$

$$=(\gamma_0 y_{i,j}^*-y_{i,j}^{(0)})+\sum_{k=1}^{N-2}(\gamma_k y_{i,j}^*+y_{i,j}^{(k-1)}-y_{i,j}^{(k)})z^k+$$

$$+(\gamma_{N-1}y_{i,j}^*+y_{i,j}^{(N-2)})z^{N-1},\ i,j=\overline{1,n}.$$

By equating the coefficients of like powers of z, we obtain the system of equations

$$\begin{cases}R_{i,j}^{(0)}=\gamma_0 y_{i,j}^*-y_{i,j}^{(0)},\\ R_{i,j}^{(k)}=\gamma_k y_{i,j}^*+y_{i,j}^{(k-1)}-y_{i,j}^{(k)},\ \ k=\overline{1,N-2},\ i,j=\overline{1,n},\\ R_{i,j}^{(N-1)}=\gamma_{N-1}y_{i,j}^*+y_{i,j}^{(N-2)},\end{cases}$$

equivalent to the system

$$\begin{cases} y_{i,j}^{(0)} = \gamma_0 y_{i,j}^* - R_{i,j}^{(0)}, \\ y_{i,j}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)}, \ k = \overline{1, N-2}, \ i, j = \overline{1, n}. \\ y_{i,j}^{(N-2)} = -\gamma_{N-1} y_{i,j}^* + R_{i,j}^{(N-1)}. \end{cases}$$

We observe that there exist coefficients

$$u_{i,j}^{(k)}, v_{i,j}^{(k)} \in \mathbb{C}, \ k = \overline{0, N-2}, \ i, j = \overline{1, n},$$

such that the relation

$$y_{i,j}^{(k)} = u_{i,j}^{(k)} y_{i,j}^* + v_{i,j}^{(k)}, \ k = \overline{0, N-2}, \ i, j = \overline{1, n},$$

holds. From the first equation we obtain

$$u_{i,j}^{(0)} = \gamma_0, \ v_{i,j}^{(0)} = -R_{i,j}^{(0)}, \ i, j = \overline{1, n},$$

and from the following $N-2$ equations we obtain the coefficients

$$y_{i,j}^{(k)} = \gamma_k y_{i,j}^* + y_{i,j}^{(k-1)} - R_{i,j}^{(k)} = \gamma_k y_{i,j}^* + u_{i,j}^{(k-1)} y_{i,j}^* + v_{i,j}^{(k-1)} - R_{i,j}^{(k)} =$$

$$= (\gamma_k + u_{i,j}^{(k-1)}) y_{i,j}^* + (v_{i,j}^{(k-1)} - R_{i,j}^{(k)}), \ k = \overline{1, N-2}, \ i, j = \overline{1, n},$$

which implies the recurrence relations

$$u_{i,j}^{(k)} = u_{i,j}^{(k-1)} + \gamma_k, \ v_{i,j}^{(k)} = v_{i,j}^{(k-1)} - R_{i,j}^{(k)}, \ k = \overline{1, N-2}, \ i, j = \overline{1, n}.$$

We obtain the direct formulas

$$u_{i,j}^{(k)} = \sum_{r=0}^{k} \gamma_r, \ v_{i,j}^{(k)} = -\sum_{r=0}^{k} R_{i,j}^{(r)}, \ k = \overline{0, N-2}, \ i, j = \overline{1, n}.$$

By substituting them in the last equation of the system, we obtain

$$u_{i,j}^{(N-2)} y_{i,j}^* + v_{i,j}^{(N-2)} = -\gamma_{N-1} y_{i,j}^* + R_{i,j}^{(N-1)},$$

equivalent to the relation

$$y_{i,j}^* \sum_{r=0}^{N-1} \gamma_r = \sum_{r=0}^{N-1} R_{i,j}^{(r)}, \ i, j = \overline{1, n},$$

which implies the formula

$$y_{i,j}^{*}=\frac{\sum\limits_{r=0}^{N-1}R_{i,j}^{(r)}}{\sum\limits_{r=0}^{N-1}\gamma_r}=\frac{R_{i,j}}{T(1)},$$

where

$$R_{i,j}=\sum_{r=0}^{N-1}R_{i,j}^{(r)}\ ,\ i,j=\overline{1,n}\ .$$

By denoting $R=(R_{i,j})_{i,j=\overline{1,n}}$, we obtain the formula

$$P^{*}=-\frac{1}{T(1)}R\ . \tag{5.11}$$

Therefore, we have theoretically justified the following algorithm of complexity $O(n^4)$ for determining the matrix of limiting transition probabilities:

Algorithm 5.2

Input: $P\in M_n(\mathbb{R})$;

Output: $P^{*}\in M_n(\mathbb{R})$;

1) *Find the coefficients of the polynomial* $\Delta(z)=\sum\limits_{k=0}^{n}\beta_k z^k$ *by using Algorithm 5.1;*
2) *Using Horner's scheme, perform* $m(1)$ *successive divisions of the polynomial* $\Delta(z)$ *by* $z-1$, *obtaining the polynomial* $T(z)$ *such that* $T(1)\neq 0$. *At the same time, store the coefficients* β_k^{*}, $k=\overline{0,N}$, *of the polynomial* $(z-1)T(z)$, *obtained after the penultimate division;*
3) *Determine the value* $T(1)$;
4) *Determine the matrices* $R^{(k)}$, $k=\overline{0,N-1}$, *by using the formula (5.10);*
5) *Compute the matrix* $R=\sum\limits_{k=0}^{N-1}R^{(k)}$;
6) *Determine the matrix* P^{*} *using the formula (5.11)*

Example 5.1 *Consider the discrete Markov stochastic system* L *with the transition probability matrix*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Determine the limiting probability matrix.

Solution. It is easy to observe that, for each positive integer number n, we have

$$P^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the stochastic system L represents a 2-periodic Markov process. In this case, the matrix sequence $(P^n)_{n=0}^{\infty}$ is divergent. However, the limiting probability matrix exists:

$$P^* = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k .$$

This matrix may be determined using Algorithm 5.2. According to this algorithm, we have the following steps:

- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; s_1 = trP = 0, \; s_2 = trP^2 = 2,$

  $\beta_0 = 1, \; \beta_1 = -s_1 = 0, \; \beta_2 = -\frac{1}{2}(s_2 + \beta_1 s_1) = -1;$

- Divide the polynomial $\beta_2 z^2 + \beta_1 z + \beta_0$ by $(z-1)$ as many times as possible:

	-1	0	1
1	-1	-1	0
1	-1	-2	

  and obtain $m(1) = 1, \; N = 2; \; \beta_0^* = 1, \; \beta_1^* = 0, \; \beta_2^* = -1; \; \gamma_0 = -1, \; \gamma_1 = -1;$

- $T(1) = \gamma_0 + \gamma_1 = -2;$

- $R^{(0)} = \beta_0^* I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; R^{(1)} = \beta_1^* + PR^{(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$

- $R = R^{(0)} + R^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$;
- $P^* = -\frac{1}{T(1)} R = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$.

In this way, we obtained the limiting probability matrix

$$P^* = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

The rows of this matrix are equal, which means that the limiting probabilities do not depend on the initial state of the system. Even if the system L is not ergodic (since for each positive integer number n the matrix P^n has entries equal to zero), the limiting distribution $\pi^* = (0.5, 0.5)$ of the states of the system exists. □

Example 5.2 *Consider the Markov process L with the transition matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Determine the limiting probability matrix of the system.

Solution. We determine the limiting probability matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},\ P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4/9 & 4/9 & 1/9 \end{pmatrix},\ P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 13/27 & 13/27 & 1/27 \end{pmatrix} \Rightarrow$$

$$\Rightarrow\ s_1 = tr\, P = 7/3,\ s_2 = tr\, P^2 = 19/9,\ s_3 = tr\, P^3 = 55/27\ \Rightarrow$$

$$\Rightarrow\ \beta_0 = 1,\ \beta_1 = -s_1 = -7/3,\ \beta_2 = -(s_2 + \beta_1 s_1)/2 = 5/3,$$

$$\beta_3 = -(s_3 + \beta_1 s_2 + \beta_2 s_1)/3 = -1/3\ \Rightarrow$$

	-1/3	5/3	-7/3	1
1	-1/3	4/3	-1	0
1	-1/3	1	0	
1	-1/3	2/3		

$$\Rightarrow \beta_0^* = -1,\ \beta_1^* = 4/3,\ \beta_2^* = -1/3;\ \gamma_0 = 1,\ \gamma_1 = -1/3 \Rightarrow$$

$$\Rightarrow T(1) = \gamma_0 + \gamma_1 = 2/3;$$

$$R^{(0)} = \beta_0^* I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ R^{(1)} = \beta_1^* I + PR^{(0)} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ -1/3 & -1/3 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow R = R^{(0)} + R^{(1)} = \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & -2/3 & 0 \\ -1/3 & -1/3 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow P^* = -\frac{1}{T(1)} R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Therefore, the limiting probability matrix of the system L is

$$P^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$ □

5.3 Determining differential components using homogeneous linear recurrences

It was shown in Section 5.2 that the entries of the matrix $P(t)$ can be represented in the form (5.7), i.e.

$$p_{i,j}(t) = \sum_{y \in \mathbb{C} \setminus D} \sum_{k=0}^{m(y)-1} \frac{t^k}{y^t} \beta_{i,j,k}(y),\ \forall t > deg(B_{i,j}(z)),\ i, j = \overline{1,n},$$

where

$$D = \{z \in \mathbb{C} \mid | I_n - zP | \neq 0\},\ \beta_{i,j,k}(y) \in \mathbb{C},\ \forall y \in \mathbb{C} \setminus D,\ k = \overline{0, m(y)-1},$$

$m(y)$ is the multiplicity of the root y of the polynomial $\Delta(z) = | I_n - zP |$ and $B_{i,j}(z)$ is a polynomial of maximal degree $n-1$, $i, j = \overline{1,n}$. Denoting by

$$\beta_k(y) = (\beta_{i,j,k}(y))_{i,j=\overline{1,n}},\ \forall y \in \mathbb{C} \setminus D,\ k = \overline{0, m(y)-1},$$

we obtain

$$P(t)=\sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\frac{t^k}{y^t}\beta_k(y),\ \forall t\geq n. \tag{5.12}$$

We mention that the set $\mathbb{C}\setminus D$ is formed by the reciprocals of the nonzero eigenvalues of the matrix P, the multiplicity of each element being equal to that of the corresponding eigenvalue. Therefore, the relation (5.12) represents the decomposition of the transition matrix $P(t)$ with respect to the eigenvalues of the matrix P. The problem of determining the differential components is:

Problem 5.2 *Determine the matrix coefficients* $\beta_k(y),\ \forall y\in\mathbb{C}\setminus D,\ k=\overline{0,m(y)-1}$.

Consider the discrete stochastic system L with a finite set of states V ($|V|=n$) and transition time equal to one. Let $P=(p_{i,j})_{i,j=\overline{1,n}}$ be the transition probability matrix of the system L. The $t-$step transition matrix of the system is $P(t)=P^t,\ \forall t\geq 0$. We take into account the problem of determining the matrix coefficients $\beta_k(y),\ \forall y\in\mathbb{C}\setminus D,\ k=\overline{0,m(y)-1}$.

Consider the matrix sequence $a=(P(t))_{t=0}^{\infty}$. It is easy to observe that the recurrence relation

$$a_t=Pa_{t-1},\ \forall t\geq 1,$$

holds. Therefore,

$$a\in Rol^*[M_n(\mathbb{R})][1],\ q=(P)\in G^*[M_n(\mathbb{R})][1](a).$$

From Theorem 1.28, we have

$$a\in Rol^*[\mathbb{R}][n],\ \Delta(z)=\left|I_n-zP\right|\in H^*[\mathbb{R}][n](a).$$

Let $r=deg\Delta(z)$. Consider the subsequence $\overline{a}=(P(t))_{t=n-r}^{\infty}$ of a. We have

$$\overline{a}\in Rol[\mathbb{R}][r],\ \Delta(z)\in H[\mathbb{R}][r](\overline{a}).$$

By writing these relations entrywise, we obtain

$$\overline{a}_{i,j}\in Rol[\mathbb{R}][r],\ \Delta(z)\in H[\mathbb{R}][r](\overline{a}_{ij}),\ i,j=\overline{1,n}.$$

Based on the direct formula for homogeneous linear recurrences, we have

$$p_{i,j}(t) = a_{i,j}(t) = \overline{a}_{i,j}(t-n+r) = I_r^{[\overline{a}_{i,j}]}(B^T)^{-1}(\beta_{t-n+r})^T,$$
$$i,j = \overline{1,n},\ \forall t \geq n-r, \tag{5.13}$$

where

$$\beta_t = \left(\frac{t^k}{y^t}\right)_{\substack{y\in\mathbb{C}\setminus D,\\ k=\overline{0,m(y)-1}}},\ \forall t \geq 0,\ B = (\beta_j)_{j=\overline{0,r-1}},\ 0^0 \equiv 1. \tag{5.14}$$

The way to determine the initial state of the sequences $\overline{a}_{i,j}$, $i,j = \overline{1,n}$, is obvious:

$$I_r^{[\overline{a}_{i,j}]} = (\overline{a}_{i,j}(t))_{t=0}^{r-1} = (a_{i,j}(t))_{t=n-r}^{n-1} = (p_{i,j}(t))_{t=n-r}^{n-1},\ i,j = \overline{1,n}. \tag{5.15}$$

By denoting

$$I_r^{[\overline{a}_{i,j}]}(B^T)^{-1} = (\gamma_{i,j,s}(y))_{y\in\mathbb{C}\setminus D,\, s=\overline{0,m(y)-1}},\ i,j = \overline{1,n}, \tag{5.16}$$

the formula (5.13) is transformed in the following way:

$$p_{i,j}(t) = \sum_{y\in\mathbb{C}\setminus D}\sum_{s=0}^{m(y)-1}\frac{(t-n+r)^s}{y^{t-n+r}}\ \gamma_{i,j,s}(y) =$$

$$= \sum_{y\in\mathbb{C}\setminus D}\sum_{s=0}^{m(y)-1}\sum_{k=0}^{s} C_s^k (r-n)^{s-k} y^{n-r}\gamma_{i,j,s}(y)\frac{t^k}{y^t} =$$

$$= \sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\frac{t^k}{y^t}\sum_{s=k}^{m(y)-1} C_s^k (r-n)^{s-k} y^{n-r}\gamma_{i,j,s}(y) =$$

$$= \sum_{y\in\mathbb{C}\setminus D}\sum_{k=0}^{m(y)-1}\frac{t^k}{y^t}\beta_{i,j,k}(y),\ i,j = \overline{1,n},\ \forall t \geq n-r, \tag{5.17}$$

where

$$\beta_{i,j,k}(y) = y^{n-r}\sum_{s=k}^{m(y)-1} C_s^k (r-n)^{s-k}\gamma_{i,j,s}(y),$$
$$\forall y \in \mathbb{C}\setminus D,\ k = \overline{0,m(y)-1},\ i,j = \overline{1,n}. \tag{5.18}$$

By writing the relations (5.17) in matrix form, we obtain the decomposition (5.12), the entries of the matrices $\beta_k(y)$ ($y \in \mathbb{C} \setminus D$, $k = \overline{0, m(y)-1}$) being determined according to (5.18).

In this way, we theoretically justified the following algorithm which decomposes the $t-$step transition matrix of the discrete stochastic systems with transition time equal to one:

Algorithm 5.3

Input: the transition matrix P;

Output: the matrix coefficients $\beta_k(y)$ ($y \in \mathbb{C} \setminus D$, $k = \overline{0, m(y)-1}$).

1) *Compute the coefficients of the polynomial $\Delta(z)$ based on the characteristic polynomial of the matrix P. Algorithm 5.1, based on the Leverrier method, was described in Section 5.2;*
2) *Find the complex solutions of the equation $\Delta(z) = 0$ by using an efficient numerical method. Thus, the set of solutions $\mathbb{C} \setminus D$ is obtained;*
3) *If the multiplicities of the obtained solutions are not known yet, they can be determined according to Horner's scheme by successively dividing the polynomial $\Delta(z)$ as many times as possible by the binomial $(z-y)$, $\forall y \in \mathbb{C} \setminus D$. The multiplicity $m(y)$ of each solution $y \in \mathbb{C} \setminus D$ is equal to the number of divisions performed;*
4) *By using the formula (5.14), obtain the matrix B;*
5) *By applying an efficient numerical method, determine the matrix $(B^T)^{-1}$;*
6) *Compute the values C_s^k, $s = \overline{0, \max\limits_{y \in \mathbb{C} \setminus D} m(y)-1}$, $k = \overline{0, s}$, using Pascal's triangle:*

   $C_s^0 = C_s^s = 1$, $C_s^k = C_{s-1}^{k-1} + C_{s-1}^k$ ($k = \overline{1, s-1}$);

7) *Recursively, determine the values $(r-n)^s$, $s = \overline{0, \max\limits_{y \in \mathbb{C} \setminus D} m(y)-1}$;*
8) *For each $i, j = \overline{1, n}$, the following steps are performed:*
   a. *Determine the initial state $I_r^{[\overline{a}_{i,j}]}$ using the formula (5.15);*

b. *Compute the values* $\gamma_{i,j,s}(y)$, $y \in \mathbb{C} \setminus D$, $s = \overline{0, m(y)-1}$, *according to (5.16);*

c. *For each* $y \in \mathbb{C} \setminus D$, $k = \overline{0, m(y)-1}$, *using the formula (5.18) and the values obtained at the steps* $6)-7)$, *find the entry* $\beta_{i,j,k}(y)$ *of the matrix* $\beta_k(y)$.

Remark 5.1 *The first two steps of this algorithm can be substituted with any efficient method for determining the eigenvalues of the matrix* P, *the set* $\mathbb{C} \setminus D$ *containing the reciprocals of the non-zero eigenvalues, with the same multiplicities.*

Remark 5.2 *If the eigenvalues of the matrix* P *are known, then this algorithm consists only of the steps* $3)-8)$, *the complexity being* $O(n^4)$.

Remark 5.3 *The matrix* $\beta_0(1)$ *represents the limiting probability matrix of the stochastic system* L, *the complexity of its determination being the same* $-$ $O(n^4)$.

Remark 5.4 *We obtained the decomposition*

$$P(t) = \sum_{y \in \mathbb{C} \setminus D} \sum_{k=0}^{m(y)-1} \beta_k(y) \frac{t^k}{y^t}, \ \forall t \geq n-r.$$

For $t = \overline{0, n-r-1}$, *the decomposition can be written in the form*

$$P(t) = L(t) + \sum_{y \in \mathbb{C} \setminus D} \sum_{k=0}^{m(y)-1} \beta_k(y) \frac{t^k}{y^t}, \tag{5.19}$$

where $L(t)$ *is a constant matrix for a fixed value of* t. *Knowing the matrices* $\beta_k(y)$, $\forall y \in \mathbb{C} \setminus D$, $k = \overline{0, m(y)-1}$, *we can determine these matrices by using the formula* (5.19), *taking into account the relation* $P(t) = P^t$, $\forall t \geq 0$.

Remark 5.5 *It was mentioned in [56] and [137] that the matrices* $L(t)$, $t = \overline{0, n-r-1}$ *and the matrix coefficients* $\beta_k(y)$, $y \in (\mathbb{C} \setminus D) \setminus \{1\}$, $k = \overline{0, m(y)-1}$, *are differential matrices, i.e. the sum of entries from each row is zero. The unique non-differential matrix coefficient in the decomposition is the stochastic matrix* $\beta_0(1)$ *and the other matrices* $\beta_k(1)$, $k = \overline{1, m(1)-1}$, *are zero.*

Example 5.3 *Consider the Markov process* L *with the transition matrix*

$$P=\begin{pmatrix}1 & 0 & 0\\ 0 & 0.5 & 0.5\\ 0.5 & 0 & 0.5\end{pmatrix}.$$

Find the limiting probability matrix and the differential components of this system.

Solution. We will compute the limiting probability matrix and the differential components of the stochastic system L by using the method based on the theory of homogeneous linear recurrences. We have:

$$P=\begin{pmatrix}1 & 0 & 0\\ 0 & 0.5 & 0.5\\ 0.5 & 0 & 0.5\end{pmatrix},\ P^2=\begin{pmatrix}1 & 0 & 0\\ 0.25 & 0.25 & 0.5\\ 0.75 & 0 & 0.25\end{pmatrix},\ P^3=\begin{pmatrix}1 & 0 & 0\\ 0.5 & 0.125 & 0.375\\ 0.875 & 0 & 0.125\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ s_1=tr\,P=2\,,\ s_2=tr\,P^2=1.5\,,\ s_3=tr\,P^3=1.25\ \Rightarrow$$

$$\Rightarrow\ \overline{\beta}_0=1\,,\ \overline{\beta}_1=-s_1=-2\,,\ \overline{\beta}_2=-(s_2+\overline{\beta}_1 s_1)/2=1.25\,,$$

$$\overline{\beta}_3=-(s_3+\overline{\beta}_1 s_2+\overline{\beta}_2 s_1)/3=-0.25\ \Rightarrow$$

$$\Rightarrow\ \Delta(z)=\sum_{k=0}^{3}\overline{\beta}_k z^k=1-2z+1.25z^2-0.25z^3=(1-z)(1-0.5z)^2\ \Rightarrow$$

$$\Rightarrow\ \mathbb{C}\setminus D=\{z\in\mathbb{C}\mid\Delta(z)=0\}=\{1,2\}\,,\ m(1)=1,\ m(2)=2\,,\ r=n=3\ \Rightarrow$$

$$\Rightarrow\ \beta_0=(1,1,0)\,,\ \beta_1=(1,0.5,0.5)\,,\ \beta_2=(1,0.25,0.5)\ \Rightarrow$$

$$\Rightarrow\ B=\begin{pmatrix}1 & 1 & 0\\ 1 & 0.5 & 0.5\\ 1 & 0.25 & 0.5\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ (B^T)^{-1}=\begin{pmatrix}1 & 0 & -2\\ -4 & 4 & 6\\ 4 & -4 & -4\end{pmatrix}.$$

$$C_0^0=C_1^0=C_1^1=1\,,\ (r-n)^0=0^0=1,\ (r-n)^1=0^1=0\,.$$

$$\Gamma_{11}=I_3^{[\overline{a}_{11}]}(B^T)^{-1}=(1,1,1)\begin{pmatrix}1 & 0 & -2\\ -4 & 4 & 6\\ 4 & -4 & -4\end{pmatrix}=(1,0,0)\,;$$

$$\Gamma_{12} = (0,0,0)(B^T)^{-1} = (0,0,0)\,;\ \Gamma_{13} = (0,0,0)(B^T)^{-1} = (0,0,0)\,;$$

$$\Gamma_{21} = (0,0,0.25)(B^T)^{-1} = (1,-1,-1)\,;\ \Gamma_{22} = (1,0.5,0.25)(B^T)^{-1} = (0,1,0)\,;$$

$$\Gamma_{23} = (0,0.5,0.5)(B^T)^{-1} = (0,0,1)\,;\ \Gamma_{31} = (0,0.5,0.75)(B^T)^{-1} = (1,-1,0)\,;$$

$$\Gamma_{32} = (0,0,0)(B^T)^{-1} = (0,0,0)\,;\ \Gamma_{33} = (1,0.5,0.25)(B^T)^{-1} = (0,1,0)\,.$$

$$\beta_{ijk}(y) = \sum_{s=k}^{m(y)-1} 0^{s-k}\gamma_{ijs}(y) = \gamma_{ijk}(y)\,,\ \forall y \in \mathbb{C}\setminus D\,,\ k = \overline{0, m(y)-1}\,,\ i,j = \overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_0(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \beta_0(2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ \beta_1(2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the transition probability matrix can be written in the form

$$P(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\left(\frac{1}{2}\right)^t + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t\left(\frac{1}{2}\right)^t,\ \forall t \geq 0\,.$$

The limiting probability matrix is

$$P^* = \beta_0(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the differential components are

$$\beta_0(2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},\ \beta_1(2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

Example 5.4 *Consider the Markov process L with the transition matrix*

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

Find the limiting probability matrix and the differential components of the system.

Solution. Let us compute the limiting probability matrix and the differential components of this Markov process.

$$P=\begin{pmatrix}1/2 & 0 & 1/2\\ 1/2 & 1/4 & 1/4\\ 1/4 & 1/2 & 1/4\end{pmatrix},\ P^2=\begin{pmatrix}3/8 & 1/4 & 3/8\\ 7/16 & 3/16 & 3/8\\ 7/16 & 1/4 & 5/16\end{pmatrix},$$

$$P^3=\begin{pmatrix}13/32 & 1/4 & 11/32\\ 13/32 & 15/64 & 23/64\\ 27/64 & 7/32 & 23/64\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ s_1=tr\,P=1,\ s_2=tr\,P^2=7/8,\ s_3=tr\,P^3=1\ \Rightarrow$$

$$\Rightarrow\ \overline{\beta}_0=1,\ \overline{\beta}_1=-s_1=-1,\ \overline{\beta}_2=-(s_2+\overline{\beta}_1 s_1)/2=1/16,$$

$$\overline{\beta}_3=-(s_3+\overline{\beta}_1 s_2+\overline{\beta}_2 s_1)/3=-1/16\ \Rightarrow$$

$$\Rightarrow\ \Delta(z)=\sum_{k=0}^{3}\overline{\beta}_k z^k=1-z+\frac{1}{16}z^2-\frac{1}{16}z^3=-\frac{1}{16}(z-1)(z-4i)(z+4i)\ \Rightarrow$$

$$\Rightarrow\ \mathbb{C}\setminus D=\{z\in\mathbb{C}\mid\Delta(z)=0\}=\{1,-4i,4i\},\ m(1)=m(-4i)=m(4i)=1,\ r=n=3\ \Rightarrow$$

$$\Rightarrow\ \beta_0=(1,1,1),\ \beta_1=(1,\,i/4,\,-i/4),\ \beta_2=(1,-1/16,-1/16)\ \Rightarrow$$

$$\Rightarrow\ B=\begin{pmatrix}1 & 1 & 1\\ 1 & i/4 & -i/4\\ 1 & -1/16 & -1/16\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ (B^T)^{-1}=\frac{1}{17}\begin{pmatrix}1 & 8+2i & 8-2i\\ 0 & -34i & 34i\\ 16 & -8+32i & -8-32i\end{pmatrix}.$$

$$C_0^0=1,\ (r-n)^0=0^0=1.$$

$$\Gamma_{11}=I_3^{[\overline{a}_{11}]}(B^T)^{-1}=(1,1/2,3/8)(B^T)^{-1}=(7/17,5/17-3i/17,5/17+3i/17);$$

$$\Gamma_{12}=(0,0,1/4)(B^T)^{-1}=(4/17,-2/17+8i/17,-2/17-8i/17);$$

$$\Gamma_{13}=(0,1/2,3/8)(B^T)^{-1}=(6/17,-3/17-5i/17,-3/17+5i/17);$$

$$\Gamma_{21}=(0,1/2,7/16(B^T)^{-1}=(7/17,-7/34-3i/17,-7/34+3i/17);$$

$$\Gamma_{22}=(1,1/4,3/16)(B^T)^{-1}=(4/17,13/34-i/34,13/34+i/34);$$

$$\Gamma_{23}=(0,1/4,3/8)(B^T)^{-1}=(6/17,-3/17+7i/34,-3/17-7i/34);$$

$$\Gamma_{31} = (0,1/4,7/16)(B^T)^{-1} = (7/17,-7/34+11i/34,-7/34-11i/34);$$

$$\Gamma_{32} = (0,1/2,1/4)(B^T)^{-1} = (4/17,-2/17-9i/17,-2/17+9i/17);$$

$$\Gamma_{33} = (1,1/4,5/16)(B^T)^{-1} = (6/17,11/34+7i/34,11/34-7i/34).$$

$$\beta_{l,j,k}(y) = \sum_{s=k}^{m(y)-1} 0^{s-k}\gamma_{l,j,s}(y) = \gamma_{l,j,k}(y),\ \forall y \in \mathbb{C}\setminus D,\ k = \overline{0,m(y)-1},\ l,j = \overline{1,3} \Rightarrow$$

$$\Rightarrow \beta_0(1) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix},$$

$$\beta_0(-4i) = \begin{pmatrix} 5/17-3i/17 & -2/17+8i/17 & -3/17-5i/17 \\ -7/34-3i/17 & 13/34-i/34 & -3/17+7i/34 \\ -7/34+11i/34 & -2/17-9i/17 & 11/34+7i/34 \end{pmatrix},$$

$$\beta_0(4i) = \begin{pmatrix} 5/17+3i/17 & -2/17-8i/17 & -3/17+5i/17 \\ -7/34+3i/17 & 13/34+i/34 & -3/17-7i/34 \\ -7/34-11i/34 & -2/17+9i/17 & 11/34-7i/34 \end{pmatrix}.$$

Therefore, the transition probability matrix can be decomposed in the form

$$P(t) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix} +$$

$$+\begin{pmatrix} 5/17-3i/17 & -2/17+8i/17 & -3/17-5i/17 \\ -7/34-3i/17 & 13/34-i/34 & -3/17+7i/34 \\ -7/34+11i/34 & -2/17-9i/17 & 11/34+7i/34 \end{pmatrix}(-4i)^{-t} +$$

$$+\begin{pmatrix} 5/17+3i/17 & -2/17-8i/17 & -3/17+5i/17 \\ -7/34+3i/17 & 13/34+i/34 & -3/17-7i/34 \\ -7/34-11i/34 & -2/17+9i/17 & 11/34-7i/34 \end{pmatrix}(4i)^{-t},\ \forall t \geq 0.$$

The limiting probability matrix is

$$P^* = \beta_0(1) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix}$$

and the differential components are

$$\beta_0(-4i)=\begin{pmatrix} 5/17-3i/17 & -2/17+8i/17 & -3/17-5i/17 \\ -7/34-3i/17 & 13/34-i/34 & -3/17+7i/34 \\ -7/34+11i/34 & -2/17-9i/17 & 11/34+7i/34 \end{pmatrix},$$

$$\beta_0(4i)=\begin{pmatrix} 5/17+3i/17 & -2/17-8i/17 & -3/17+5i/17 \\ -7/34+3i/17 & 13/34+i/34 & -3/17-7i/34 \\ -7/34-11i/34 & -2/17+9i/17 & 11/34-7i/34 \end{pmatrix}. \qquad \square$$

5.4 Finding the differential components by separating the eigenvalues

This algorithm represents a generalization of the method used to determine the limiting probability matrix. We will separate all eigenvalues of the matrix P successively, not only the eigenvalue 1. In this way, we will obtain, in the end, all matrix coefficients in the decomposition (5.12), presented above.

Similarly to the case of determining the limiting probability matrix ([81], [94]), we will determine the matrix

$$H(z)=(I_n-zP)^{-1}.$$

The only difference is that we will not simplify the fraction $H(z)$ by cancelling the factor $(z-1)^{m(1)-1}$. Therefore, we obtain

$$H(z)=\frac{1}{\Delta(z)}\sum_{k=0}^{n-1}R^{(k)}z^k, \tag{5.20}$$

where the matrix coefficients $R^{(k)}$, $k=\overline{0,n-1}$, are determined in the following way:

$$R^{(0)}=\beta_0 I_n;\ R^{(k)}=\beta_k I_n+PR^{(k-1)},\ k=\overline{1,n-1}. \tag{5.21}$$

The values β_k, $k=\overline{0,n}$, represent the coefficients of the polynomial $\Delta(z)$, computed using Algorithm 5.1.

According to (5.4), the entries of the matrix $H(z)$ are written in the following form:

$$H_{i,j}(z) = B_{i,j}(z) + \sum_{y \in \mathbb{C} \setminus D} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k},\ i,j = \overline{1,n}. \tag{5.22}$$

The relation (5.22) can be simplified and written in the form

$$H_{i,j}(z) = B_{i,j}(z) + \frac{Q_{i,j}(z)}{\Delta(z)},\ i,j = \overline{1,n}, \tag{5.23}$$

where $Q_{i,j}(z) \in \mathbb{C}[z]$ and

$$deg(Q_{i,j}(z)) < deg(\Delta(z)) = r,\ i,j = \overline{1,n}.$$

By writing (5.20) entrywise and substituting it in (5.23), we obtain

$$\sum_{k=0}^{n-1} R_{i,j}^{(k)} z^k = B_{i,j}(z)\Delta(z) + Q_{i,j}(z),\ i,j = \overline{1,n}.$$

Therefore, the polynomial

$$B_{i,j}(z) = \sum_{k=0}^{n-1-r} b_{i,j,k} z^k$$

is the quotient and the polynomial

$$Q_{i,j}(z) = \sum_{k=0}^{r-1} q_{i,j,k} z^k$$

is the remainder upon division of the polynomial $\sum_{k=0}^{n-1} R_{i,j}^{(k)} z^k$ by the polynomial $\Delta(z)$ and they can be determined by applying the following algorithm of division with remainder of the polynomials:

Algorithm 5.4

For $i,j = \overline{1,n}$, the following steps are performed:

- $q_{i,j,k} = R_{i,j}^{(k)},\ k = \overline{0,n-1}$;
- *For $k = n-1, n-2, \ldots, r$ the following formulas are applied:*
  - $b_{i,j,k-r} = \dfrac{q_{i,j,k}}{\beta_r}$;
  - $q_{i,j,k-t} = q_{i,j,k-t} - b_{i,j,k-r}\beta_{r-t},\ t = \overline{0,r}$.

Consider $\mu \in \mathbb{C} \setminus D$ with $m(\mu) = m$, i.e. μ^{-1} is a nonzero eigenvalue of multiplicity m. According to (5.22) and (5.23), by separating the root μ, we obtain

$$\frac{Q_{i,j}(z)}{\Delta(z)} = \sum_{k=1}^{m} \frac{\alpha_{i,j,k}(\mu)}{(z-\mu)^k} + \sum_{y \in (\mathbb{C} \setminus D) \setminus \{\mu\}} \sum_{k=1}^{m(y)} \frac{\alpha_{i,j,k}(y)}{(z-y)^k}, \; i,j = \overline{1,n}. \tag{5.24}$$

Consider the polynomials

$$\Delta(z) = (z-\mu)^m D(z),$$

$$D(z) = \sum_{k=0}^{r-m} d_k z^k.$$

We denote $deg(D(z)) = M$. The relation (5.24) takes the form

$$\frac{Q_{i,j}(z)}{\Delta(z)} = \frac{F_{i,j}(z)}{(z-\mu)^m} + \frac{E_{i,j}(z)}{D(z)}, \; i,j = \overline{1,n},$$

where

$$E_{i,j}(z) = \sum_{k=0}^{M-1} e_{i,j,k} z^k,$$

$$F_{i,j}(z) = \sum_{k=0}^{m-1} f_{i,j,k} z^k \in \mathbb{C}[z], \; i,j = \overline{1,n}.$$

Bringing to a common denominator, we obtain

$$Q_{i,j}(z) = F_{i,j}(z) D(z) + E_{i,j}(z)(z-\mu)^m, \; i,j = \overline{1,n}.$$

Denoting

$$\xi(k) = C_m^k (-\mu)^{-k}, \; k = \overline{0,m}$$

and using the Newton binomial expansion, we obtain the formula

$$(z-\mu)^m = \sum_{k=0}^{m} C_m^k (-\mu)^{m-k} z^k = (\xi(m))^{-1} \sum_{k=0}^{m} \xi(k) z^k.$$

Next, we apply the following transformations:

$$\sum_{t=0}^{r-1} q_{i,j,t} z^t = \sum_{k=0}^{m-1} f_{i,j,k} z^k \sum_{s=0}^{M} d_s z^s + (\xi(m))^{-1} \sum_{s=0}^{M-1} e_{i,j,s} z^s \sum_{k=0}^{m} \xi(k) z^k =$$

$$= \sum_{k=0}^{m-1}\sum_{s=0}^{M} f_{i,j,k} d_s z^{k+s} + (\xi(m))^{-1} \sum_{k=0}^{m}\sum_{s=0}^{M-1} \xi(k) e_{i,j,s} z^{k+s} =$$

$$= \sum_{t=0}^{r-1} z^t \left[\sum_{\substack{k+s=t \\ 0\le k\le m-1 \\ 0\le s\le M}} f_{i,j,k} d_s + (\xi(m))^{-1} \sum_{\substack{k+s=t \\ 0\le k\le m \\ 0\le s\le M-1}} \xi(k) e_{i,j,s} \right].$$

Equating the corresponding coefficients, we obtain

$$q_{i,j,t} = \sum_{\substack{0\le k\le m-1 \\ 0\le t-k\le M}} d_{t-k} f_{i,j,k} + (\xi(m))^{-1} \sum_{\substack{0\le s\le t \\ t-m\le s\le M-1}} \xi(t-s) e_{i,j,s} =$$

$$= \sum_{k=0}^{m-1} d_{t-k} I_{\{0\le x\le M\}}(t-k) f_{i,j,k} + (\xi(m))^{-1} \sum_{s=0}^{t} \xi(t-s) I_{\{t-m\le x\le M-1\}}(s) e_{i,j,s},$$

where $I_A(x)$ is the indicator function of the set A:

$$I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

For $t \le M-1$, the above formula can be written in the form

$$q_{i,j,t} = \sum_{k=0}^{m-1} d_{t-k} I_{\{x\le t\}}(k) f_{i,j,k} + (\xi(m))^{-1} \sum_{s=0}^{t} \xi(t-s) I_{\{x\ge t-m\}}(s) e_{i,j,s} =$$

$$= \sum_{k=0}^{m-1} d_{t-k} I_{\{x\le t\}}(k) f_{i,j,k} + (\xi(m))^{-1} e_{i,j,t} + (\xi(m))^{-1} \sum_{s=0}^{t-1} \xi(t-s) I_{\{x\ge t-m\}}(s) e_{i,j,s},$$

which implies

$$e_{i,j,t} = \xi(m) \left[q_{i,j,t} - \sum_{k=0}^{m-1} d_{t-k} I_{\{x\le t\}}(k) f_{i,j,k} - (\xi(m))^{-1} \sum_{s=0}^{t-1} \xi(t-s) I_{\{x\ge t-m\}}(s) e_{i,j,s} \right].$$

We note that the following representation holds:

$$e_{i,j,t} = w_{i,j,t} + \sum_{k=0}^{m-1} x_{t,k} f_{i,j,k},\ i,j = \overline{1,n}.$$

Next, we determine the coefficients $w_{i,j,t}$ and $x_{t,k}$ from this representation. We have

$$w_{i,j,t} + \sum_{k=0}^{m-1} x_{t,k} f_{i,j,k} =$$

$$= \xi(m) q_{i,j,t} - \sum_{k=0}^{m-1} \xi(m) d_{t-k} I_{\{x \le t\}}(k) f_{i,j,k} -$$

$$- \sum_{s=0}^{t-1} \xi(t-s) I_{\{x \ge t-m\}}(s) \left[w_{i,j,s} + \sum_{k=0}^{m-1} x_{s,k} f_{i,j,k} \right] =$$

$$= \left[\xi(m) q_{i,j,t} - \sum_{s=0}^{t-1} \xi(t-s) I_{\{x \ge t-m\}}(s) w_{i,j,s} \right] +$$

$$+ \sum_{k=0}^{m-1} f_{i,j,k} \left[-\xi(m) d_{t-k} I_{\{x \le t\}}(k) - \sum_{s=0}^{t-1} \xi(t-s) I_{\{x \ge t-m\}}(s) x_{s,k} \right],$$

which implies

$$\begin{cases} x_{t,k} = -\xi(m) d_{t-k} I_{\{x \le t\}}(k) - \sum\limits_{s=max\{0,t-m\}}^{t-1} \xi(t-s) x_{s,k}, \ k = \overline{0, m-1}, \\ w_{i,j,t} = \xi(m) q_{i,j,t} - \sum\limits_{s=max\{0,t-m\}}^{t-1} \xi(t-s) w_{i,j,s}, \ t = \overline{0, M-1}, \ i, j = \overline{1, n}. \end{cases} \quad (5.25)$$

For $t \ge M$ we have the transformations

$$q_{i,j,t} = \sum_{k=0}^{m-1} d_{t-k} I_{\{0 \le x \le M\}}(t-k) f_{i,j,k} +$$

$$+ (\xi(m))^{-1} \cdot \sum_{s=0}^{M-1} \xi(t-s) I_{\{x \ge t-m\}}(s) \cdot \left[w_{i,j,s} + \sum_{k=0}^{m-1} x_{s,k} f_{i,j,k} \right] =$$

$$= (\xi(m))^{-1} \sum_{s=0}^{M-1} \xi(t-s) I_{\{x \ge t-m\}}(s) w_{i,j,s} +$$

$$+ \sum_{k=0}^{m-1} f_{i,j,k} \left[d_{t-k} I_{\{0 \le x \le M\}}(t-k) + (\xi(m))^{-1} \sum_{s=0}^{M-1} \xi(t-s) I_{\{x \ge t-m\}}(s) x_{s,k} \right],$$

equivalent to

$$\sum_{k=0}^{m-1} r_{t,k} f_{i,j,k} = s_{i,j,t}, \ t = \overline{M, r-1}, \ i, j = \overline{1, n}, \quad (5.26)$$

where

$$\begin{cases} r_{t,k} = d_{t-k} I_{\{0\le x\le M\}}(t-k) + (\xi(m))^{-1} \sum\limits_{s=max\{0,t-m\}}^{M-1} \xi(t-s) x_{s,k}, \\ s_{i,j,t} = q_{i,j,t} - (\xi(m))^{-1} \sum\limits_{s=max\{0,t-m\}}^{M-1} \xi(t-s) w_{i,j,s}, \ k = \overline{0,m-1}. \end{cases} \quad (5.27)$$

Let us find the values $\alpha_{i,j,k}(\mu)$, $k = \overline{1,m}$, $i, j = \overline{1,n}$. According to (5.24), we have

$$\frac{F_{i,j}(z)}{(z-\mu)^m} = \sum_{k=1}^{m} \frac{\alpha_{i,j,k}(\mu)}{(z-\mu)^k} = \frac{1}{(z-\mu)^m} \sum_{k=1}^{m} \alpha_{i,j,k}(\mu)(z-\mu)^{m-k},$$

which implies

$$\sum_{s=0}^{m-1} f_{i,j,s} z^s = \sum_{k=1}^{m} \alpha_{i,j,k}(\mu)(z-\mu)^{m-k} = \sum_{k=0}^{m-1} \alpha_{i,j,m-k}(\mu)(z-\mu)^k =$$

$$= \sum_{k=0}^{m-1} \alpha_{i,j,m-k}(\mu) \sum_{s=0}^{k} C_k^s (-\mu)^{k-s} z^s = \sum_{s=0}^{m-1} z^s \sum_{k=s}^{m-1} \alpha_{i,j,m-k}(\mu) C_k^s (-\mu)^{k-s},$$

hence

$$f_{i,j,s} = \sum_{k=s}^{m-1} C_k^s (-\mu)^{k-s} \alpha_{i,j,m-k}(\mu), \ s = \overline{0,m-1}, \ i, j = \overline{1,n}.$$

By substituting this relation into (5.26), we obtain

$$s_{i,j,t} = \sum_{k=0}^{m-1} r_{t,k} \sum_{s=k}^{m-1} C_s^k (-\mu)^{s-k} \alpha_{i,j,m-s}(\mu) = \sum_{s=0}^{m-1} \alpha_{i,j,m-s}(\mu) \sum_{k=0}^{s} C_s^k (-\mu)^{s-k} r_{t,k} =$$

$$= \sum_{s=1}^{m} \alpha_{i,j,s}(\mu) \sum_{k=0}^{m-s} C_{m-s}^k (-\mu)^{m-s-k} r_{t,k} = \sum_{s=1}^{m} r_{t,s}^* \alpha_{i,j,s}(\mu), \ t = \overline{M, r-1}, \ i, j = \overline{1,n},$$

where

$$r_{t,s}^* = \sum_{k=0}^{m-s} C_{m-s}^k (-\mu)^{m-s-k} r_{t,k}, \ t = \overline{M, r-1}, \ s = \overline{1,m}. \quad (5.28)$$

The solution of the obtained system is

$$\alpha_{i,j}(\mu) = (R^*)^{-1} S_{i,j}, \ i, j = \overline{1,n}, \quad (5.29)$$

where

$$\alpha_{i,j}(\mu) = ((\alpha_{i,j,s}(\mu))_{s=\overline{1,m}})^T, \ S_{i,j} = ((s_{i,j,t})_{t=\overline{M,r-1}})^T, \ R^* = (r_{t,s}^*)_{t=\overline{M,r-1}, s=\overline{1,m}}.$$

The functions

$$\nu_k(z)=(1-z)^{-k},\ \forall k\geq 1,$$

were introduced in Section 5.2. It was noted that these functions satisfy the recurrence

$$\nu_{k+1}(z)=\frac{d\nu_k(z)}{kdz},\ \forall k\geq 1.$$

Also, it was proved that

$$\nu_k(z)=\sum_{t=0}^{\infty}T_{k-1}(t)z^t,\ \forall k\geq 1,$$

where the coefficient $T_{k-1}(t)$ is a polynomial of degree which does not exceed $k-1$. It was obtained that the entries of the matrices $\beta_{i,j,k}(y)$ represent the coefficients of the polynomial in the variable t:

$$W_{i,j}(y,t)=\sum_{k=0}^{m(y)-1}(-y)^{-k-1}\alpha_{i,j,k+1}(y)T_k(t),\ \forall y\in\mathbb{C}\setminus D,\ i,j=\overline{1,n}.$$

Consider the polynomials

$$T_k(t)=\sum_{s=0}^{k}u_s^{(k)}t^s,\ \forall k\geq 0.$$

It follows that

$$\nu_{k+1}(z)=\frac{d}{k\,dz}\sum_{t=0}^{\infty}T_{k-1}(t)z^t=\frac{1}{k}\sum_{t=1}^{\infty}tT_{k-1}(t)z^{t-1}=\frac{1}{k}\sum_{t=0}^{\infty}(t+1)T_{k-1}(t+1)z^t,$$

equivalent to

$$T_k(t)=\frac{1}{k}(t+1)T_{k-1}(t+1)=\frac{1}{k}(t+1)\sum_{s=0}^{k-1}u_s^{(k-1)}(t+1)^s=$$

$$=\frac{1}{k}\sum_{s=0}^{k-1}u_s^{(k-1)}(t+1)^{s+1}=\frac{1}{k}\sum_{s=0}^{k-1}u_s^{(k-1)}\sum_{l=0}^{s+1}C_{s+1}^l t^l=$$

$$=\frac{1}{k}\sum_{s=0}^{k-1}u_s^{(k-1)}\left(1+\sum_{l=1}^{s+1}C_{s+1}^l t^l\right)=\frac{1}{k}\sum_{s=0}^{k-1}u_s^{(k-1)}+\frac{1}{k}\sum_{l=1}^{k}t^l\sum_{s=l-1}^{k-1}u_s^{(k-1)}C_{s+1}^l,$$

which implies

$$u_0^{(0)} = 1,\ u_0^{(k)} = \frac{1}{k}\sum_{s=0}^{k-1} u_s^{(k-1)},$$
$$u_l^{(k)} = \frac{1}{k}\sum_{s=l-1}^{k-1} C_{s+1}^{l} u_s^{(k-1)},\ \forall k \geq 1,\ l = \overline{1,k}. \tag{5.30}$$

Therefore, we have the formula for computing the coefficients in the decomposition:

$$\beta_{i,j,k}(y) = \sum_{s=k+1}^{m(y)} (-y)^{-s} \alpha_{i,j,s}(y) u_k^{(s-1)},$$
$$y \in \mathbb{C} \setminus D,\ k = \overline{0, m(y)-1},\ i, j = \overline{1,n}. \tag{5.31}$$

As a result, we justified the following algorithm for finding differential components:

Algorithm 5.5

Input: the transition probability matrix P;

Output: the matrix coefficients $\beta_k(y)$ ($y \in \mathbb{C} \setminus D$, $k = \overline{0, m(y)-1}$).

1)-3) Perform the steps $1)-3)$ from Algorithm 5.3;

4) *According to (5.21), determine the matrices $R^{(k)}$, $k = \overline{0,n-1}$;*

5) *Find the values $q_{i,j,k}$, $k = \overline{0,r-1}$, $i, j = \overline{1,n}$, by using Algorithm 5.4;*

6) *Compute C_s^k, $s = \overline{1, \max\limits_{y \in \mathbb{C} \setminus D} m(y)}$, $k = \overline{0,s}$, according to Pascal's triangle;*

7) *Based on (5.30), determine the values $u_l^{(k)}$, $k = \overline{0, \max\limits_{y \in \mathbb{C} \setminus D} m(y)-1}$, $l = \overline{0,k}$;*

8) *For each $\mu \in \mathbb{C} \setminus D$, perform the following steps:*

   a. *Determine the values $\xi(k) = C_m^k (-\mu)^{-k}$, $k = \overline{0,m}$ ($m = m(\mu)$);*

   b. *Using Horner's scheme, perform m successive divisions of the polynomial $\Delta(z)$ by $z - \mu$, obtaining the coefficients d_k, $k = \overline{0,r-m}$, of the quotient;*

   c. *Compute the values $x_{t,k}$, $t = \overline{0,M-1}$, $k = \overline{0,m-1}$, based on (5.25);*

   d. *Compute the values $r_{t,k}$, $t = \overline{M,r-1}$, $k = \overline{0,m-1}$, based on (5.27);*

   e. *Determine the entries of the matrix R^* according to (5.28);*

f. Using an efficient numerical method, compute the matrix $(R^*)^{-1}$;

g. For $i, j = \overline{1,n}$, *perform the following steps:*

g_1) *Compute the values* $w_{i,j,t}$, $t = \overline{0, M-1}$, *according to (5.25);*

g_2) *Compute the values* $s_{i,j,t}$, $t = \overline{M, r-1}$, *using the formula (5.27);*

g_3) *Determine the vector* $\alpha_{i,j}(\mu)$ *according to the relation (5.29);*

g_4) *By using (5.31), compute* $\beta_{i,j,k}(\mu)$ *and store its value in the corresponding position in the matrix* $\beta_k(\mu)$, $k = \overline{0, m(\mu)-1}$.

The remarks made for Algorithm 5.3 are also valid for Algorithm 5.5. Additionally, we can note two further aspects:

Remark 5.6 *Unlike Algorithm 5.3, Algorithm 5.5 can also be applied when some eigenvalues are unknown. In this case, the set* $\mathbb{C} \setminus D$ *is replaced by the set of reciprocals of the known non-zero eigenvalues, the algorithm finding only the matrix coefficients corresponding to the elements of this set.*

Remark 5.7 *If we apply Algorithm* 5.5 *for the set of known eigenvalues* $\{1\}$, *then we obtain the limiting probability matrix* $\beta_0(1)$ *of the discrete stochastic system* L. *Therefore, Algorithm 5.2 represents an enhanced particular case of Algorithm* 5.5.

Example 5.5 *Consider the Markov process* L *with the transition matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

Determine the limiting probability matrix and the differential components.

Solution. We will determine the limiting probability matrix and the differential components of the stochastic system L. We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix},\ P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 0.25 & 0.5 \\ 0.75 & 0 & 0.25 \end{pmatrix},\ P^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.125 & 0.375 \\ 0.875 & 0 & 0.125 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \beta_0 = 1,\ \beta_1 = -2,\ \beta_2 = 1.25,\ \beta_3 = -0.25 \Rightarrow$$

$$\Rightarrow \mathbb{C} \setminus D = \{1,2\},\ m(1)=1,\ m(2)=2,\ r=n=3 \Rightarrow$$

$$\Rightarrow R^{(0)} = \beta_0 I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^{(1)} = \beta_1 I + PR^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1.5 & 0.5 \\ 0.5 & 0 & -1.5 \end{pmatrix},$$

$$R^{(2)} = \beta_2 I + PR^{(1)} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0.25 & 0.5 & -0.5 \\ -0.25 & 0 & 0.5 \end{pmatrix};$$

$$q_{i,j,k} = R_{i,j}^{(k)},\ i,j=\overline{1,3},\ k=\overline{0,2};$$

$$C_1^0 = C_1^1 = C_2^0 = C_2^2 = 1,\ C_2^1 = C_1^0 + C_1^1 = 2;$$

$$u_0^{(0)} = u_0^{(1)} = u_1^{(1)} = 1.$$

For $\mu = 1$, we have

$$m = m(\mu) = 1,\ M = r - m = 2;\ \xi(0) = 1,\ \xi(1) = -1;$$

	-0.25	1.25	-2	1
1	-0.25	1	-1	0

$$\Rightarrow d_0 = -1,\ d_1 = 1,\ d_2 = -0.25 \Rightarrow$$

$$\Rightarrow x_{0,0} = -\xi(1)d_0 = -1,\ x_{1,0} = -\xi(1)d_1 - \xi(1)x_{0,0} = 0 \Rightarrow$$

$$\Rightarrow r_{2,0} = d_2 - \xi(1)x_{1,0} = -0.25 \Rightarrow r_{2,1}^* = r_{2,0} = -0.25 \Rightarrow$$

$$\Rightarrow R^* = (-0.25) \Rightarrow (R^*)^{-1} = (-4).$$

$$w_{i,j,0} = -q_{i,j,0} = -R_{i,j}^{(0)},\ w_{i,j,1} = -q_{i,j,1} + w_{i,j,0} = -R_{i,j}^{(0)} - R_{i,j}^{(1)},\ i,j=\overline{1,3} \Rightarrow$$

$$\Rightarrow s_{i,j,2} = q_{i,j,2} - w_{i,j,1} = R_{i,j}^{(0)} + R_{i,j}^{(1)} + R_{i,j}^{(2)} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0.25 & 0 & 0 \\ 0.25 & 0 & 0 \end{pmatrix}_{i,j},\ i,j=\overline{1,3} \Rightarrow$$

$$\Rightarrow\ \alpha_{i,j}(1)=(-4)s_{i,j,2}=\begin{pmatrix}-1&0&0\\-1&0&0\\-1&0&0\end{pmatrix}_{i,j},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_{i,j,0}(1)=-\alpha_{i,j}(1)=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}_{i,j},\ i,j=\overline{1,3}\ \Rightarrow\ \beta_0(1)=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}.$$

For $\mu=2$, we have

$$m=m(\mu)=2,\ M=r-m=1;\ \xi(0)=1,\ \xi(1)=-1,\ \xi(2)=0.25;$$

	-0.25	1.25	-2	1
2	-0.25	0.75	-0.5	0
2	-0.25	0.25	0	

$$\Rightarrow\ d_0=0.25,\ d_1=-0.25\ \Rightarrow\ x_{0,0}=-\xi(2)d_0=-0.0625,\ x_{0,1}=0\ \Rightarrow$$

$$\Rightarrow\ r_{1,0}=0,\ r_{1,1}=0.25,\ r_{2,0}=-0.0625,\ r_{2,1}=-0.25\ \Rightarrow$$

$$\Rightarrow\ r^*_{1,1}=0.25,\ r^*_{1,2}=0,\ r^*_{2,1}=-0.125,\ r^*_{2,2}=-0.0625\ \Rightarrow$$

$$\Rightarrow\ R^*=\begin{pmatrix}0.25&0\\-0.125&-0.0625\end{pmatrix}\Rightarrow\ (R^*)^{-1}=\begin{pmatrix}4&0\\-8&-16\end{pmatrix}.$$

$$w_{i,j,0}=0.25q_{i,j,0}=0.25R^{(0)}_{i,j},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ s_{i,j,1}=q_{i,j,1}+4w_{i,j,0}=R^{(0)}_{i,j}+R^{(1)}_{i,j},$$

$$s_{i,j,2}=q_{i,j,2}-w_{i,j,0}=R^{(2)}_{i,j}-0.25R^{(0)}_{i,j},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ S_{i,j}=\begin{pmatrix}R^{(0)}_{i,j}+R^{(1)}_{i,j}\\R^{(2)}_{i,j}-0.25R^{(0)}_{i,j}\end{pmatrix},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \alpha_{i,j}(2)=(R^*)^{-1}S_{i,j}=\begin{pmatrix}4R^{(0)}_{i,j}+4R^{(1)}_{i,j}\\-4R^{(0)}_{i,j}-8R^{(1)}_{i,j}-16R^{(2)}_{i,j}\end{pmatrix},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_{i,j,0}(2)=-0.5\alpha_{i,j,1}(2)+0.25\alpha_{i,j,2}(2)=-3R^{(0)}_{i,j}-4R^{(1)}_{i,j}-4R^{(2)}_{i,j},$$

$$\beta_{i,j,1}(2)=0.25\alpha_{i,j,2}(2)=-R_{i,j}^{(0)}-2R_{i,j}^{(1)}-4R_{i,j}^{(2)},\ i,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_0(2)=\begin{pmatrix}0&0&0\\-1&1&0\\-1&0&1\end{pmatrix},\ \beta_1(2)=\begin{pmatrix}0&0&0\\-1&0&1\\0&0&0\end{pmatrix}.$$

Therefore, we obtained the same decomposition:

$$P(t)=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\-1&1&0\\-1&0&1\end{pmatrix}\left(\frac{1}{2}\right)^t+\begin{pmatrix}0&0&0\\-1&0&1\\0&0&0\end{pmatrix}t\left(\frac{1}{2}\right)^t,\ \forall t\geq 0.$$

So, the limiting probability matrix is

$$P^*=\beta_0(1)=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix}$$

and the differential components are

$$\beta_0(2)=\begin{pmatrix}0&0&0\\-1&1&0\\-1&0&1\end{pmatrix},\ \beta_1(2)=\begin{pmatrix}0&0&0\\-1&0&1\\0&0&0\end{pmatrix}.$$ □

Example 5.6 *Consider the Markov process L with the transition matrix*

$$P=\begin{pmatrix}1/2&0&1/2\\1/2&1/4&1/4\\1/4&1/2&1/4\end{pmatrix}.$$

Determine the limiting probability matrix and the differential components.

Solution. We will determine the limiting probability matrix and the differential components of the stochastic system L. We have:

$$P=\begin{pmatrix}1/2&0&1/2\\1/2&1/4&1/4\\1/4&1/2&1/4\end{pmatrix},\ P^2=\begin{pmatrix}3/8&1/4&3/8\\7/16&3/16&3/8\\7/16&1/4&5/16\end{pmatrix},$$

$$P^3=\begin{pmatrix}13/32&1/4&11/32\\13/32&15/64&23/64\\27/64&7/32&23/64\end{pmatrix}\Rightarrow$$

$$\Rightarrow\ s_1 = tr\,P = 1,\ s_2 = tr\,P^2 = 7/8,\ s_3 = tr\,P^3 = 1\ \Rightarrow$$

$$\Rightarrow\ \beta_0 = 1,\ \beta_1 = -s_1 = -1,\ \beta_2 = -(s_2 + \beta_1 s_1)/2 = 1/16,$$

$$\beta_3 = -(s_3 + \beta_1 s_2 + \beta_2 s_1)/3 = -1/16\ \Rightarrow$$

$$\Rightarrow\ \Delta(z) = \sum_{k=0}^{3} \beta_k z^k = 1 - z + \frac{1}{16}z^2 - \frac{1}{16}z^3 = -\frac{1}{16}(z-1)(z-4i)(z+4i)\ \Rightarrow$$

$$\Rightarrow\ \mathbb{C} \setminus D = \{z \in \mathbb{C} \mid \Delta(z) = 0\} = \{1, -4i, 4i\},$$

$$m(1) = m(-4i) = m(4i) = 1,\ r = n = 3\ \Rightarrow$$

$$\Rightarrow\ R^{(0)} = \beta_0 I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^{(1)} = \beta_1 I + PR^{(0)} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -3/4 & 1/4 \\ 1/4 & 1/2 & -3/4 \end{pmatrix},$$

$$R^{(2)} = \beta_2 I + PR^{(1)} = \begin{pmatrix} -1/16 & 1/4 & -1/8 \\ -1/16 & 0 & 1/8 \\ 3/16 & -1/4 & 1/8 \end{pmatrix};$$

$$q_{l,j,k} = R_{l,j}^{(k)},\ l, j = \overline{1,3},\ k = \overline{0,2};$$

$$C_1^0 = C_1^1 = 1;\ u_0^{(0)} = 1.$$

For $\mu = 1$, we have $m = m(\mu) = 1,\ M = r - m = 2;\ \xi(0) = 1,\ \xi(1) = -1;$

	-1/16	1/16	-1	1
1	-1/16	0	-1	0

$$\Rightarrow\ d_0 = -1,\ d_1 = 0,\ d_2 = -1/16\ \Rightarrow$$

$$\Rightarrow\ x_{0,0} = -\xi(1)d_0 = -1,\ x_{1,0} = -\xi(1)d_1 - \xi(1)x_{0,0} = -1\ \Rightarrow$$

$$\Rightarrow\ r_{2,0} = d_2 - \xi(1)x_{1,0} = -17/16\ \Rightarrow\ r_{2,1}^* = r_{2,0} = -17/16\ \Rightarrow$$

$$\Rightarrow\ R^* = (-17/16)\ \Rightarrow\ (R^*)^{-1} = (-16/17).$$

$$w_{l,j,0} = -q_{l,j,0} = -R_{l,j}^{(0)},$$

$$w_{l,j,1} = -q_{l,j,1} + w_{l,j,0} = -R_{l,j}^{(0)} - R_{l,j}^{(1)},\ l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow s_{l,j,2} = q_{l,j,2} - w_{l,j,1} = R_{l,j}^{(0)} + R_{l,j}^{(1)} + R_{l,j}^{(2)} = \begin{pmatrix} 7/16 & 1/4 & 3/8 \\ 7/16 & 1/4 & 3/8 \\ 7/16 & 1/4 & 3/8 \end{pmatrix}_{l,j},\ l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow \alpha_{l,j}(1) = \left(-\frac{16}{17}\right) s_{l,j,2} = \begin{pmatrix} -7/17 & -4/17 & -6/17 \\ -7/17 & -4/17 & -6/17 \\ -7/17 & -4/17 & -6/17 \end{pmatrix}_{l,j},\ l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow \beta_{l,j,0}(1) = -\alpha_{l,j}(1) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix}_{l,j},\ l, j = \overline{1,3} \Rightarrow$$

$$\beta_0(1) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix}.$$

For $\mu = -4i$, we have

$$m = m(\mu) = 1,\ M = r - m = 2;\ \xi(0) = 1,\ \xi(1) = -i/4;$$

	-1/16	1/16	-1	1
-4i	-1/16	1/16+i/4	-i/4	0

$$\Rightarrow d_0 = -i/4,\ d_1 = 1/16 + i/4,\ d_2 = -1/16 \Rightarrow$$

$$\Rightarrow x_{0,0} = -\xi(1) d_0 = 1/16,\ x_{1,0} = -\xi(1) d_1 - \xi(1) x_{0,0} = -1/16 + i/32 \Rightarrow$$

$$\Rightarrow r_{2,0} = d_2 + (\xi(1))^{-1} (\xi(1) x_{1,0}) = -1/8 + i/32 \Rightarrow$$

$$\Rightarrow r_{2,1}^* = r_{2,0} = -1/8 + i/32 \Rightarrow$$

$$\Rightarrow R^* = (-1/8 + i/32) \Rightarrow (R^*)^{-1} = (-\frac{32}{17}(4+i)).$$

$$w_{l,j,0} = \xi(1) q_{l,j,0} = -\frac{i}{4} R_{l,j}^{(0)},$$

$$w_{l,j,1} = -\frac{i}{4} q_{l,j,1} - \xi(1) w_{l,j,0} = \frac{1}{16} R_{l,j}^{(0)} - \frac{i}{4} R_{l,j}^{(1)},\ l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow\ s_{l,j,2}=q_{l,j,2}-w_{l,j,1}=-\frac{1}{16}R^{(0)}_{l,j}+\frac{i}{4}R^{(1)}_{l,j}+R^{(2)}_{l,j}=$$

$$=\begin{pmatrix}-1/8-i/8 & 1/4 & -1/8+i/8\\ -1/16+i/8 & -1/16-3i/16 & 1/8+i/16\\ 3/16+i/16 & -1/4+i/8 & 1/16-3i/16\end{pmatrix}_{l,j},\ l,j=\overline{1,3}\ \Rightarrow$$

$$\alpha_{l,j}(-4i)=-\frac{32}{17}(4+i)s_{l,j,2}=\frac{1}{17}\begin{pmatrix}12+20i & -32-8i & 20-12i\\ 12-14i & 2+26i & -14-12i\\ -22-14i & 36-8i & -14+22i\end{pmatrix}_{l,j},\ l,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_{l,j,0}(-4i)=-\frac{i}{4}\alpha_{l,j,1}(-4i)=$$

$$=-\frac{i}{4}\cdot\frac{1}{17}\begin{pmatrix}12+20i & -32-8i & 20-12i\\ 12-14i & 2+26i & -14-12i\\ -22-14i & 36-8i & -14+22i\end{pmatrix}_{l,j},\ l,j=\overline{1,3}\ \Rightarrow$$

$$\Rightarrow\ \beta_0(-4i)=\begin{pmatrix}5/17-3i/17 & -2/17+8i/17 & -3/17-5i/17\\ -7/34-3i/17 & 13/34-i/34 & -3/17+7i/34\\ -7/34+11i/34 & -2/17-9i/17 & 11/34+7i/34\end{pmatrix}.$$

For $\mu=4i$, we have

$$m=m(\mu)=1,\ M=r-m=2;\ \xi(0)=1,\ \xi(1)=i/4;$$

	-1/16	1/16	-1	1
4i	-1/16	1/16-i/4	i/4	0

$$\Rightarrow\ d_0=i/4,\ d_1=1/16-i/4,\ d_2=-1/16\ \Rightarrow$$

$$\Rightarrow\ x_{0,0}=-\xi(1)d_0=1/16,\ x_{1,0}=-\xi(1)d_1-\xi(1)x_{0,0}=-1/16-i/32\ \Rightarrow$$

$$\Rightarrow\ r_{2,0}=d_2+(\xi(1))^{-1}(\xi(1)x_{1,0})=-1/8-i/32\ \Rightarrow\ r^*_{2,1}=r_{2,0}=-1/8-i/32\ \Rightarrow$$

$$\Rightarrow\ R^*=(-1/8-i/32)\ \Rightarrow\ (R^*)^{-1}=(-\frac{32}{17}(4-i)).$$

$$w_{l,j,0}=\xi(1)q_{l,j,0}=\frac{i}{4}R^{(0)}_{l,j},$$

$$w_{l,j,1}=\frac{i}{4}q_{l,j,1}-\xi(1)w_{l,j,0}=\frac{1}{16}R^{(0)}_{l,j}+\frac{i}{4}R^{(1)}_{l,j},\ l,j=\overline{1,3}\ \Rightarrow$$

$$s_{l,j,2} = q_{l,j,2} - w_{l,j,1} = -\frac{1}{16} R_{l,j}^{(0)} - \frac{i}{4} R_{l,j}^{(1)} + R_{l,j}^{(2)} =$$

$$= \begin{pmatrix} -1/8 + i/8 & 1/4 & -1/8 - i/8 \\ -1/16 - i/8 & -1/16 + 3i/16 & 1/8 - i/16 \\ 3/16 - i/16 & -1/4 - i/8 & 1/16 + 3i/16 \end{pmatrix}_{l,j}, \; l, j = \overline{1,3} \Rightarrow$$

$$\alpha_{l,j}(4i) = -\frac{32}{17}(4-i) s_{l,j,2} = \frac{1}{17} \begin{pmatrix} 12 - 20i & -32 + 8i & 20 + 12i \\ 12 + 14i & 2 - 26i & -14 + 12i \\ -22 + 14i & 36 + 8i & -14 - 22i \end{pmatrix}_{l,j}, \; l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow \beta_{l,j,0}(4i) = \frac{i}{4} \alpha_{l,j,1}(4i) = \frac{i}{4} \cdot \frac{1}{17} \begin{pmatrix} 12 - 20i & -32 + 8i & 20 + 12i \\ 12 + 14i & 2 - 26i & -14 + 12i \\ -22 + 14i & 36 + 8i & -14 - 22i \end{pmatrix}_{l,j}, \; l, j = \overline{1,3} \Rightarrow$$

$$\Rightarrow \beta_0(4i) = \begin{pmatrix} 5/17 + 3i/17 & -2/17 - 8i/17 & -3/17 + 5i/17 \\ -7/34 + 3i/17 & 13/34 + i/34 & -3/17 - 7i/34 \\ -7/34 - 11i/34 & -2/17 + 9i/17 & 11/34 - 7i/34 \end{pmatrix}.$$

Therefore, we obtained the same decomposition of the $t-$step transition matrix. The limiting probability matrix is

$$P^* = \beta_0(1) = \begin{pmatrix} 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \\ 7/17 & 4/17 & 6/17 \end{pmatrix}$$

and the differential components are

$$\beta_0(-4i) = \begin{pmatrix} 5/17 - 3i/17 & -2/17 + 8i/17 & -3/17 - 5i/17 \\ -7/34 - 3i/17 & 13/34 - i/34 & -3/17 + 7i/34 \\ -7/34 + 11i/34 & -2/17 - 9i/17 & 11/34 + 7i/34 \end{pmatrix},$$

$$\beta_0(4i) = \begin{pmatrix} 5/17 + 3i/17 & -2/17 - 8i/17 & -3/17 + 5i/17 \\ -7/34 + 3i/17 & 13/34 + i/34 & -3/17 - 7i/34 \\ -7/34 - 11i/34 & -2/17 + 9i/17 & 11/34 - 7i/34 \end{pmatrix}.$$

It can be noted that these two differential components $\beta_0(-4i)$ and $\beta_0(4i)$, associated with nonreal roots $-4i$ and $4i$, that are reciprocal conjugates of each other and are roots of the polynomial $\Delta(z)$, are also reciprocal conjugates. Taking into ac-

count this property, the obtained decomposition of the matrix $P(t)$ can be written in a real form. Considering the notation

$$U=\begin{pmatrix} 5/17 & -2/17 & -3/17 \\ -7/34 & 13/34 & -3/17 \\ -7/34 & -2/17 & 11/34 \end{pmatrix},\ V=\begin{pmatrix} 3/17 & -8/17 & 5/17 \\ 3/17 & 1/34 & -7/34 \\ -11/34 & 9/17 & -7/34 \end{pmatrix},$$

it follows that

$$\beta_0(-4i)=U-iV,\ \beta_0(4i)=U+iV.$$

The decomposition of $P(t)$ is

$$P(t)=P^*+(U-iV)(i/4)^t+(U+iV)(-i/4)^t=$$

$$=P^*+(U-iV)(i/4)^t+\overline{(U-iV)(i/4)^t}=$$

$$=P^*+2\,\mathrm{Re}((U-iV)(i/4)^t)=P^*+2(U\,\mathrm{Re}((i/4)^t)+V\,\mathrm{Im}((i/4)^t)),\ \forall t\geq 0.$$

Since

$$\frac{i}{4}=\frac{1}{4}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right),$$

we have

$$\left(\frac{i}{4}\right)^t=4^{-t}\left(\cos\frac{t\pi}{2}+i\sin\frac{t\pi}{2}\right),$$

which implies

$$\mathrm{Re}\left(\frac{i}{4}\right)^t=4^{-t}\cos\frac{t\pi}{2},\ \mathrm{Im}\left(\frac{i}{4}\right)^t=4^{-t}\sin\frac{t\pi}{2},$$

i.e. the decomposition of $P(t)$ takes the real form

$$P(t)=P^*+2U\cdot 4^{-t}\cos\frac{t\pi}{2}+2V\cdot 4^{-t}\sin\frac{t\pi}{2},\ \forall t\geq 0.$$

In the end, we obtain

$$P(t)=\frac{1}{17}\left(\begin{pmatrix} 7 & 4 & 6 \\ 7 & 4 & 6 \\ 7 & 4 & 6 \end{pmatrix}+\begin{pmatrix} 10 & -4 & -6 \\ -7 & 13 & -6 \\ -7 & -4 & 11 \end{pmatrix}\cdot 4^{-t}\cos\frac{t\pi}{2}+\right.$$

$$+\left(\begin{pmatrix} 6 & -16 & 10 \\ 6 & 1 & -7 \\ -11 & -18 & -7 \end{pmatrix}\cdot 4^{-t}\sin\frac{t\pi}{2}\right),\ \forall t\geq 0\,. \qquad \square$$

5.5 Enhanced method for determining the limiting probability matrix

By combining Algorithm 5.1 and Algorithm 5.2, the algorithm developed in Section 5.2 for determining the limiting probability matrix can be described in the following way:

Algorithm 5.6

Input: the transition probability matrix $P\in\mathcal{M}_n(\mathbb{R})$;

Output: the limiting probability matrix $Q\in\mathcal{M}_n(\mathbb{R})$;

1. *Set* $\beta_0=1$ *and* $P^0=I_n$;
2. *For each* $k=\overline{1,n}$, *compute:*

$$P^k=PP^{k-1},\ s_k=trP^k,\ \beta_k=-\frac{1}{k}\left(s_k+\sum_{j=1}^{k-1}\beta_j s_{k-j}\right);$$

3. *Using Horner's scheme, divide* $\Delta(z)=\sum_{k=0}^{n}\beta_k z^k$ *by* $(z-1)^{m(1)}$ *and obtain the quotient* $V(z)$, *where* $m(1)$ *is the multiplicity of the root* $z_0=1$ *of the polynomial* $\Delta(z)$. *Additionally, store the coefficients* $\beta_k^*, k=\overline{0,N}$, *of the polynomial* $V'(z)=(z-1)V(z)$, *obtained at the last step;*
4. *Compute* $S^{(0)}=\beta_0^* I$ *and* $S^{(k)}=\beta_k^* I_n+PS^{(k-1)}, k=\overline{1,N-1}$;
5. *Determine the matrix* $S=\sum_{k=0}^{N-1}S^{(k)}$ *and the value* $V(1)$;
6. *Determine the limiting probability matrix* $Q=-S/V(1)$.

According to the methods and computation procedures from Section 4.1, we may conclude that Algorithm 5.6 has the complexity $O(n^{\omega+1})$.

Next, we propose a new approach for determining the limiting probability matrix in the Markov processes. We will provide a theoretical basis for a modification of Algorithm 5.6 that reduces the complexity of the new algorithm for finding the limiting probability matrix Q to $O(n^3)$.

Let $K_P(z)$ be the characteristic polynomial of the transition probability matrix P. Then, according to Algorithm 5.6, we have

$$\Delta(z) = (-1)^n z^n K_P(z^{-1}) = (z-1)^{m(1)} V(z).$$

Replacing z with z^{-1}, we obtain

$$(-1)^n z^{-n} K_P(z) = (z^{-1}-1)^{m(1)} V(z^{-1}),$$

which implies

$$V(z^{-1}) = \frac{K_P(z)}{(-z)^n (z^{-1}-1)^{m(1)}}. \tag{5.32}$$

Additionally, according to step 4 of Algorithm 5.6, we have

$$S^{(k)} = \sum_{j=0}^{k} \beta^*_{k-j} P^j, \; k = \overline{0, N-1}. \tag{5.33}$$

Taking into account the relations

$$V'(z) = (z-1)V(z), \;\; V(z) = \sum_{k=0}^{N-1} \gamma_k z^k, \;\; V'(z) = \sum_{k=0}^{N} \beta^*_k z^k$$

and dividing the polynomial $V'(z)$ by $z-1$ using Horner's scheme, we obtain

$$\gamma_k = \sum_{j=k+1}^{N} \beta^*_j, \; k = \overline{0, N-1}.$$

Since

$$\sum_{j=0}^{N} \beta^*_j = V'(1) = 0,$$

we have

$$\gamma_k = -\sum_{j=0}^{k} \beta^*_j, \; k = \overline{0, N-1}. \tag{5.34}$$

Therefore, by using the equalities (5.33) and (5.34), we obtain the following formula:

$$S = \sum_{k=0}^{N-1} S^{(k)} = -\sum_{k=0}^{N-1} \gamma_{N-1-k} P^k. \tag{5.35}$$

Now, using the equalities

$$V(z^{-1}) = \sum_{k=0}^{N-1} \gamma_k z^{-k} = z^{1-N} \sum_{k=0}^{N-1} \gamma_k z^{N-1-k} = z^{1-N} \sum_{k=0}^{N-1} \gamma_{N-1-k} z^k,$$

and applying the relation (5.32), we obtain

$$\sum_{k=0}^{N-1} \gamma_{N-1-k} z^k = z^{N-1} V(z^{-1}) = z^{n-m(1)} V(z^{-1}) = \frac{z^{n-m(1)} K_P(z)}{(-z)^n (z^{-1}-1)^{m(1)}} = (-1)^{N-1} \frac{K_P(z)}{(z-1)^{m(1)}}.$$

Taking into account that $K_P(z) \vdots (z-1)^{m(1)}$, we can write $K_P(z)$ as

$$K_P(z) = (z-1)^{m(1)} T(z).$$

The previous relation becomes

$$\sum_{k=0}^{N-1} \gamma_{N-1-k} z^k = (-1)^{N-1} T(z). \tag{5.36}$$

The formula (5.36) is an equality of two polynomials. Replacing z with P, we obtain the equality of two matrix polynomials

$$\sum_{k=0}^{N-1} \gamma_{N-1-k} P^k = (-1)^{N-1} T(P). \tag{5.37}$$

The formula (5.35) becomes

$$S = (-1)^N T(P), \tag{5.38}$$

i.e. the limiting probability matrix is

$$Q = -\frac{S}{V(1)} = \frac{(-1)^{N-1}}{V(1)} T(P). \tag{5.39}$$

Therefore, the limiting probability matrix Q is the product of the matrix $R = T(P)$ with a constant $c = \dfrac{(-1)^{N-1}}{V(1)}$. Since the limiting probability matrix Q is a stochastic matrix, we can determine the constant c through the normalization operation, i.e. c^{-1} represents the sum of the entries from an arbitrary row of the matrix R.

Therefore, the results presented above provide a theoretical foundation for the following algorithm for determining the limiting probability matrix Q:

Algorithm 5.7

Input: the transition probability matrix $P \in \mathcal{M}_n(\mathbb{R})$;

Output: the limiting probability matrix $Q \in \mathcal{M}_n(\mathbb{R})$;

1. *Determine the characteristic polynomial* $K_P(z) = |P - zI_n|$;
2. *Divide the polynomial* $K_P(z)$ *by* $(z-1)^{m(1)}$, *where* $m(1)$ *is the multiplicity of the eigenvalue* $z_0 = 1$, *obtaining the quotient* $T(z)$;
3. *Compute the matrix* $R = T(P)$;
4. *Determine the limiting probability matrix* Q *by dividing the matrix* R *by the sum of the entries in an arbitrary row.*

For Algorithm 5.7 to be efficient, it is necessary to employ the optimization techniques described in Section 4.1. Thus, in order to achieve a time complexity of $O(n^3)$ for determining the limiting probability matrix Q, it is necessary to use the fastest matrix multiplication algorithm of complexity $O(n^\omega)$, a characteristic polynomial computation algorithm of complexity $O(n^\omega)$ in step 1, Horner's scheme of complexity $O(n \cdot m(1))$ in step 2 and a matrix polynomial evaluation algorithm of complexity $O(n^{\max\{\omega+1/2,\,3\}})$ in step 3.

Thus, the theoretical complexity of the new algorithm is $O(n^3)$. However, in practice, this complexity cannot be attained for the reasons discussed in Section 4.1. When Strassen's matrix multiplication algorithm with $\omega \approx 2.802$ is employed, an approximate time complexity of $O(n^{3.302})$ is obtained. In practice, further improvement of this complexity is achieved through parallelization, according to the techniques described in Section 4.1.

Example 5.7 *Consider the Markov process* L *with transition probability matrix*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Determine the limiting probability matrix.

Solution. We have

$$P^{2t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ P^{2t+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \forall t \geq 0,$$

i.e. the process is 2-periodic.

Therefore, in this case, the limit $\lim_{t\to\infty} P^t$ does not exist, but the limiting probability matrix Q exists and it can be determined using Algorithm 5.7. If we apply this algorithm, we have:

$$K_P(z) = |P - zI_n| = \begin{vmatrix} -z & 1 \\ 1 & -z \end{vmatrix} = (z-1)(z+1) \Rightarrow$$

$$\Rightarrow T(z) = z+1 \Rightarrow T(P) = I_2 + P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow$$

$$\Rightarrow Q = \frac{1}{1+1} T(P) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We obtained the same result as in Example 5.1. We may note that, even if the Markov process is not ergodic, the rows of the limiting probability matrix are identical and the vector of limiting probabilities is $\pi^* = (0.5, 0.5)$, i.e., regardless of the initial state of the Markov process, the probability of being in each of the two states is 0.5 in the limiting distribution. □

Example 5.8 *Let L be a discrete Markov process with transition probability matrix*

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{pmatrix}.$$

Determine the limiting probability matrix.

Solution. We will find the limiting probability matrix Q by applying Algorithm 5.7:

$$K_P(z) = |P - z\,I_n| = \begin{vmatrix} 0.5-z & 0.5 \\ 0.4 & 0.6-z \end{vmatrix} = (z-1)(z-0.1) \Rightarrow$$

$$\Rightarrow T(z) = z - 0.1 \Rightarrow T(P) = -0.1 \cdot I_2 + P = \begin{pmatrix} 0.4 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \Rightarrow$$

$$\Rightarrow Q = \frac{1}{0.4 + 0.5} T(P) = \begin{pmatrix} \frac{4}{9} & \frac{5}{9} \\ \frac{4}{9} & \frac{5}{9} \end{pmatrix}.$$

In this case, the rows of the limiting probability matrix are identical and the Markov process is ergodic, with the vector of limiting probabilities $\pi^* = \left(\frac{4}{9}, \frac{5}{9}\right)$. □

Example 5.9 *Let L be a Markov process with the transition probability matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Determine the limiting probability matrix.

Solution. Let us apply Algorithm 5.7 for finding the limiting probability matrix Q:

$$K_P(z) = |P - zI_n| = \begin{vmatrix} 1-z & 0 & 0 \\ 0 & 1-z & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} - z \end{vmatrix} = (z-1)^2 (1/3 - z) \Rightarrow$$

$$\Rightarrow T(z) = \frac{1}{3} - z \Rightarrow T(P) = \frac{1}{3} I_3 - P = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow Q = \frac{1}{-\frac{2}{3} + 0 + 0} \cdot T(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Therefore, we obtained the same result as in Example 5.2. In this case, the rows of the limiting probability matrix Q are different. It is easy to observe that, for this example, the limit $\lim_{t\to\infty} P^t$ exists and it is equal to Q. □

5.6 Enhanced method for determining the differential components

The aim of this section is to provide a theoretical foundation for a new algorithm of complexity $O(n^{\omega+1})$ to determine the differential components of Markov processes with a finite set of states ([101]), where $O(n^{\omega})$ is the complexity of a matrix multiplication algorithm. We will show that such an algorithm can be developed when the roots of the characteristic polynomial are known.

According to Section 5.3, based on (5.17) and [66], we have the decomposition (5.12), where the coefficients

$$\beta_k(y),\ y \in (\mathbb{C} \setminus \mathcal{D}) \setminus \{1\},\ k = \overline{0, m(y)-1},$$

represent the differential components of the given Markov process. Since the set $\mathbb{C} \setminus \mathcal{D}$ consists of reciprocals of the eigenvalues of the transition probability matrix P, the decomposition (5.12) is equivalent to the decomposition

$$P(t) = \sum_{y \in \sigma(P)} \sum_{k=0}^{m(y)-1} t^k y^t \overline{\beta}_k(y),\ \forall t \geq n - r,$$

where

- $K_P(z) = |P - zI|$ is the characteristic polynomial of the transition probability matrix P;
- $\sigma(P) = \{z \in \mathbb{C} \mid K_P(z) = 0\}$ is the spectrum of P;
- $r = deg(K_P(z))$;
- $m(y)$ is the multiplicity of the root y of the polynomial $K_P(z)$;
- $\overline{\beta}_k(y) = \beta_k(y^{-1}),\ \forall y \in \sigma(P)$.

In particular, we have

$$P(t)=\sum_{y\in\sigma(P)}\sum_{k=0}^{m(y)-1}t^k y^t\overline{\beta}_k(y),\ t=\overline{n-r,n-1}.$$

This relation can be written entrywise in the form:

$$p_{i,j}(t)=\sum_{y\in\sigma(P)}\sum_{k=0}^{m(y)-1}t^k y^t\overline{\beta}_{i,j,k}(y),\ t=\overline{n-r,n-1},\ i,j=\overline{1,n}.$$

If we denote

$$\beta_t=(t^k y^t)_{(y,k)\in\sigma(P)\times\{0,\ldots,m(y)-1\}},\forall t\geq 0,\ B=(\beta_j)_{j=\overline{0,r-1}},\ 0^0\equiv 1, \tag{5.40}$$

and

$$p_{i,j}=(p_{i,j}(t))_{t=n-r}^{n-1},\ \overline{\beta}_{i,j}=(\overline{\beta}_{i,j,k}(y))_{(y,k)\in\sigma(P)\times\{0,\ldots,m(y)-1\}},i,j=\overline{1,n},$$

we obtain

$$p_{i,j}=\overline{\beta}_{i,j}B^T,\ i,j=\overline{1,n},$$

where B^T is the transpose of the matrix B. Since β_j, $j=\overline{1,r}$, are linearly independent, the matrix B is invertible. In conclusion,

$$\overline{\beta}_{i,j}=p_{i,j}(B^T)^{-1},\ i,j=\overline{1,n}.$$

Finally, if we consider

$$\Delta=(\overline{\beta}_{i,j})_{(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n\}},\ \Pi=(p_{i,j})_{(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n\}},$$

then we obtain

$$\Delta=\Pi(B^T)^{-1}. \tag{5.41}$$

Since Π is an $n^2\times r$ matrix and B is an $r\times r$ matrix, the resulting matrix Δ is an $n^2\times r$ matrix, which contains all the entries of r coefficient matrices $\overline{\beta}_k(y),\ y\in\sigma(P)$, each of size $n\times n$. A row of Δ corresponding to the index pair (i,j) represents the entries in position (i,j) of the limit and differential matrices. This means that the columns of Δ represent the flattened version of the limit and differential matrices. In a similar way, this applies to the matrix Π whose columns represent the flattened version of the powers P^t, $t=\overline{n-r,n-1}$.

Now, if we analyze the computational complexity of calculating Δ according to (5.41), then we can notice the following. Better performance in calculating (5.41) can be obtained if we split the rows of Δ and Π into approximately $\left[\frac{n^2}{r}\right]$ parts of dimension r. After that the equation (5.41) can be regarded as about $\left[\frac{n^2}{r}\right]$ matrix multiplications of dimension $r \times r$. This means that we have the complexity $O(r^{\omega-1}n^2)$ for determining Δ. Since, in the worst case, we have $r = n$, the complexity of Δ according to formula (5.41) is $O(n^{\omega+1})$.

In addition to this complexity, we need to take into account the complexity of calculating the matrices Π, B, B^T and $(B^T)^{-1}$. For obtaining the matrix Π we need to have all matrices P^t, $t = \overline{n-r, n-1}$, i.e. in the worst case we need $O(n^{\omega+1})$ operations. If the eigenvalues of the matrix P are known, then the matrices B and B^T can be computed by using $O(n^2)$ elementary operations and the computation of $(B^T)^{-1}$ takes $O(n^{\omega})$ arithmetic operations. Taking all the above remarks into account, we can conclude that the complexity of the entire algorithm is $O(n^{\omega+1})$. Therefore, having the eigenvalues of the transition matrix P, we can compute the matrix of limiting state probabilities and the differential matrices using $O(n^{\omega+1})$ arithmetic operations.

Consequently, based on the results mentioned above, we can propose the following algorithm for determining the differential matrices.

Algorithm 5.8

Input: the transition probability matrix $P \in \mathcal{M}_n(\mathbb{R})$*;*

Output: the limiting and the differential matrices;

1. *Calculate* $P^0 = I_n, P^1 = P^0 P, P^2 = P^1 P, \ldots P^{n-1} = P^{n-2} P$;
2. *Build the matrix* Π*, by collecting entries from the powers of the matrix P;*
3. *Determine the matrix* B *according to* (5.40)*;*
4. *Compute the matrix* $(B^T)^{-1}$;

5. *Determine the matrix* Δ *using formula* (5.41)*;*

6. *Build the limit and the differential matrices, by collecting entries from* Δ.

Example 5.10 *Consider the Markov process* L *with the transition probability matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}.$$

Determine the limiting probability matrix and the differential components.

Solution. According to Algorithm 5.8, we have

$$1)\ P^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix},\ P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 0.25 & 0.5 \\ 0.75 & 0 & 0.25 \end{pmatrix};$$

$$2\text{-}4)\ K_P(z) = |P - z I_n| = \begin{vmatrix} 1-z & 0 & 0 \\ 0 & 0.5-z & 0.5 \\ 0.5 & 0 & 0.5-z \end{vmatrix} = (1-z)(0.5-z)^2 \Rightarrow$$

$$\Rightarrow r = deg(K_P(z)) = 3,\ \sigma(P) = \{1, 0.5\},\ m(1) = 1,\ m(0.5) = 2 \Rightarrow$$

$$\Rightarrow \beta_0 = (1,1,0),\ \beta_1 = (1, 0.5, 0.5),\ \beta_2 = (1, 0.25, 0.5) \Rightarrow B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \\ 1 & 0.25 & 0.5 \end{pmatrix} \Rightarrow$$

$$\Rightarrow\ (B^T)^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ -4 & 4 & 6 \\ 4 & -4 & -4 \end{pmatrix};\qquad \Pi = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -- & -- & -- \\ 0 & 0 & 0.25 \\ 1 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \\ -- & -- & -- \\ 0 & 0.5 & 0.75 \\ 0 & 0 & 0 \\ 1 & 0.5 & 0.25 \end{pmatrix};$$

5) $\Delta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -- & -- & -- \\ 0 & 0 & 0.25 \\ 1 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \\ -- & -- & -- \\ 0 & 0.5 & 0.75 \\ 0 & 0 & 0 \\ 1 & 0.5 & 0.25 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ -4 & 4 & 6 \\ 4 & -4 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -- & -- & -- \\ 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -- & -- & -- \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$

6) $\overline{\beta}_0(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \overline{\beta}_0(0.5) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \overline{\beta}_1(0.5) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$

Therefore, the t−step transition probability matrix can be represented as follows:

$$P(t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \left(\frac{1}{2}\right)^t + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t \left(\frac{1}{2}\right)^t, \forall t \geq 0. \qquad \square$$

Example 5.11 *Consider the Markov process L with the transition probability matrix*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Determine the limiting probability matrix and the differential components.

Solution. According to Algorithm 5.8, we have

1) $P^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2-4) $K_P(z) = |P - z I_n| = \begin{vmatrix} -z & 1 \\ 1 & -z \end{vmatrix} = z^2 - 1 = (z-1)(z+1) \Rightarrow$

$\Rightarrow r = deg(K_P(z)) = 2, \ \sigma(P) = \{1, -1\}, \ m(1) = m(-1) = 1 \Rightarrow$

$\Rightarrow \beta_0 = (1,1), \ \beta_1 = (1,-1) \Rightarrow B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow$

$$\Rightarrow (B^T)^{-1} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}; \qquad \Pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -- & -- \\ 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$5)\ \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -- & -- \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \\ -- & -- \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix};$$

$$6)\ \overline{\beta}_0(1) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},\ \overline{\beta}_0(-1) = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}.$$

Therefore, the $t-$step transition probability matrix can be represented as follows:

$$P(t) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} + \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} (-1)^t, \forall t \geq 0. \qquad \square$$

6 STOCHASTIC SYSTEMS WITH A FINAL SEQUENCE OF STATES

In this chapter the discrete stochastic systems with a final sequence of states are studied. For such systems several random variables are defined: the evolution time, the first alert moments and the total alert time. Their main probabilistic characteristics, such as the distribution law, expectation and variance, standard deviation, moments, dominant value and median are studied.

In what follows, we present numerical methods for determining these characteristics. They are based on the properties of homogeneous linear recurrences, studied in the previous chapters. Also, the formula for the cardinality of the set of numbers with a fixed number of digits and a fixed sum of digits is applied for probabilistic characterization of the total alert time of the stochastic systems with final critical state.

The analysis is initially conducted under the assumption that the states are independent. We also consider the case in which the transition time is a random variable. Further, the results obtained are extended for the stochastic systems with a final sequence of states and interdependent states. Finally, the cases when the transition time is rational or real are studied.

6.1 Stochastic systems with a final sequence of states. Fundamental definitions

The stochastic systems with a final sequence of states are a generalization of the Markov processes. The main problems formulated for such systems concern the determination of the probabilistic characteristics of several descriptive random variables.

Consider a stochastic system L with the set of states $V = \{v_1, v_2, \ldots, v_\omega\}$, where $|V| = \omega < \infty$. The state of the system at each discrete time $t = 0,1,2,\ldots$ is

$v(t) \in V$. The transition time from one state u to the state v at discrete time t is $\theta(u,v)$, $\forall u,v \in V$.

We define a probability function $p^* : V \to [0,1]$ on the set V, where $p^*(v)$ represents the probability that the system L starts its evolution from the state $v \in V$. The vector $P^* = ((p^*(v_j))_{j=1}^{\omega})^T$ represents the initial distribution of the discrete stochastic system L.

The transition of the system from an arbitrary state $u \in V$ to another state $v \in V$ at each discrete time t is performed with probability $p(u,v) \in [0,1]$. If the system does not have the possibility to pass from the state u to the state v, then it is considered that the probability of this transition is $p(u,v) = 0$. The matrix $P = (p(u,v))_{u,v \in V}$ represents the transition probability matrix of the discrete stochastic system L. The completion of the evolution of the system is conditioned by the consecutive passage of the system through a fixed sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$.

Definition 6.1 *The moment in time T, when the evolution of the stochastic system L is complete, is called the evolution time of the discrete stochastic system L.*

As we can easily note, the evolution time T is a random variable. We consider the problem of determining the main characteristics (distribution law, expectation, variance, standard deviation, moments) of the evolution time T.

If $\theta(u,v) = 1$, $\forall u,v \in V$, then the stochastic system L has unit transition time, otherwise it has non-unit transition time. The case $p(u,v) = p^*(v)$, $\forall u,v \in V$, means that the states of the system are independent, otherwise the states of the system are interdependent, i.e. the next state of the system depends on the current state.

If the equality $x_1 = x_2 = \ldots = x_m = v^* \in V$ is satisfied, then the system represents a stochastic system with final critical state. For such systems, we introduce the definitions of final critical state, first alert moments and the total alert time.

Definition 6.2 *The state* $v^* \in V$ *is called a critical state of degree* m *and the value* $p = p^*(v^*)$ *is called the probability of occurrence of the critical state.*

Definition 6.3 *The moment in time at which, for the first time, the system has remained in its critical state* k *times, is called the moment* μ_k *of the first alert of order* k, $k = \overline{1,m}$.

Definition 6.4 *The time period* η, *during which the system remains in its critical state throughout its evolution, is called the total alert time.*

For these stochastic systems, we will develop methods for determining the main probabilistic characteristics of the evolution time, moments of the first alert and total alert time. The obtained algorithms will be based on the properties of the homogeneous linear recurrences and will use the characteristic and the generating functions for probabilistic characterization of the random variables.

6.2 Systems with independent states, unit transition time and final sequence of states

Consider a discrete stochastic system L with a final sequence of states, independent states and unit transition time ([83]). The definition of the system L and its dynamics were introduced in Section 6.1.

According to Section 6.1, a probability function $p^* : V \to [0,1]$ is defined on the set V, where $p^*(v)$ represents the probability that the system L starts its evolution from the state v. The transition of the system from the state $u \in V$ to the state $v \in V$ at time t is performed with probability $p_{(u,v)}(t) = p^*(v)$ which does not depend on the current state.

The stochastic system completes its evolution when it consecutively passes through a given sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$. Let T be its evolution time. Consider the problem of determining the main probabilistic characteristics of the evolution time.

Denote

$$X_k = \{x_k\},\ \overline{X_k} = V \setminus X_k,$$

$$\pi_k = p^*(x_k),\ w_k = \prod_{j=1}^{k} \pi_j,\ k = \overline{1,m}.$$

Let $a = rep(T)$, where T is the evolution time of the discrete stochastic system L. The obvious inequality $T \geq m-1$ implies the relations $a_n = 0,\ n = \overline{0,m-2}$. If $w_m = 0$, then $a_n = 0,\ \forall n \in \mathbb{N}$ and $\mathbb{P}(T = +\infty) = 1$, i.e. the stochastic system L has an infinite evolution, which implies that the moments of the evolution time are infinite. Further, we consider the case $w_m \neq 0$.

If $T = m-1$, then $v(j) = x_{j+1},\ j = \overline{0,m-1}$ and we have $a_{m-1} = w_m$. It follows that the first m terms of the sequence a are the components of the vector

$$I_m^{[a]} = (0,0,\ldots,0,w_m) \in \mathbb{R}^m. \tag{6.1}$$

Consider $\forall n \in \mathbb{Z}$. Let $S(V) = \{A \mid A \subseteq V\}$. We denote

$$P_\Phi(n) = P(T = n, v(j) \in \Phi_j, j = \overline{0,t-1}),\ \forall \Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t,\ t \in \mathbb{N}.$$

We introduce the following functions on $\mathbb{Z}$:

$$\alpha_k(n) = P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X_k})}(n);$$

$$\beta_k(n) = P_{(X_1, X_2, \ldots, X_k)}(n);$$

$$\gamma_k(n) = P_{(X_2, X_3, \ldots, X_k)}(n),\ k = \overline{1,m}.$$

We have

$$\beta_k(n) = P_{(X_1, X_2, \ldots, X_k)}(n) = a_n - \sum_{j=1}^{k} \alpha_j(n),\ k = \overline{1,m}. \tag{6.2}$$

Consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2,3,\ldots,s\} \mid x_{t-1+j} = x_j, j = \overline{1,s+1-t}\},\ s = \overline{1,m}.$$

The minimum of each set is denoted by

$$t_s = \min_{k \in T_s} k\,,\ s = \overline{1,m}\,. \tag{6.3}$$

It represents the smallest index in the sequence $(x_1, x_2, \ldots, x_s)$ from which, when $(x_1, x_2, \ldots, x_s)$ is superimposed on itself, the overlapping components coincide. Using (6.2), we obtain

$$\gamma_s(n) = P_{(X_2, X_3, \ldots, X_s)}(n) =$$

$$= \pi_2 \pi_3 \ldots \pi_{t_s - 1} P_{(X_{t_s}, X_{t_s+1}, \ldots, X_s)}(n - t_s + 2) =$$

$$= \frac{w_{t_s - 1}}{\pi_1} \beta_{s+1-t_s}(n - t_s + 2) =$$

$$= \frac{w_{t_s - 1}}{\pi_1} \left(a_{n-t_s+2} - \sum_{j=1}^{s+1-t_s} \alpha_j (n - t_s + 2) \right),\ s = \overline{1,m}\,. \tag{6.4}$$

In particular, we have

$$\gamma_1(n) = a_n\,,\ n = \overline{0,\infty}\,. \tag{6.5}$$

The values $\alpha_k(n)$, $k = \overline{1,m}$, are determined in the following way:

$$\alpha_1(n) = P_{(\overline{X_1})}(n) = (1 - \pi_1) a_{n-1}\,, \tag{6.6}$$

$$\alpha_k(n) = P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X_k})}(n) = \pi_1 P_{(X_2, X_3, \ldots, X_{k-1}, \overline{X_k})}(n-1) =$$

$$= \pi_1 (P_{(X_2, X_3, \ldots, X_{k-1})}(n-1) - P_{(X_2, X_3, \ldots, X_k)}(n-1)) =$$

$$= \pi_1 (\gamma_{k-1}(n-1) - \gamma_k(n-1))\,,\ k = \overline{2,m}\,. \tag{6.7}$$

For determining the distribution $a = rep(T)$, we obtain the recurrence formula

$$a_n = \sum_{j=1}^{m} \alpha_j(n) = (1 - \pi_1) a_{n-1} + \sum_{j=2}^{m} \pi_1 (\gamma_{j-1}(n-1) - \gamma_j(n-1)) =$$

$$= (1 - \pi_1) a_{n-1} + \pi_1 (a_{n-1} - \gamma_m(n-1)) = a_{n-1} - \pi_1 \gamma_m(n-1)\,,\ \forall n \geq m\,. \tag{6.8}$$

According to formulas (6.4) − (6.7), using the mathematical induction method, we can prove that there exist coefficients

$$u_{jk}, v_{jk} \in \mathbb{R},\ j = \overline{1,m},\ k = \overline{0, j-1},$$

such that

$$\alpha_j(n) = \sum_{k=0}^{j-1} u_{jk} a_{n-1-k},\ \gamma_j(n-1) = \sum_{k=0}^{j-1} v_{jk} a_{n-1-k},\ j = \overline{1,m},\ \forall n \in \mathbb{Z}. \tag{6.9}$$

From (6.5) and (6.6), we obtain

$$u_{1,0} = 1 - \pi_1,\ v_{1,0} = 1. \tag{6.10}$$

Also, by using the representation (6.9), the formula (6.4) takes the form

$$\gamma_s(n-1) = \frac{w_{t_s-1}}{\pi_1}\left(a_{(n-1)-t_s+2} - \sum_{j=1}^{s+1-t_s} \sum_{k=0}^{j-1} u_{jk} a_{n-t_s-k} \right) =$$

$$= \frac{w_{t_s-1}}{\pi_1}\left(a_{(n-1)-(t_s-2)} - \sum_{k=t_s-1}^{s-1} a_{n-1-k} \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,k-t_s+1} \right) = \sum_{k=0}^{s-1} v_{sk} a_{n-1-k},$$

where

$$\begin{cases} v_{sk} = 0,\ k = \overline{0, t_s-3},\ v_{s,t_s-2} = \dfrac{w_{t_s-1}}{\pi_1}, \\ v_{sk} = -\dfrac{w_{t_s-1}}{\pi_1} \displaystyle\sum_{j=k-t_s+2}^{s-t_s+1} u_{j,k-t_s+1},\ k = \overline{t_s-1, s-1} \end{cases},\ s = \overline{1,m}, \tag{6.11}$$

and the formula (6.7) becomes

$$\alpha_s(n) = \pi_1\left(\sum_{k=0}^{s-2} v_{s-1,k} a_{n-1-k} - \sum_{k=0}^{s-1} v_{sk} a_{n-1-k} \right) = \sum_{k=0}^{s-1} u_{sk} a_{n-1-k},$$

where

$$\begin{aligned} u_{sk} &= \pi_1 (v_{s-1,k} - v_{sk}),\ k = \overline{0, s-2}, \\ u_{s,s-1} &= -\pi_1 v_{s,s-1},\ s = \overline{2,m}. \end{aligned} \tag{6.12}$$

The formula (6.8) takes the form

$$a_n = a_{n-1} - \pi_1 \sum_{k=0}^{m-1} v_{mk} a_{n-1-k} = \sum_{k=0}^{m-1} q_k a_{n-1-k},\ \forall n \geq m,$$

where

$$q_0 = 1 - \pi_1 v_{m,0},\ q_k = -\pi_1 v_{m,k},\ k = \overline{1, m-1}. \tag{6.13}$$

Therefore,

$$a = rep(T) \in Rol[\mathbb{R}][m],\ q = (q_k)_{k=0}^{m-1} \in G[\mathbb{R}][m](a).$$

Taking into account the formula (6.1), we obtain $\dim[\mathbb{R}](a) = m$, i.e. the distribution of the evolution time of the discrete stochastic system L represents an $m-$minimal homogeneous linear recurrence.

In this way, we established the following algorithm, which allows us to determine the generating vector and the initial state of the distribution of the evolution time. The complexity of this algorithm is $O(m^3)$. Further, having the generating vector q and the initial state $I_m^{[rep(T)]}$, we can probabilistically describe the random variable T by using Algorithm 4.3 or Algorithm 4.4.

Algorithm 6.1

Input: V, $X \in V^m$, $p^* : V \to [0,1]$;

Output: $q \in G[\mathbb{R}][m](rep(T))$, $I_m^{[rep(T)]}$;

1) *Determine* $\pi_k = p^*(x_k)$, $w_k = \prod_{j=1}^{k} \pi_j$, $k = \overline{1, m}$;

2) *If* $w_m = 0$, *then* $I_1^{[rep(T)]} = (0)$, $q = (1) \in G[\mathbb{R}][1](rep(T))$
and stop the algorithm; otherwise go to step 3;

3) *Determine* $I_m^{[rep(T)]}$ *according to the formula (6.1);*

4) *Compute the values* $u_{1,0}$ *and* $v_{1,0}$ *according to the formula (6.10);*

5) *For each* $s = \overline{2, m}$, *compute the coefficients* u_{sk} *and* v_{sk}, $k = \overline{0, s-1}$:

   a) *determine the coefficient* t_s *according to the formula (6.3);*

   b) *compute the coefficients* v_{sk}, $k = \overline{0, s-1}$, *using the formula (6.11);*

   c) *determine the coefficients* u_{sk}, $k = \overline{0, s-1}$, *according to the formula (6.12);*

6) *Determine the components of the generating vector* q *using the formula (6.13).*

By using the generating vector q determined by Algorithm 6.1 and applying Algorithm 4.3 for $n=2$, we obtain simplified formulas of order $O(m)$ for the expectation and variance:

$$\mathbb{E}(T)=\overline{m}=m-2+\frac{1+\Delta_1(q)}{w_m},$$
$$\mathbb{V}(T)=(m-1)(m-2)+\overline{m}(\overline{m}-2m+3)+\frac{\Delta_1(q)+\Delta_2(q)}{w_m}, \tag{6.14}$$

where the functions

$$\Delta_j(q)=\sum_{k=1}^{m-1}k^j q_k\ ,\ \forall j\in\mathbb{N}^*,$$

were defined in Section 4.3. Also, the dominant value of the evolution time T is $\mu(T)=m-1$, since we have

$$a_n=P(T=n)\le\sum_{v\in V}p^*(v)P(T=n-1)=a_{n-1}\sum_{v\in V}p^*(v)=a_{n-1},\ \forall n\ge m,$$

which implies $a_n\le a_{m-1},\ \forall n\ge m$, and $a_n=0<a_{m-1}$ for $n<m-1$.

Next, we simplify Algorithm 6.1 by eliminating u_{sk}, $s=\overline{1,m}$, $k=\overline{0,s-1}$.

Theorem 6.1 *The following formula holds:*

$$v_{sk}=\begin{cases}0, & 0\le k\le t(s)-3\\ \dfrac{w_{t(s)-1}}{\pi_1}, & k=t(s)-2\\ -\dfrac{w_{t(s)-1}}{\pi_1}(1-\pi_1), & t(s-t(s)+1)=2\ and\ k=t(s)-1\\ -\dfrac{w_{t(s)-1}}{\pi_1}, & t(s-t(s)+1)\ge 3\ and\ k=t(s)-1\\ w_{t(s)-1}v_{s-t(s)+1,k-t(s)+1}, & t(s)\le k\le s-1\end{cases}.$$

Proof. The particular case $k\le t(s)-2$ follows immediately from Algorithm 6.1. Further, we consider the case $k\ge t(s)$. We have:

$$v_{sk}=-\frac{w_{t(s)-1}}{\pi_1}\sum_{j=k-t(s)+2}^{s-t(s)+1}u_{j,k-t(s)+1}=$$

$$= -\frac{w_{t(s)-1}}{\pi_1}\left(u_{k-t(s)+2,k-t(s)+1} + \sum_{j=k-t(s)+3}^{s-t(s)+1} u_{j,k-t(s)+1} \right) =$$

$$= -\frac{w_{t(s)-1}}{\pi_1}\left(-\pi_1 v_{k-t(s)+2,k-t(s)+1} + \sum_{j=k-t(s)+3}^{s-t(s)+1} \pi_1 (v_{j-1,k-t(s)+1} - v_{j,k-t(s)+1}) \right) =$$

$$= -\frac{w_{t(s)-1}}{\pi_1}\left(-\pi_1 v_{k-t(s)+2,k-t(s)+1} + \pi_1 (v_{k-t(s)+2,k-t(s)+1} - v_{s-t(s)+1,k-t(s)+1}) \right) =$$

$$= w_{t(s)-1} v_{s-t(s)+1,k-t(s)+1} .$$

When $k = t(s)-1$, we obtain

$$v_{sk} = -\frac{w_{t(s)-1}}{\pi_1} \sum_{j=1}^{s-t(s)+1} u_{j0} = -\frac{w_{t(s)-1}}{\pi_1}\left(u_{1,0} + \sum_{j=2}^{s-t(s)+1} u_{j0} \right) =$$

$$= -\frac{w_{t(s)-1}}{\pi_1}\left(1 - \pi_1 + \pi_1 (v_{1,0} - v_{s-t(s)+1,0}) \right) = -\frac{w_{t(s)-1}}{\pi_1}\left(1 - \pi_1 v_{s-t(s)+1,0} \right).$$

If $t(s-t(s)+1) = 2$, then

$$v_{s-t(s)+1,0} = \frac{w_{t(s-t(s)+1)-1}}{\pi_1} = \frac{w_1}{\pi_1} = 1,$$

which implies the formula

$$v_{s,t(s)-1} = -\frac{w_{t(s)-1}}{\pi_1}\left(1 - \pi_1 \right).$$

But, if $t(s-t(s)+1) \geq 3$, then

$$v_{s-t(s)+1,0} = 0$$

and we have the relation

$$v_{s,t(s)-1} = -\frac{w_{t(s)-1}}{\pi_1} \ .$$

Thus, the statement of the theorem is true. □

Using this theorem, Algorithm 6.1 can be rewritten in the following way.

Algorithm 6.2

Input: $X=(x_j)_{j=1}^{m}\in V^m$, $\pi_j=p^*(x_j)$, $j=\overline{1,m}$;

Output: $I_m^{[a]}$, $q\in G[\mathbb{R}][m](a)$.

1) *Determine the values* $w_k=\prod_{j=1}^{k}\pi_j$, $k=\overline{1,m}$;

2) *Obtain the initial state* $I_m^{[a]}=(0,0,\ldots,0,w_m)$;

3) *For each* $s=\overline{1,m}$, *execute the following steps:*

   a. *Compute the values*

   $$t(s)=\min(\{t\in\{2,3,\ldots,s+1\}\mid x_{t-1+j}=x_j,\ j=\overline{1,s+1-t}\});$$

   b. *Obtain the parameters* $v_{s,k}$, $k=\overline{0,s-1}$, *using Theorem 6.1;*

4) *Compute the components*

$$q_0=1-\pi_1v_{m,0},\ q_k=-\pi_1v_{m,k},\ k=\overline{1,m-1},$$

of the generating vector $q\in G[\mathbb{R}][m](a)$.

Theorem 6.2 *The expectation of the evolution time* T *of the stochastic system* L *is*

$$\mathbb{E}(T)=-1+(m+w_m^{-1})+\frac{1}{w_m}\sum_{k=0}^{m-1}(k+1)z_{mk},$$

where

$$z_{sk}=-\pi_1v_{sk},\ s=\overline{1,m},\ k=\overline{0,s-1}.$$

Proof. By applying Algorithm 6.2, we obtain the generating function

$$G^{[a]}(z)=\frac{w_mz^{m-1}}{H_m^{[q]}(z)}.$$

Since

$$G^{[a]}(1)=\sum_{n=0}^{\infty}P(T=n)=1,$$

we have the relation

$$H_m^{[q]}(1)=w_m.$$

The derivative of the generating function is

$$\frac{\partial G^{[a]}(z)}{\partial z} = \frac{(m-1)w_m z^{m-2} H_m^{[q]}(z) - w_m z^{m-1} \dfrac{\partial H_m^{[q]}(z)}{\partial z}}{(H_m^{[q]}(z))^2},$$

which implies the formula

$$\mathbb{E}(T) = \frac{\partial G^{[a]}(z)}{\partial z}\bigg|_{z=1} = (m-1) - \frac{1}{w_m}\frac{\partial H_m^{[q]}(z)}{\partial z}\bigg|_{z=1} =$$

$$= (m-1) - \frac{1}{w_m}\frac{\partial\left(1 - \sum\limits_{k=0}^{m-1} q_k z^{k+1}\right)}{\partial z}\bigg|_{z=1} = (m-1) + \frac{1}{w_m}\sum_{k=0}^{m-1}(k+1)q_k .$$

Since Algorithm 6.2 gives

$$q_0 = 1 - \pi_1 v_{m,0},\ q_k = -\pi_1 v_{m,k},\ k = \overline{1, m-1},$$

we obtain

$$\mathbb{E}(T) = (m-1) + \frac{1}{w_m}\left(1 + \sum_{k=0}^{m-1}(k+1)(-\pi_1 v_{m,k})\right) =$$

$$= -1 + (m + w_m^{-1}) + \frac{1}{w_m}\sum_{k=0}^{m-1}(k+1)z_{mk}.$$

Therefore, the proof of the theorem is complete. □

Example 6.1 *Let L be a stochastic system with the set of states $V = \{v_1, v_2\}$ and the final sequence of states $X = (v_2, v_1)$. The transition time is one. The states of the system are independent and the initial distribution of the states is*

$$p^*(v_1) = 0.3,\ p^*(v_2) = 0.7 .$$

Determine the expectation, the variance, the standard deviation and the moments of order 1 and 2 of the evolution time.

Solution. Let T be the evolution time of the system and $a = rep(T)$. According to the developed algorithm, we have the formulas

$$\pi_1 = p^*(v_2) = 0.7,\ \pi_2 = p^*(v_1) = 0.3,\ w_1 = \pi_1 = 0.7,\ w_2 = \pi_1\pi_2 = 0.21 .$$

Therefore, $a \in Rol[\mathbb{R}][2]$ with the initial state $I_2^{[a]} = (0, 0.21)$.

We have

$$u_{1,0} = 1 - \pi_1 = 0.3\,,\ v_{1,0} = 1\,.$$

Since $t_2 = 3$, we obtain

$$v_{2,0} = 0\,,\ v_{2,1} = \frac{w_2}{\pi_1} = 0.3\,,$$

$$u_{2,0} = \pi_1(v_{1,0} - v_{2,0}) = 0.7\,,\ u_{2,1} = -\pi_1 v_{2,1} = -0.21\,.$$

Therefore, the generating vector of the sequence a is $q = (q_0, q_1)$, where

$$q_0 = 1 - \pi_1 v_{2,0} = 1\,,\ q_1 = -\pi_1 v_{2,1} = -0.21\,.$$

The expectation, the variance, the standard deviation and the moments of order 1 and 2 of the evolution time of the system are determined as follows:

$$\nu_1(T) = \mathbb{E}(T) = \overline{m} = m - 2 + \frac{1 + \Delta_1(q)}{w_m} = \frac{1 + q_1}{w_2} \approx 3.761905\,,$$

$$\mathbb{V}(T) = (m-1)(m-2) + \overline{m}(\overline{m} - 2m + 3) + \frac{\Delta_1(q) + \Delta_2(q)}{w_m} =$$

$$= \overline{m}(\overline{m} - 1) + \frac{q_1 + q_1}{w_2} \approx 8.390024\,,$$

$$\sigma(T) = \sqrt{\mathbb{V}(T)} \approx 2.896554\,,$$

$$\nu_2(T) = \mathbb{V}(T) + \nu_1^2(T) \approx 22.541953\,. \qquad \square$$

6.3 Stochastic systems with final critical state

We study now a particular case of the discrete stochastic systems considered above. Consider the case when the components of the final sequence are equal, i.e. the equality

$$x_1 = x_2 = \ldots = x_m = v^* \in V$$

holds. For such stochastic systems, the definitions of final critical state, moments of the first alert and the total alert time were introduced in Section 6.1.

Similarly to the general case, the evolution time T is the time elapsed from the start of the system's evolution until its termination, that is, it is the time instant at which the system completes its evolution. To find the probabilistic characteristics of the evolution time, we can also apply Algorithm 4.4 and Algorithm 6.1.

Applying Algorithm 6.1 for this particular case, we obtain the simplified formula for calculating the components of the generating vector:

$$q = (q_k)_{k=0}^{m-1} \in G[\mathbb{R}][m](rep(T)),$$

$$q_k = (1-p)p^k,\ k = \overline{0, m-1},$$

where p is the probability of the critical state. The complexity of the algorithm becomes $O(m)$. In the case $m = 1$ we obtain

$$q = (1-p) \in G[\mathbb{R}][1](rep(T)),$$

i.e. the evolution time has a geometric distribution with ratio $r = 1-p$.

The moments μ_k of the first alert of order k, $k = \overline{1, m}$, represent the moments of time when, for the first time, the system consecutively remains k times in its critical state. We have the sequence of inequalities

$$0 \le \mu_1 < \mu_2 < \ldots < \mu_m = T.$$

To determine the main probabilistic characteristics of these random variables we will consider m discrete stochastic systems similar to the given system. Therefore, we will have L_k, $k = \overline{1, m}$, all with the same set of states V, the same critical state v^* and the same probability $p = p^*(v^*)$ of occurrence of critical state, but with different degrees. Therefore, the critical state of the system L_k will be of degree k, $k = \overline{1, m}$. We observe that the moment μ_k of the first alert of order k represents the evolution time of the system L_k, $k = \overline{1, m}$.

The total alert time is the period of time η during which the system is in its critical state during its evolution. The value $\gamma = \eta + 1$ represents the number of visits

to the critical state. Let L_k, $k=\overline{1,m}$, be the systems defined above. Consider that the total alert time of the system L_k is η_k and the number of visits to the critical state is

$$\gamma_k = \eta_k + 1,\ k=\overline{1,m}.$$

As mentioned above, the evolution time of the system L_k is the moment μ_k of the first alert of order k of the system L, $k=\overline{1,m}$.

Theorem 6.3 *The formula* $\eta_m = \mu_{m-1}+1$, $\forall m>1$, *holds, where* $\eta_1 = 0$.

Proof. The formula $\eta_1 = 0$ is obvious (since the unique visit of the critical state happens when the system stops). Let $m \geq 2$. We observe that the random variables η_m and $\mu_{m-1}+1$ have the same set of possible values. We will prove that they follow the same distribution law.

Let $p = p^*(v^*)$ be the probability of occurrence of the critical state, $q = 1-p$ and $a^{(m)} = rep(\gamma_m)$. Obviously, for $k<m$ we have $a_k^{(m)} = 0$. We will represent the evolution of the system in the form of binary sequences, where 1 means the occurrence of the critical state and 0 means the occurrence of another state of the system. These sequences satisfy the property:

"*the sequence* $\{1\}^m$ *appears only at the end*".

Assume that $\mu_m = n-1 \geq m$, i.e. the length of the binary sequences is $n>m$. Obviously, the $(n-m)$−th digit is 0 (immediately before the sequence $\{1\}^m$). Therefore, the sequence composed of the first $(n-m-1)$ digits has the property that it does not contain the sequence $\{1\}^m$.

Denote by $f_m(n,k)$ the number of binary sequences of length n which contain exactly k ones and do not contain the sequence $\{1\}^m$. These sequences contain $(n-k)$ zeros. Label the zeros with numbers from 1 to $(n-k)$. We will also introduce the label 0 for marking the beginning of the sequence.

Consider all decompositions

$$k = \sum_{j=0}^{n-k} t_j ,$$

where t_j represents the number of ones after the j−th zero, $j = \overline{0, n-k}$. We have

$$0 \le t_j < m , \ j = \overline{0, n-k} .$$

We associate each decomposition with the number $\overline{t_0 t_1 \ldots t_{n-k}}$ written in base m. Therefore, we have $B_m(k, n-k+1)$ such decompositions. As a result, we obtain

$$f_m(n,k) = B_m(k, n-k+1) , \ \forall n > m , \ 0 \le k \le n .$$

By concatenating them with the sequence $(0, \{1\}^m)$, we obtain all possible evolutions, with the evolution time $\mu_m \ge m$. Therefore,

$$P(\mu_m = n-1, \gamma_m = k) = f_m(n-m-1, k-m) p^k q^{n-k} =$$

$$= B_m(k-m, n-k) p^k q^{n-k} ,$$

$$a_k^{(m)} = P(\gamma_m = k) = \sum_{n=k+1}^{\infty} P(\mu_m = n-1, \gamma_m = k) =$$

$$= \sum_{n=k+1}^{\infty} B_m(k-m, n-k) p^k q^{n-k} =$$

$$= p^k \sum_{n=1}^{\infty} B_m(k-m, n) q^n = p^k G^{[d^{(k-m)}]}(q) , \ \forall k > m ,$$

where

$$d_t^{(s)} = B_m(s,t) , \ s = \overline{0, \infty} , \ t = \overline{0, \infty} .$$

From Theorem 4.14 we have the formula

$$G^{[d^{(s)}]}(z) = \frac{z}{1-z} \sum_{k=0}^{m-2} G^{[d^{(s-1-k)}]}(z) , \ \forall s \ge m ,$$

$$G^{[d^{(s)}]}(z) = \frac{z}{(1-z)^{s+1}} , \ s = \overline{0, m-1} .$$

For $k = m$ we obtain

$$a_m^{(m)} = P(\gamma_m = m) = \sum_{n=m}^{\infty} P(\mu_m = n-1, \gamma_m = m) =$$

$$= \sum_{n=m}^{\infty} p^m q^{n-m} = p^m \sum_{n=0}^{\infty} q^n = p^{m-1}.$$

Let $k \geq 2m$. We have the following sequence of equalities:

$$a_k^{(m)} = p^k G^{[d^{(k-m)}]}(q) = \frac{p^k q}{1-q} \sum_{j=0}^{m-2} G^{[d^{(k-m-1-j)}]}(q) =$$

$$= \sum_{j=0}^{m-2} qp^j (p^{k-1-j} G^{[d^{(k-m-1-j)}]}(q)) = \sum_{j=0}^{m-2} qp^j a_{k-1-j}^{(m)}.$$

For $m < k < 2m$ we obtain

$$a_k^{(m)} = p^k G^{[d^{(k-m)}]}(q) = p^k \frac{q}{p^{k-m+1}} = qp^{m-1}.$$

However, since

$$\sum_{j=0}^{m-2} qp^j a_{k-1-j}^{(m)} = \sum_{j=0}^{k-m-1} qp^j a_{k-1-j}^{(m)} =$$

$$= qp^{k-m-1} a_m^{(m)} + \sum_{j=0}^{k-m-2} qp^j a_{k-1-j}^{(m)} = qp^{k-2} + q^2 p^{m-1} \sum_{j=0}^{k-m-2} p^j =$$

$$= qp^{k-2} + q^2 p^{m-1} \frac{1-p^{k-m-1}}{1-p} = qp^{m-1} = a_k^{(m)},$$

we have

$$a_m^{(m)} = w_{m-1},$$

$$a_k^{(m)} = \sum_{j=0}^{m-2} qp^j a_{k-1-j}^{(m)} = \sum_{j=0}^{m-2} r_j a_{k-1-j}^{(m)}, \ \forall k > m,$$

where

$$w_{m-1} = p^{m-1}, \ r_j = qp^j, \ j = \overline{0, m-2}.$$

Let $\psi_m = \gamma_m - 2 = \eta_m - 1$ and $c = rep(\psi_m)$. It follows that

$$c_k^{(m)} = a_{k+2}^{(m)}, \ k = \overline{0, \infty},$$

hence

$$c_k^{(m)} = \sum_{j=0}^{m-2} r_j c_{k-1-j}^{(m)}, \ \forall k \geq m-1.$$

It implies that

$$c = (c_k^{(m)})_{k=0}^{\infty} \in Rol[(0;1)][m-1],$$

$$r = (r_j)_{j=\overline{0,\, m-2}} \in G[(0;1)][m-1](c),$$

$$I_{m-1}^{[c]} = (0, 0, \ldots, 0, w_{m-1}).$$

By applying the results concerning the distribution of the moments of the first alert, we have $c = rep(\mu_{m-1})$. Therefore, $rep(\eta_m - 1) = rep(\mu_{m-1})$, which implies:

$$rep(\eta_m) = rep(\mu_{m-1} + 1), \ \forall m > 1.$$

In conclusion, the random variables η_m and $\mu_{m-1} + 1$ follow the same distribution law. This means that they are equal, i.e. $\eta_m = \mu_{m-1} + 1$, $\forall m > 1$. □

Theorem 6.4 *The moment of order n of the total alert time is*

$$\nu_n(\eta_m) = \sum_{k=0}^{n} C_n^k \nu_k(\mu_{m-1}),$$

the dominant value is $\mu(\eta_m) = m-1$, the median is $m_0(\eta_m) = m_0(\mu_{m-1}) + 1$ and the expectation, the variance and the standard deviation can be determined according to relation (4.1).

Proof. For the moment of order $n \in \mathbb{N}^*$ we have the sequence of equalities:

$$\nu_n(\eta_m) = \mathbb{E}((\eta_m)^n) = \mathbb{E}((\mu_{m-1} + 1)^n) = \mathbb{E}\left(\sum_{k=0}^{n} C_n^k (\mu_{m-1})^k \right) = \sum_{k=0}^{n} C_n^k \nu_k(\mu_{m-1}).$$

The dominant value is

$$\mu(\eta_m) = \mu(\mu_{m-1} + 1) = (m-2) + 1 = m-1.$$

The remaining formulas are clear from what was mentioned above. □

Example 6.2 *Consider a stochastic system L with final critical state of degree 4 and the probability of occurrence of the critical state $p = 0.8$. Determine the expectation,*

the variance and the standard deviation of the evolution time, moments of the first alert and total alert time.

Solution. Let T be the evolution time of the stochastic system L. It has already been proved that the distribution of the random variable T is a homogeneous linear recurrence with the initial state

$$I_4^{[T]} = (0,0,0,w_4)$$

and the generating vector

$$q = (q_k)_{k=0}^{3} \in G[\mathbb{R}][4](rep(T)),$$

whose components are

$$w_k = p^k,\ q_k = (1-p)p^k,\ k = \overline{0,3}.$$

Therefore, we have the initial state

$$I_4^{[T]} = (0,0,0,0.4096)$$

and the generating vector

$$q = (0.2, 0.16, 0.128, 0.1024) \in G[\mathbb{R}][4](rep(T)).$$

Assume that μ_k is the moment of the first alert of order k, $k = \overline{1,4}$. We have $\mu_k \in Rol[\mathbb{R}][k]$, $k = \overline{1,4}$, and the initial state and the generating vector of them are

$$I_k^{[\mu_k]} = (0,\ldots,0,w_k) \in \mathbb{R}^k,$$

$$q^{(k)} = (q_j)_{j=0}^{k-1} \in G[\mathbb{R}][k](\mu_k).$$

The expectation and the variance of the moment of the first alert of order k, $k = \overline{1,4}$, are determined according to the obtained formulas:

$$\mathbb{E}(\mu_k) = \overline{m}_k = k - 2 + \frac{1 + \Delta_1(q^{(k)})}{w_k},$$

$$\mathbb{V}(\mu_k) = (k-1)(k-2) + \overline{m}_k(\overline{m}_k - 2k + 3) + \frac{\Delta_1(q^{(k)}) + \Delta_2(q^{(k)})}{w_k}.$$

By performing calculations, we obtain:

k	w_k	q_{k-1}	$\Delta_1(q^{(k)})$	$\Delta_2(q^{(k)})$	$\mathbb{E}(\mu_k)$	$\mathbb{V}(\mu_k)$	$\sigma(\mu_k)$
1	0.8	0.2	0	0	0.25	0.3125	0.559017
2	0.64	0.16	0.16	0.16	1.8125	1.9727	1.404513
3	0.512	0.128	0.416	0.672	3.7656	7.00806	2.647273
4	0.41	0.102	0.723	1.594	6.2070	19.14833	4.375881

Table 6.1 The characteristics of the moments of the first alert

For computing the evolution time T, the moments of the first alert μ_k, $k=\overline{1,4}$ and the total alert time $\eta=\mu_{m-1}+1$, we obtain the expectation, the variance and the standard deviation.

ξ	$\mathbb{E}(\xi)$	$\mathbb{V}(\xi)$	$\sigma(\xi)$
μ_1	0.25	0.3125	0.559017
μ_2	1.8125	1.97265625	1.404513
μ_3	3.765625	7.008056641	2.647273
μ_4	6.20703125	19.14833069	4.375881
T	6.20703125	19.14833069	4.375881
η	4.765625	7.008056641	2.647273

Table 6.2 The characteristics of the stochastic system

6.4 Systems with a final sequence of states and random transition time

Let L_Θ be a stochastic system with the set of states $V=\{v_1,v_2,\ldots,v_\omega\}$, $|V|=\omega\le\infty$. The state of the system at each step $n=0,1,2,\ldots$ is $v(n)\in V$. The transition time $\theta_{(u,v)}(t)$ from the state $u\in V$ to the state $v\in V$ is a positive random vari-

able, whose distribution law does not depend on the states through which the system passes and does not depend on the time when the transition is performed.

The initial distribution $P^* = ((p^*(v_j))_{j=1}^{\omega})^T$ of the discrete stochastic system L_Θ is also given. The transition of the system from the state $u \in V$ to the state $v \in V$ at time t is performed with probability $p_{(u,v)}(t) = p^*(v)$, which does not depend on the current state of the system.

The system completes its evolution when it consecutively passes through a fixed sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m.$$

Let T_Θ be the evolution time of the discrete stochastic system L_Θ. The aim is to determine the main probabilistic characteristics of the evolution time T_Θ.

Let θ be a random variable with the same distribution law as the transition time of the system. We assume that all required probabilistic characteristics of the random variable θ are known. In the general case, their determination is not so easy, being possible only for certain distribution laws.

The stochastic system L_Θ is associated with the related stochastic system L_1, with unit transition time. Let T_Θ be the evolution time of the system L_Θ and let T_1 be the evolution time of the associated system L_1. We note that T_1 represents the number of transitions performed by L_Θ during its evolution.

Let $a = rep(T_1)$ and

$$\theta_j = \theta_{(v(j),v(j+1))}(\tau_j),\ j = \overline{0, T_1 - 1},$$

where τ_j is the moment in time when the transition is performed. By the law of total probability, we have

$$F_{T_\Theta}(x) = \sum_{k=0}^{\infty} a_k F_{\sum_{j=0}^{k-1} \theta_j}(x).$$

Substituting this relation into the formula for the characteristic function and taking into account that the random variables θ_j, $j=\overline{0,T_1-1}$, are independent, we obtain

$$\Phi_{T_\Theta}(t)=\int_{-\infty}^{+\infty}e^{itx}d\sum_{k=0}^{\infty}a_k F_{\sum_{j=0}^{k-1}\theta_j}(x)=\sum_{k=0}^{\infty}a_k\int_{-\infty}^{+\infty}e^{itx}dF_{\sum_{j=0}^{k-1}\theta_j}(x)=$$

$$=\sum_{k=0}^{\infty}a_k\Phi_{\sum_{j=0}^{k-1}\theta_j}(t)=\sum_{k=0}^{\infty}a_k\prod_{j=0}^{k-1}\Phi_{\theta_j}(t)=\sum_{k=0}^{\infty}a_k(\Phi_\theta(t))^k=G_{T_1}(\Phi_\theta(t)). \tag{6.15}$$

Using the properties of generating and characteristic functions, we obtain

$$\Phi_\theta^{(k)}(0)=i^k\nu_k(\theta),\ G_{T_1}^{(k)}(\Phi_\theta(0))=G_{T_1}^{(k)}(1),\ k=1,2,\ldots. \tag{6.16}$$

The values $\nu_k(\theta)$ are known and the values $G_{T_1}^{(k)}(1)$, $k=\overline{1,\infty}$, can be obtained by applying Algorithm 6.1 and Algorithm 4.4.

By using the values obtained from (6.16) as input for Algorithm 4.2, we can determine the values $\Phi_{T_\Theta}^{(k)}(0)$, $k=1,2,\ldots$. Further, having these values, we can compute the moments of the evolution time T_Θ using the formula

$$\nu_k(T_\Theta)=\frac{1}{i^k}\Phi_{T_\Theta}^{(k)}(0),\ k=\overline{1,\infty}. \tag{6.17}$$

In particular, we obtain

$$\begin{aligned}&\mathbb{E}(T_\Theta)=\mathbb{E}(\theta)\mathbb{E}(T_1);\\ &\mathbb{V}(T_\Theta)=\mathbb{E}(T_1)\mathbb{V}(\theta)+(\mathbb{E}(\theta))^2\mathbb{V}(T_1).\end{aligned} \tag{6.18}$$

In this way, we have established theoretically the following algorithm:

Algorithm 6.3

Input: V, $X\in V^m$, $p^*:V\to[0,1]$, $\nu_k(\theta)$, $k=\overline{1,n}$;

Output: $\mathbb{E}(T_\Theta)$, $\mathbb{V}(T_\Theta)$, $\sigma(T_\Theta)$, $\nu_k(T_\Theta)$, $k=\overline{1,n}$;

1) Determine $\mathbb{E}(\theta)$ and $\mathbb{V}(\theta)$ according to the formula (4.1);

2) Find $\mathbb{E}(T_1)$, $\mathbb{V}(T_1)$, $G_{T_1}^{(k)}(1)$, $k=\overline{1,n}$, by using Algorithm 6.1 and Algorithm 4.4;

3) *Determine the values* $\mathbb{E}(T_\Theta)$ *and* $\mathbb{V}(T_\Theta)$ *using the formula (6.18);*

4) *Obtain* $\sigma(T_\Theta)$ *from the relation (4.1);*

5) *Determine the values* $\Phi_\theta^{(k)}(0)$, $k=\overline{1,n}$, *according to the formula (6.16);*

6) *Obtain the values* $\Phi_{T_\Theta}^{(k)}(0)$, $k=\overline{1,n}$, *by applying Algorithm 4.2;*

7) *Determine* $\nu_k(T_\Theta)$, $k=\overline{1,n}$, *according to the formula (6.17).*

If the moments are not required, then this algorithm is reduced to the first 4 steps, the complexity being $O(m^3)$. Next, we study a few particular cases.

We have already established that $a=rep(T_1)\in Rol[\mathbb{R}][m]$ and we presented Algorithm 6.1 for determining the generating vector $q\in G[\mathbb{R}][m](rep(T_1))$. By using the formulas (4.19), (6.1) and (6.15), the function $\Phi_{T_\Theta}(t)$ becomes

$$\Phi_{T_\Theta}(t)=G_{T_1}(\Phi_\theta(t))=\frac{w_m(\Phi_\theta(t))^{m-1}}{H_m^{[q]}(\Phi_\theta(t))}. \tag{6.19}$$

If θ is a nonnegative integer random variable, then T_Θ is such a random variable too and

$$G_{T_\Theta}(z)=G_{T_1}(G_\theta(z))=\frac{w_m(G_\theta(z))^{m-1}}{H_m^{[q]}(G_\theta(z))}. \tag{6.20}$$

Consider the particular case $c=rep(\theta)\in Rol[\mathbb{R}][s]$. Let $r\in G[\mathbb{R}][s](c)$. According to the properties of the homogeneous linear recurrences, we have

$$G_\theta(z)=\frac{A(z)}{H_s^{[r]}(z)}, \tag{6.21}$$

where

$$A(z)=G_s^{[c]}(z)-z\sum_{k=0}^{s-1}r_k z^k G_{s-1-k}^{[c]}(z) \tag{6.22}$$

is a polynomial of degree $\deg(A(z))<s$.

Substituting (6.21) in (6.20) and multiplying the numerator and denominator by $(H_s^{[r]}(z))^{m-1}$, we obtain that $G_{T_\Theta}(z)$ is a rational fraction with the numerator

$$U(z) = w_m(A(z))^{m-1} \tag{6.23}$$

and denominator

$$V(z) = (H_s^{[r]}(z))^{m-1} H_m^{[q]}\left(\frac{A(z)}{H_s^{[r]}(z)}\right) =$$

$$= (H_s^{[r]}(z))^{m-1} - \sum_{k=0}^{m-1} q_k (A(z))^{k+1} (H_s^{[r]}(z))^{m-k-2} . \tag{6.24}$$

For $k = \overline{0, m-1}$, we have the inequality

$$\deg((A(z))^{k+1}(H_s^{[r]}(z))^{m-k-2}) < s(k+1) + s(m-k-2) =$$

$$= s(m-1) = \deg((H_s^{[r]}(z))^{m-1}) .$$

Therefore,

$$M = \deg(V(z)) = s(m-1) > (m-1)\deg(A(z)) = \deg(U(z)),$$

which implies that $G_{T_\Theta}(z)$ is a regular rational fraction. Additionally, we obtain

$$V(0) = (H_s^{[r]}(0))^{m-1} H_m^{[q]}(G_\theta(0)) = H_m^{[q]}(P(\theta = 0)) = H_m^{[q]}(0) = 1 \neq 0 .$$

By applying Theorem 1.3, we have

$$\begin{gathered} rep(T_\Theta) \in Rol[\mathbb{R}][M], \\ V(z) \in H[\mathbb{R}][M](T_\Theta), \\ I_M^{[rep(T_\Theta)]} = (c_0, c_1, \ldots, c_{M-1}), \end{gathered} \tag{6.25}$$

where

$$c_k = \sum_{j=0}^{k-1} q_j c_{k-1-j} + u_k , \; k = \overline{0, M-1}, \tag{6.26}$$

and the coefficients q_k and u_k, $k = \overline{0, M-1}$, are given by the relations

$$\begin{gathered} u = (u_0, u_1, \ldots, u_{M-1}) = (U(z) \mid z)^{[M-1]}, \\ H_M^{[q]}(z) = V(z) . \end{gathered} \tag{6.27}$$

Thus, if $rep(\theta) \in Rol[\mathbb{R}]$, we can determine the probabilistic characteristics of the evolution time T_Θ by applying Algorithm 4.3 or Algorithm 4.4, using the initial

state $I_M^{[rep(T_\Theta)]}$ and the generating vector $q \in G[\mathbb{R}](rep(T_\Theta))$ as input, computed using formulas (6.22) − (6.27). This method is a more efficient alternative to Algorithm 6.3 in this case.

Example 6.3 *Let L be a stochastic system with final critical state of degree 4 and the probability $p = 0.8$ of occurrence of the critical state. The transition time follows a geometric distribution with ratio $p^* = 0.9$. Determine its expectation, variance and standard deviation.*

Solution. The stochastic system L is associated with the system L_1 with unit transition time. The expectation, the variance and the standard deviation of the evolution time T_1 of the system L_1 were determined in Example 6.2.

Let θ be a random variable with the same distribution as the transition time of the stochastic system L. We have

$$\mathbb{E}(\theta) = (p^*)^{-1} = \frac{10}{9},$$

$$\mathbb{V}(\theta) = (1-p^*)(p^*)^{-2} = \frac{10}{81}.$$

The expectation of the evolution time T is

$$\mathbb{E}(T) = \mathbb{E}(\theta)\mathbb{E}(T_1)$$

and the variance is

$$\mathbb{V}(T) = \mathbb{E}(T_1)\mathbb{V}(\theta) + (\mathbb{E}(\theta))^2\mathbb{V}(T_1).$$

We obtain:

$\mathbb{E}(\theta)$	$\mathbb{V}(\theta)$	$\mathbb{E}(T_1)$	$\mathbb{V}(T_1)$	$\mathbb{E}(T)$	$\mathbb{V}(T)$	$\sigma(T)$
1.1111	0.1235	6.207	19.1483	6.8967	24.4062	4.9403

Table 6.3 The characteristics of the evolution time of the stochastic system

6.5 Systems with a final sequence of states and interdependent states

Let L be a stochastic system with the set of states $V = \{v_1, v_2, \ldots, v_\omega\}$, where $|V| = \omega < \infty$. The state of the system at each discrete time $t = 0,1,2,\ldots$ is $v(t) \in V$. The transition time of the system from the state u to the state v at each discrete time t is one.

On the set V the probability function $p^* : V \to [0,1]$ is defined, where $p^*(v)$ represents the probability that the system L starts its evolution from the state $v \in V$. The vector $P^* = ((p^*(v_j))_{j=1}^{\omega})^T$ represents the initial distribution of the discrete stochastic system L.

The transition of the system from an arbitrary state $u \in V$ to another state $v \in V$ at each discrete time t is performed with probability $p(u,v) \in [0,1]$. If the system does not have the possibility to pass from the state u to the state v, then it is considered that this transition is performed with probability $p(u,v) = 0$. The matrix $P = (p(u,v))_{u,v \in V}$ represents the transition probability matrix of the discrete stochastic system L.

The stochastic system stops when it consecutively passes through a fixed sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m.$$

Assume that the stochastic system L has completed its evolution at time T. The value T represents the evolution time of the system L. We consider the problem of finding the main probabilistic characteristics of the evolution time T ([84]).

Let $a = rep(T)$ be the distribution of the evolution time T of the stochastic system L. We consider the sets

$$X_j = \{x_j\},\ \overline{X}_j = V \setminus X_j,\ j = \overline{1,m}.$$

Also, we denote

$$\pi_{ij} = p(x_i, x_j),\ \pi_j = p^*(x_j),\ \omega_j = \prod_{k=3}^{j} \pi_{k-1,k},\ i, j = \overline{1,m}. \tag{6.28}$$

If $\exists\, j \in \{2,3,\ldots,m\}$ such that $\pi_{j-1,j} = 0$, then the stochastic system never reaches the final sequence of states, i.e. $a_n = 0$, $\forall n \in \mathbb{N}$, $\mathbb{P}(T = +\infty) = 1$ and the system L has an infinite evolution, hence the moments of the evolution time T are also infinite. Further, we consider $\pi_{j-1,\,j} \neq 0$, $j = \overline{2,m}$.

Consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2,3,\ldots,s\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\},\ s = \overline{1,m}.$$

The minimum of each set is denoted by

$$t_s = \min_{k \in T_s} k,\ s = \overline{1,m}. \tag{6.29}$$

It represents the smallest index in the sequence $(x_1, x_2, \ldots, x_s)$ from which, when $(x_1, x_2, \ldots, x_s)$ is superimposed on itself, the overlapping components coincide.

Let $\forall n \in \mathbb{Z}$ and $S(V) = \{A \mid A \subseteq V\}$. We consider the hypothesis *H:*

"The initial state of the system is known".

We introduce the notation

$$P_\Phi(n) = P(T = n, v(j) \in \Phi_j, j = \overline{0,t-1} \mid H),\ \forall \Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t,\ t \in \mathbb{N},$$

and, in the case $|\,\Phi_j\,| = 1$, the expression Φ_j is replaced by its element. Based on this notation, we introduce the following functions on $\mathbb{Z}$:

$$\alpha_k(n) = P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}(n),$$

$$\beta_k(n) = P_{(X_1, X_2, \ldots, X_k)}(n),$$

$$\gamma_k(n) = P_{(X_2, X_3, \ldots, X_k)}(n),\ k = \overline{1,m}.$$

Initially, we consider the case $m \geq 2$. We easily obtain the following relation:

$$\beta_k(n) = P_{(X_1, X_2, \ldots, X_k)}(n) = P(n) - \sum_{j=1}^{k} P_{(X_1, X_2, \ldots, \overline{X}_j)}(n) =$$

$$= P(n) - \sum_{j=1}^{k} \alpha_j(n),\ k = \overline{1,m}. \tag{6.30}$$

From the definition, we directly obtain

$$\gamma_1(n) = P(n),\ \forall n \in \mathbb{Z}. \tag{6.31}$$

Let $s \geq 2$. For $t_s \leq s$ we have

$$\gamma_s(n) = P_{(X_2,X_3,\ldots,X_s)}(n) = \pi_{2,3}\pi_{3,4}\ldots\pi_{t_{s-1},t_s} P_{(X_{t_s},\ldots,X_s)}(n - t_s + 2) =$$

$$= \omega_{t_s} P_{(X_1,\ldots,X_{s+1-t_s})}(n - t_s + 2) = \omega_{t_s}\beta_{s+1-t_s}(n - t_s + 2) =$$

$$= \omega_{t_s}\left(P(n - t_s + 2) - \sum_{j=1}^{s+1-t_s} \alpha_j(n - t_s + 2) \right), \tag{6.32}$$

and, in the case $t_s = s+1$, we obtain

$$\gamma_s(n) = P_{(X_2,X_3,\ldots,X_s)}(n) = \omega_s \sum_{y\in V} p(x_s, y) P_{(y)}(n - s + 1) =$$

$$= \sum_{y\in V} \omega_s p(x_s, y) P_{(y)}(n - t_s + 2). \tag{6.33}$$

We determine the values $\alpha_k(n)$, $k = \overline{1,m}$. We have

$$\alpha_1(n) = P_{(\overline{X}_1)}(n) = \sum_{x\in V\setminus\{x_1\}} P_{(x)}(n) =$$

$$= \sum_{x\in V\setminus\{x_1\}} \sum_{y\in V} P_{(x,y)}(n) = \sum_{x\in V\setminus\{x_1\}} \sum_{y\in V} p(x,y) P_{(y)}(n-1) =$$

$$= \sum_{y\in V} P_{(y)}(n-1) \sum_{x\in V\setminus\{x_1\}} p(x,y) = \sum_{y\in V} \psi_1(y) P_{(y)}(n-1), \tag{6.34}$$

where

$$\psi_1(y) = \sum_{x\in V\setminus\{x_1\}} p(x,y),\ \forall y \in V. \tag{6.35}$$

For $k = 2$, we obtain

$$\alpha_2(n) = P_{(X_1,\overline{X}_2)}(n) = \sum_{y\neq x_2} P_{(x_1,y)}(n) =$$

$$= \sum_{y \neq x_2} p(x_1, y) P_{(y)}(n-1) = \sum_{y \neq x_2} p(x_1, y) P_{(y)}(n-1), \tag{6.36}$$

and for $k \geq 3$ we have

$$\alpha_k(n) = P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X_k})}(n) = \pi_{1,2} P_{(X_2, X_3 \ldots, X_{k-1}, \overline{X_k})}(n-1) =$$

$$= \pi_{1,2} (P_{(X_2, X_3 \ldots, X_{k-1})}(n-1) - P_{(X_2, X_3 \ldots, X_k)}(n-1)) =$$

$$= \pi_{1,2} (\gamma_{k-1}(n-1) - \gamma_k(n-1)). \tag{6.37}$$

From the obvious relation

$$P(n) = \sum_{k=1}^{m} \alpha_k(n) = \alpha_1(n) + \alpha_2(n) + \sum_{k=3}^{m} \alpha_k(n), \ \forall n \geq m, \tag{6.38}$$

by using the relations (6.30), (6.36) and (6.37), we obtain the formula

$$P_{(x_1)}(n) = \beta_1(n) = P(n) - \alpha_1(n) = \alpha_2(n) + \sum_{k=3}^{m} \alpha_k(n) =$$

$$= \sum_{y \neq x_2} p(x_1, y) P_{(y)}(n-1) + \sum_{k=3}^{m} \pi_{1,2} (\gamma_{k-1}(n-1) - \gamma_k(n-1)) =$$

$$= \sum_{y \neq x_2} p(x_1, y) P_{(y)}(n-1) + \pi_{1,2} (\gamma_2(n-1) - \gamma_m(n-1)) =$$

$$= \sum_{y \in V} p(x_1, y) P_{(y)}(n-1) - \pi_{1,2} \gamma_m(n-1), \ \forall n \geq m. \tag{6.39}$$

For $x \neq x_1$ the following relation holds:

$$P_{(x)}(n) = \sum_{y \in V} p(x, y) P_{(y)}(n-1) = \sum_{y \in V} p(x, y) P_{(y)}(n-1). \tag{6.40}$$

We observe that there exist

$$u_{jk}(y), \ v_{jk}(y) \in \mathbb{R}, \ j = \overline{1, m}, \ k = \overline{0, j-1}, \ \forall y \in V,$$

such that $\alpha_j(n)$ and $\gamma_j(n-1), \ j = \overline{1, m}, \ \forall n \in \mathbb{Z}$, can be written in the form

$$\alpha_j(n) = \sum_{k=0}^{j-1} \sum_{y \in V} u_{j,k}(y) P_{(y)}(n-1-k), \tag{6.41}$$

$$\gamma_j(n-1)=\sum_{k=0}^{j-1}\sum_{y\in V}v_{j,k}(y)P_{(y)}(n-1-k).$$

For $n<m-1$ these relations are obvious and they are true for $\forall u_{jk}(y),\, v_{jk}(y)\in\mathbb{R}$. Let us prove them for $n\geq m$ using mathematical induction on j. In this way, we obtain the formulas for the coefficients

$$u_{jk}(y),\ v_{jk}(y),\ j=\overline{1,m},\ k=\overline{0,j-1},\ \forall y\in V.$$

For $j=1$ we have the formulas

$$\alpha_1(n)=\sum_{y\in V}\psi_1(y)P_{(y)}(n-1)=\sum_{k=0}^{0}\sum_{y\in V}u_{10}(y)P_{(y)}(n-1-k),$$

$$\gamma_1(n-1)=a_{n-1}=\sum_{y\in V}P_{(y)}(n-1)=\sum_{k=0}^{0}\sum_{y\in V}v_{10}(y)P_{(y)}(n-1-k),$$

where

$$u_{10}(y)=\psi_1(y),\ v_{10}(y)=1,\ \forall y\in V. \tag{6.42}$$

For $j=2$ we obtain the relations

$$\alpha_2(n)=\sum_{y\neq x_2}p(x_1,y)P_{(y)}(n-1)=\sum_{k=0}^{1}\sum_{y\in V}u_{2k}(y)P_{(y)}(n-1-k),$$

$$\gamma_2(n-1)=P_{(x_2)}(n-1)=\sum_{k=0}^{1}\sum_{y\in V}v_{2k}(y)P_{(y)}(n-1-k),$$

where

$$\begin{cases}u_{20}(x_2)=0, & u_{20}(y)=p(x_1,y),\ \forall y\neq x_2, & u_{21}(y)=0,\ \forall y\in V\\ v_{20}(x_2)=1, & v_{20}(y)=0,\ \forall y\neq x_2, & v_{21}(y)=0,\ \forall y\in V\end{cases}. \tag{6.43}$$

Thus, the relation is true for $\forall j\in\{1,2\}$. Let it be true for $j=\overline{1,s-1},\ s\geq 3,\ \forall n<r$. We have

$$\alpha_s(r)=\pi_{1,2}(\gamma_{s-1}(r-1)-\gamma_s(r-1))=$$

$$=\pi_{1,2}\left(\sum_{k=0}^{s-2}\sum_{y\in V}v_{s-1,k}(y)P_{(y)}(r-1-k)-\sum_{k=0}^{s-1}\sum_{y\in V}v_{sk}(y)P_{(y)}(r-1-k)\right)=$$

$$= \sum_{k=0}^{s-1} \sum_{y \in V} u_{sk}(y) P_{(y)}(r-1-k),$$

where

$$\begin{cases} u_{sk}(y) = \pi_{1,2}(v_{s-1,k}(y) - v_{sk}(y)), k = \overline{0, s-2}, \\ u_{s,s-1}(y) = -p(x_1, x_2) v_{s,s-1}(y), \end{cases}, \ \forall y \in V. \tag{6.44}$$

If $t_s \le s$, then

$$\gamma_s(r-1) = \omega_{t_s} \left(\sum_{y \in V} P_{(y)}(r - t_s + 1) - \sum_{j=1}^{s+1-t_s} \alpha_j (r - t_s + 1) \right) =$$

$$= \omega_{t_s} \left(\sum_{y \in V} P_{(y)}(r - t_s + 1) - \sum_{j=1}^{s+1-t_s} \sum_{k=0}^{j-1} \sum_{y \in V} u_{jk}(y) P_{(y)}(r - t_s - k) \right) =$$

$$= \omega_{t_s} \left(\sum_{y \in V} P_{(y)}(r - 1 - (t_s - 2)) - \sum_{k=t_s-1}^{s-1} \sum_{y \in V} P_{(y)}(r - 1 - k) \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,k-t_s+1}(y) \right) =$$

$$= \sum_{k=0}^{s-1} \sum_{y \in V} v_{s,k}(y) P_{(y)}(r-1-k),$$

where

$$\begin{cases} v_{sk}(y) = 0, \ k = \overline{0, t_s - 3}, \quad v_{s,t_s-2}(y) = \omega_{t_s}, \\ v_{sk}(y) = -v_{s,t_s-2}(y) \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,\,k-t_s+1}(y), \ k = \overline{t_s - 1, s-1}, \end{cases} \quad \forall y \in V, \tag{6.45}$$

and, in the case $t_s = s+1$, we obtain

$$\gamma_s(r-1) = \sum_{y \in V} \omega_s p(x_s, y) P_{(y)}(r - 1 - (t_s - 2)) = \sum_{k=0}^{s-1} \sum_{y \in V} v_{s,k}(y) P_{(y)}(r-1-k),$$

where

$$v_{sk}(y) = 0, \ k = \overline{0, s-2}, \ v_{s,s-1}(y) = \omega_s p(x_s, y), \ \forall y \in V. \tag{6.46}$$

In this way, we have proved the truth of the relation (6.41), obtaining the formulas (6.42) − (6.46) for finding the coefficients of the decomposition. Substituting (6.41) in (6.39) and (6.40), we have

$$P_{(x_1)}(n) = \sum_{y\in V} p(x_1,y)P_{(y)}(n-1) - \pi_{1,2}\gamma_m(n-1) =$$

$$= \sum_{y\in V} p(x_1,y)P_{(y)}(n-1) - \pi_{1,2}\sum_{k=0}^{m-1}\sum_{y\in V} v_{m,k}(y)P_{(y)}(n-1-k) =$$

$$= \sum_{k=0}^{m-1}\sum_{y\in V} w_k(x_1,y)P_{(y)}(n-1-k),$$

$$P_{(x)}(n) = \sum_{y\in V} p(x,y)P_{(y)}(n-1) = \sum_{k=0}^{m-1}\sum_{y\in V} w_k(x,y)P_{(y)}(n-1-k),\ \forall x \neq x_1,$$

where

$$\begin{cases} w_0(x_1,y) = p(x_1,y) - \pi_{1,2}v_{m,0}(y), \\ w_k(x_1,y) = -p(x_1,x_2)v_{m,k}(y),\ k = \overline{1,m-1}, \\ w_0(x,y) = p(x,y),\ w_k(x,y) = 0,\ k = \overline{1,m-1},\ \forall x \neq x_1, \end{cases} \qquad \forall y \in V. \qquad (6.47)$$

Therefore, we have the recurrence formula

$$P_{(x)}(n) = \sum_{k=0}^{m-1}\sum_{y\in V} w_k(x,y)P_{(y)}(n-1-k),\ \forall x \in V,\ \forall n \geq m.$$

By denoting

$$h_n = (P_{(x)}(n))_{x\in V}^T,\ \forall n \in \mathbb{Z},$$

$$W^{(k)} = (w_k(x,y))_{x,y\in V},\ k = \overline{0,m-1},$$

$$W = (W^{(k)})_{k=0}^{m-1},$$

the given formula takes the form

$$h_n = \sum_{k=0}^{m-1} W^{(k)} h_{n-1-k},\ \forall n \geq m.$$

This relation can be presented in the following way:

$$h = (h_n)_{n=0}^{\infty} \in Rol^*[M_\omega(\mathbb{R})][m],$$

$$W \in G^*[M_\omega(\mathbb{R})][m](h).$$

From Theorem 1.28 and from the structure of the matrices $W^{(k)}$, we obtain

$$h \in Rol^*[\mathbb{R}][m\omega]$$

with the characteristic polynomial

$$H(z) = \mid I_\omega - zG_m^{[W]}(z) \mid \in H^*[\mathbb{R}][m\omega](h)$$

of degree

$$\deg(H(z)) \le 2m + \omega - 2.$$

This implies

$$\begin{cases} h(x) = (h_n(x))_{n=0}^{\infty} = (P_{(x)}(n))_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m\omega], \\ H(z) = \mid I_\omega - zG_m^{[W]}(z) \mid \in H^*[\mathbb{R}][m\omega](h(x)), \end{cases}, \ \forall x \in V$$

and, since

$$a_n = \sum_{x \in V} p^*(x) P_{(x)}(n) = \sum_{x \in V} p^*(x) h_n(x), \ \forall n \in \mathbb{N}, \tag{6.48}$$

we finally obtain

$$a \in Rol^*[\mathbb{R}][m\omega], \ H(z) \in H^*[\mathbb{R}][m\omega](a).$$

We use only the relation $a \in Rol^*[\mathbb{C}][m\omega]$. The minimal generating vector can be determined using the rank-based order minimization method, which is also applicable to degenerate homogeneous linear recurrences. Therefore, according to this method, we have that the minimal generating vector

$$q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$$

is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T,$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}), \ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}}, \ \forall n \in \mathbb{N}^*,$$

and R is the rank of the matrix $A_{m\omega}^{[a]}$.

Only the values a_k, $k = \overline{0, 2m\omega - 1}$, are required. They are determined by using the formula (6.48) together with (6.31) − (6.40) and the initial information:

$$a_n = P(n) = P_{(x)}(n) = 0,\ \forall x \in V,\ n = \overline{0, m-2},$$

$$\alpha_k(n) = 0,\ k = \overline{1, m},\ n = \overline{0, m-1},$$

$$P(m-1) = \pi_{1,2}\omega_m,\ a_{m-1} = \pi_1 P(m-1),$$

$$P_{(x_1)}(m-1) = P(m-1),\ P_{(x)}(m-1) = 0,\ \forall x \in V \setminus \{x_1\}. \tag{6.49}$$

In the case $m = 1$, the only difference lies in the formulas for determining the probabilities $P_{(x)}(n)$, $\forall x \in V$, $\forall n \in \mathbb{N}$. It is easy to observe that these values can be computed with the help of formulas:

$$P_{(x_1)}(0) = 1,\ P_{(x_1)}(n) = 0,\ \forall n \in \mathbb{N}^*;$$

$$P_{(x)}(0) = 0,\ P_{(x)}(n) = \sum_{y \in V} p(x, y) P_{(y)}(n-1),\ \forall n \in \mathbb{N}^*,\ \forall x \in V \setminus \{x_1\}. \tag{6.50}$$

In this way, we have established theoretically the following algorithm.

Algorithm 6.4

Input: the initial distribution P^*, the transition probability matrix P and the final sequence X;

Output: $rep(T)$, $\mathbb{E}(T)$, $\mathbb{V}(T)$, $\sigma(T)$, $\nu_k(T)$, $k = \overline{1, r}$, $r \geq 2$;

1) *If $m > 1$, go to the next step. Otherwise, in the case when $m = 1$, execute the below instructions for each $n = \overline{0, 2m\omega - 1}$:*
   a) *compute $P_{(x)}(n)$, $\forall x \in V$, according to (6.50);*
   b) *determine the value a_n using the formula (6.48);*

   and go to step 9) of the algorithm;
2) *If $\exists j \in \{2, 3, \ldots, m\}$ such that $\pi_{j-1,j} = 0$, then the probabilistic characteristics of the evolution time are infinite and the system does not reach the final sequence of states. In this case, stop the algorithm; otherwise, proceed to the next step;*
3) *Determine ω_k, $k = \overline{1, m}$, by using the formula (6.28);*
4) *Obtain the values $\psi_1(y)$, $\forall y \in V$, by applying the relation (6.35);*

5) *Determine the values* t_s, $s=\overline{1,m}$, *according to the relation (6.29);*

6) *Find* a_n, $P(n)$, $\alpha_k(n)$, $P_{(x)}(n)$, $k=\overline{1,m}$, $n=\overline{0,m-1}$, $\forall x\in V$, *by using (6.49);*

7) *Compute* $\gamma_s(m-1)$, $s=\overline{1,m}$, *by using the relations (6.31) − (6.33);*

8) *For each* $n=\overline{m,2m\omega-1}$ *execute the steps:*

   a) *Determine* $\alpha_k(n)$, $k=\overline{1,m}$, *based on (6.34), (6.36) and (6.37);*

   b) *Obtain the value* $P(n)$ *by applying the relation (6.38);*

   c) *For* $\forall x\in V$, *calculate the value* $P_{(x)}(n)$ *according to (6.39) − (6.40);*

   d) *Find the values* $\gamma_s(n)$, $s=\overline{1,m}$, *by using the relations (6.31) − (6.33);*

   e) *Determine the value* a_n *according to the formula (6.48);*

9) *Build the matrix* $A_{m\omega}^{[a]}=(a_{i+j})_{i,j=\overline{0,m\omega-1}}$ *and find its rank* R;

10) *Build the matrix* $A_R^{[a]}=(a_{i+j})_{i,j=\overline{0,R-1}}$ *and the vector* $f_R^{[a]}=(a_k)_{k=R}^{2R-1}$. *Find the solution* $x=(q_{R-1},q_{R-2},\ldots,q_0)$ *of the system* $A_R^{[a]}x^T=(f_R^{[a]})^T$;

11) *Find the initial state* $I_R^{[a]}=(a_n)_{n=0}^{R-1}$ *and determine the minimal generating vector* $q=(q_k)_{k=0}^{R-1}$ *by reversing the components of* x, *obtained at the previous step;*

12) *Determine the probabilistic characteristics of the evolution time with the help of Algorithm 4.4.*

6.6 Stochastic systems with a final sequence of states and rational transition time

Consider the stochastic system L with a finite set of states $V=\{v_1,v_2,\ldots,v_\omega\}$, the initial distribution $P^*=(p^*(v_j))_{j=1}^{\omega}$ and the transition probability matrix $P=(p(u,v))_{u,v\in V}$. The transition time of the system from the state u to the state v is a rational number $\theta(u,v)$, $\forall u,v\in V$. The matrix $\Theta=(\theta(u,v))_{u,\,v\in V}$ represents the matrix of transition times between the states of the system. The system stops when it

consecutively passes through a given sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$. We consider the problem of finding the main probabilistic characteristics of the evolution time T.

It is easy to observe that this problem can be reduced to the case of discrete stochastic systems with nonnegative integer transition time by changing the unit of measurement. As the unit of measurement, we take the least common multiple μ of all denominators of the entries of the matrix Θ. The matrix $\Theta^* = \mu\Theta$ represents the transition-time matrix of the new system.

Furthermore, using the extended network method, the obtained problem is reduced to the case of discrete stochastic systems with unit transition time. This case was solved in Section 6.5, where a polynomial algorithm for describing the evolution time was proposed.

The extended network is built in the following way. For each state $v \in V$, we choose a state $u \in V$, such that

$$\theta(u,v) = \max_{z \in V^-(v)} \theta(z,v), \quad V^-(v) = \{z \in V \mid p(z,v) > 0\}.$$

We split the transition (u,v) into $\theta(u,v)$ unit transitions with the help of a sequence of intermediate states:

$$z_0(u,v) = u, \quad z_1(u,v), \quad z_2(u,v), \quad \ldots, \quad z_{\theta(u,\,v)-1}(u,v), \quad z_{\theta(u,\,v)}(u,v) = v.$$

The system starts its evolution from intermediate states

$$z_j(u,v), \quad j = \overline{1, \theta(u,v)-1},$$

with probability zero. If $\theta(u,v) > 1$, then the transition matrix P has the following transformations:

- *Add a new zero row and column for each intermediate state* $z_j(u,v), \quad j = \overline{1, \theta(u,v)-1}$;
- *Set* $p(z_j(u,v), z_{j+1}(u,v)) = 1, \quad j = \overline{1, \theta(u,v)-1}$;
- *For each* $z \in V^-(v)$ *with* $\theta(z,v) > 1$, *fix*

$$p(z, z_{\theta(u,v)-\theta(z,v)+1}(u,v)) = p(z,v).$$

After that, set $p(z,v) = 0$.

The final sequence of the new system is

$$X^* = (x_1, ((z_j(x_i, x_{i+1}))_{j=1}^{\theta(x_i,x_{i+1})-1}, x_{i+1})_{i=1}^{m-1}).$$

The solution of the initial problem is determined from the solution of the reduced problem, taking into account the multiplication factor μ. Therefore, if the evolution time in the initial problem is T and the evolution time in the reduced problem is T^*, then $T = T^*/\mu$, hence the moment of order n, the expectation, the variance and the distribution law are:

$$\nu_n(T) = \nu_n(T^*)/\mu^n,$$

$$\mathbb{E}(T) = \mathbb{E}(T^*)/\mu, \quad \mathbb{V}(T) = \mathbb{V}(T^*)/\mu^2,$$

$$\mathbb{P}(T = k/\mu) = \mathbb{P}(T^* = k), \ k = \overline{0,\infty}.$$

Example 6.4 *Consider a stochastic system* L *with the set of states* $V = \{e_1, e_2, e_3, e_4\}$ *and the final sequence of states* $X = (e_2, e_3, e_4)$. *Let* $P^* = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{8}, \frac{3}{8}\right)$ *be the initial distribution of the states. The transition probability and transition time matrices are:*

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{2}{3} & 0 \end{pmatrix}.$$

Build the extended network which reduces the stochastic system L *to unit transition time case.*

Solution. We apply the extended network method developed above. We can represent the stochastic system L graphically as the following dynamic network:

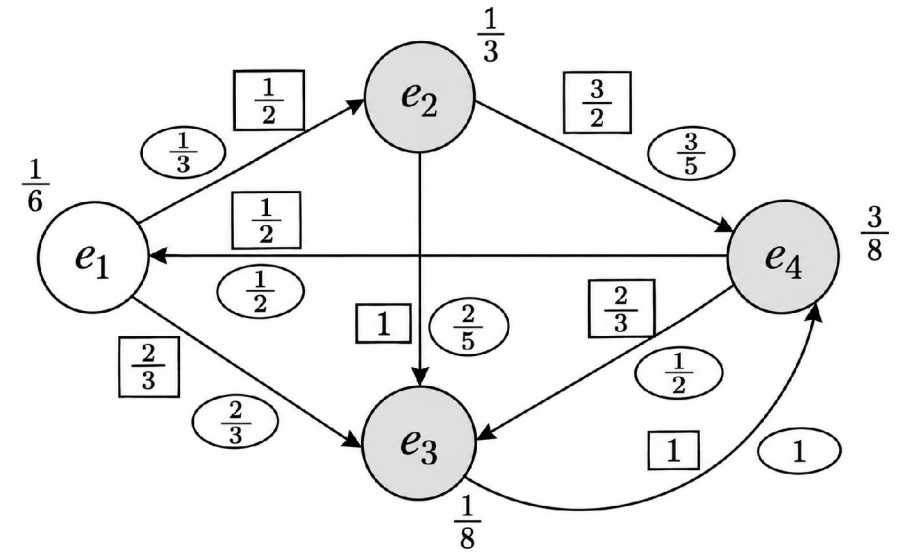

Figure 6.1 Dynamic network corresponding to the stochastic system L

By reducing the system L to the case of nonnegative integer transition time, we have:

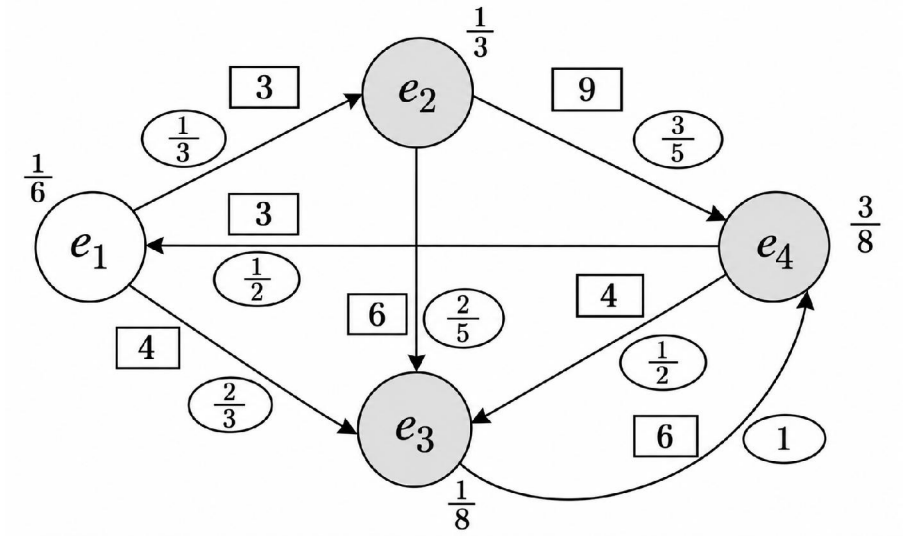

Figure 6.2 Reduced dynamic network (with positive integer transition time)

Therefore, we obtain the system L^* with the same set of states, final sequence of states, initial distribution of the states and transition probability matrix, but a different transition-time matrix (with nonnegative integer entries):

$$\Theta^* = 6\Theta = \begin{pmatrix} 0 & 3 & 4 & 0 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 6 \\ 3 & 0 & 4 & 0 \end{pmatrix}.$$

Finally, we construct the extended network:

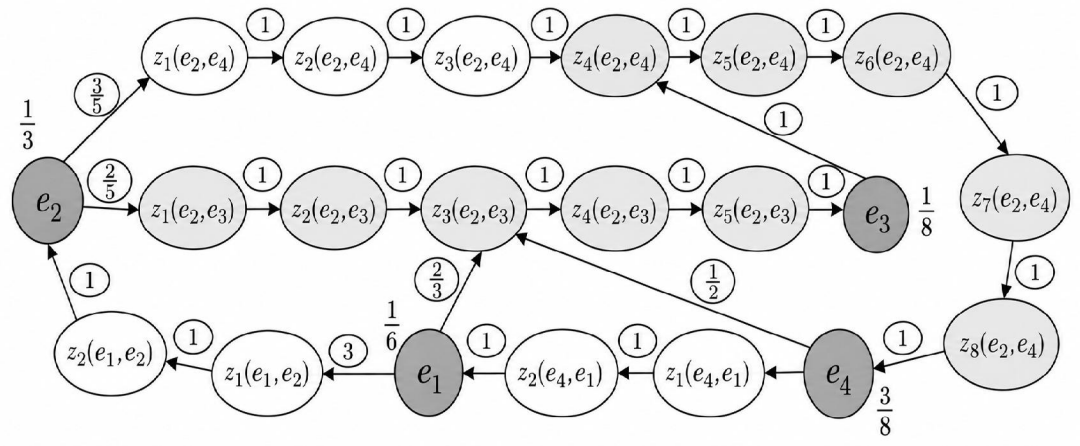

Figure 6.3 Extended network method (with unit transition time)

The final sequence of states of the new system is

$$X^* = (e_2, z_1(e_2,e_3), z_2(e_2,e_3), z_3(e_2,e_3), z_4(e_2,e_3), z_5(e_2,e_3), e_3, \\ z_4(e_2,e_4), z_5(e_2,e_4), z_6(e_2,e_4), z_7(e_2,e_4), z_8(e_2,e_4), e_4).$$ □

6.7 Systems with a final sequence of states and nonnegative real transition costs

Consider the system L with the set of states V, where $|V| = \omega < \infty$. The state of the stochastic system at each discrete time $t \in \mathbb{N}$ is $v(t) \in V$.

Let $p^*(v)$ be the probability that $v(0) = v$, for $\forall v \in V$. The transition probability and the transition cost from the state $u \in V$ to the state $v \in V$ at each discrete time t are $p(u,v)$ and $c(u,v) \in \mathbb{R}$.

The stochastic system completes its evolution when it consecutively passes through a given sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m .$$

Let T be its evolution time. Our aim is to estimate the moment of order k of the total cost C accumulated by the system in the interval of time $[0,T]$, for each $k = \overline{1,r}$.

Therefore, we consider a stochastic system L with the final sequence of states $X \in V^m$ and nonnegative real transition costs $\theta(u,v)$, $u,v \in V$ ([96]). We associate with this system a stochastic system L^* with the same set of states V, final sequence of states X, initial distribution of the states $(p^*(v))_{v \in V}$ and transition probabilities $(p(u,v))_{u,v \in V}$. The transition time of the system L^* is defined by the formula

$$\theta(u,v) = c(u,v), \ \forall u,v \in V .$$

The stochastic system L^* is a system without transition costs.

Let T^* be the evolution time of the system L^* and C the total cost of the system L. It is easy to observe that these random variables have the same distribution law. Therefore, the problem of determining the moment of order n of the total cost C of the system L is equivalent to the problem of computing the moment of order n of the evolution time T^* of the system L^*.

The particular case $\theta(u,v) \in \mathbb{Q}^+$ was studied in Section 6.6. A polynomial method for determining the distribution law and the moments of the evolution time of stochastic systems with a final sequence of states and nonnegative rational transition time was developed. The main idea was the extended network method, which reduces the case of nonnegative rational transition time to the case of nonnegative integer transition time by changing the unit of the measurement and, further, reduces the obtained problem to the case of unit transition time. This simplified case was studied in Section 6.5.

Next, we consider the case $\theta(u,v) \in \mathbb{R}^+$, i.e. the problem of finding the moment $\nu_k(T^*)$ of the evolution time T^*, $k = \overline{1,r}$, of the stochastic system L^* with the final sequence of states X and nonnegative real transition time, where r is fixed. We approximate this system with the system $L^*_{\varepsilon^*}$ with the same set of states V, final sequence of states X, initial distribution $(p^*(v))_{v \in V}$ of the states and transition probabilities $(p(u,v))_{u,v \in V}$. The transition time of the system $L^*_{\varepsilon^*}$ is defined using the formula

$$\theta^*(u,v) = \theta(u,v) + \varepsilon(u,v) \in \mathbb{Q}^+, \tag{6.51}$$

where

$$0 \le \varepsilon(u,v) \le \varepsilon^* \le 1, \ \forall u,v \in V \tag{6.52}$$

and ε^* is fixed. Let $T^*_{\varepsilon^*}$ be the evolution time of the stochastic system $L^*_{\varepsilon^*}$. Our goal is to choose $\varepsilon^* \in (0,1]$ such that

$$|\nu_k(T^*_{\varepsilon^*}) - \nu_k(T^*)| \le \varepsilon, \ k = \overline{1,r}, \tag{6.53}$$

where $\varepsilon > 0$ represents the error tolerance for computing the values $\nu_k(T^*)$, $k = \overline{1,r}$.

We associate with the system $L^*_{\varepsilon^*}$ a stochastic system L_1 having the same set of states V, final sequence of states X, initial distribution $(p^*(v))_{v \in V}$ of the states, and transition probabilities $(p(u,v))_{u,v \in V}$. The transition time of the associated system L_1 is one. Let T_1 be the evolution time of the associated system L_1. We observe that

$$T^* \le T^*_{\varepsilon^*} \le T^* + T_1 \varepsilon^*,$$

hence

$$\nu_k(T^*) \le \nu_k(T^*_{\varepsilon *}) \le \nu_k(T^* + T_1\varepsilon^*),\ k = \overline{1,r}. \tag{6.54}$$

By applying the Minkowski inequality (described in [51]), we have

$$(\nu_k(T^* + T_1\varepsilon^*))^{1/k} \le (\nu_k(T^*))^{1/k} + \varepsilon^*(\nu_k(T_1))^{1/k},\ k = \overline{1,r}. \tag{6.55}$$

From the relations (6.54) and (6.55) we obtain

$$(\nu_k(T^*))^{1/k} \le (\nu_k(T^*_{\varepsilon *}))^{1/k} \le (\nu_k(T^*))^{1/k} + \varepsilon^*(\nu_k(T_1))^{1/k},\ k = \overline{1,r},$$

which implies

$$0 \le (\nu_k(T^*_{\varepsilon *}))^{1/k} - (\nu_k(T^*))^{1/k} \le \varepsilon^*(\nu_k(T_1))^{1/k},\ k = \overline{1,r}. \tag{6.56}$$

Next, we consider the norm

$$\|\Theta\| = \max_{u,v \in V} \theta(u,v)$$

of the transition-time matrix $\Theta = (\theta(u,v))_{u,v \in V}$. From (6.51) and (6.52), we obtain

$$\theta^*(u,v) \le \|\Theta\| + 1,\ \forall u,v \in V.$$

It is easy to observe that

$$T^*_{\varepsilon *} \le (\|\Theta\| + 1)T_1,$$

which implies the inequality

$$(\nu_k(T^*_{\varepsilon *}))^{1-1/k} \le (\|\Theta\| + 1)^{k-1}(\nu_k(T_1))^{1-1/k},\ k = \overline{1,r}. \tag{6.57}$$

Using the relations (6.54), (6.55) and (6.56), we have

$$0 \le \nu_k(T^*_{\varepsilon *}) - \nu_k(T^*) =$$

$$= ((\nu_k(T^*_{\varepsilon *}))^{1/k} - (\nu_k(T^*))^{1/k}) \sum_{j=0}^{k-1} (\nu_k(T^*_{\varepsilon *}))^{(k-1-j)/k} (\nu_k(T^*))^{j/k} \le$$

$$\le \varepsilon^*(\nu_k(T_1))^{1/k} \sum_{j=0}^{k-1} (\nu_k(T^*_{\varepsilon *}))^{1-1/k} = \varepsilon^* k(\nu_k(T_1))^{1/k} (\nu_k(T^*_{\varepsilon *}))^{1-1/k} \le$$

$$\le \varepsilon^* k(\nu_k(T_1))^{1/k} (\|\Theta\| + 1)^{k-1} (\nu_k(T_1))^{1-1/k} = k(\|\Theta\| + 1)^{k-1} \nu_k(T_1)\varepsilon^*.$$

Therefore, if we choose $\varepsilon^* \in (0,1]$ such that

$$k(\|\Theta\|+1)^{k-1}\nu_k(T_1)\varepsilon^* \le \varepsilon,\ k=\overline{1,r},$$

then the relation (6.53) holds. It is necessary to have

$$0<\varepsilon^* \le \frac{\varepsilon}{k(\|\Theta\|+1)^{k-1}\nu_k(T_1)},\ k=\overline{1,r}.$$

By applying Lyapunov inequality ([51], [142]), we obtain the equivalent inequality:

$$0<\varepsilon^* \le \min_{k=1,r}\frac{\varepsilon}{k(\|\Theta\|+1)^{k-1}\nu_k(T_1)}=$$

$$=\frac{\varepsilon}{\max\limits_{k=1,r}(k(\|\Theta\|+1)^{k-1}\nu_k(T_1))}=\frac{\varepsilon}{r(\|\Theta\|+1)^{r-1}\nu_r(T_1)}.$$

Consider an arbitrary nonnegative real value ε^* in the interval

$$\left(0,\min\left\{1,\frac{\varepsilon}{r(\|\Theta\|+1)^{r-1}\nu_r(T_1)}\right\}\right].$$

If a nonnegative integer s satisfies the inequality $\varepsilon^* \ge 10^{-s}$, then we can choose

$$\varepsilon(u,v)=\frac{1-\{10^s\theta(u,v)\}}{10^s},\ \forall u,v\in V$$

and these values satisfy the relations (6.51), (6.52) and (6.53), where we denoted by $\{a\}$ the fractional part of an arbitrary real number a. From (6.51), we have

$$\theta^*(u,v)=\frac{1+\lfloor 10^s\theta(u,v)\rfloor}{10^s},\ \forall u,v\in V,$$

where $\lfloor a\rfloor$ is the integer part of an arbitrary real number a.

Therefore, under these circumstances, we can consider

$$\nu_k(C)=\nu_k(T^*)\approx\nu_k(T^*_{\varepsilon *}),\ k=\overline{1,r},$$

with the error tolerance ε. We observe that

$$\lim_{\varepsilon\to 0}\nu_k(T^*_{\varepsilon *})=\nu_k(C).$$

In this way, we have established the following algorithm for estimating the first r moments of the total cost C.

Algorithm 6.5

Input: V; $(x_j)_{j=1}^m$; ε; r; $p^*(v)$, $p(u,v)$, $c(u,v)$, $\forall u,v \in V$;

Output: $\nu_k(C)$, $k=\overline{1,r}$.

1) *Obtain the moment* $\nu_r(T_1)$ *, by using Algorithm 6.4;*

2) *Determine the norm* $\|\Theta\| = \max\limits_{u,v\in V} c(u,v)$;

3) *Compute the value* $\varepsilon^*_{\max} = \dfrac{\varepsilon}{r(\|\Theta\|+1)^{r-1}\nu_r(T_1)}$;

4) *Choose an arbitrary value* $\varepsilon^* \in (0, \min\{1, \varepsilon^*_{\max}\}]$;

5) *Choose a nonnegative integer* s, *which satisfies the condition* $\varepsilon^* \geq 10^{-s}$;

6) *Obtain the transition costs* $\theta^*(u,v) = \dfrac{1+\lfloor 10^s \theta(u,v)\rfloor}{10^s}$, $\forall u,v \in V$;

7) *Using the extended network method described in Section 6.6, determine the moments* $\nu_k(T^*_{\varepsilon^*})$, $k=\overline{1,r}$, *of the evolution time* $T^*_{\varepsilon^*}$;

8) *Estimate the moments of the total cost* C: $\nu_k(C) \approx \nu_k(T^*_{\varepsilon^*})$, $k=\overline{1,r}$.

7 COMPOSITIONS AND OPTIMIZATION PROBLEMS

In this chapter, we study several generalizations of the stochastic systems with a final sequence of states. We analyze the ordered and unordered sequential compositions and excludable and nonexcludable parallel compositions of stochastic systems with a final sequence of states. For these models we study the main probabilistic characteristics of the evolution time (cost): distribution law, expectation, variance, standard deviation, moments, dominant value and median.

Further, we study the problem of optimization of the evolution time in the case of independent states and in the case of interdependent states. We use the geometric and the signomial programming methods for determining the optimal initial distribution and the optimal transition probability matrix, which minimize the expectation of the evolution time.

7.1 Sequential compositions of stochastic systems

Consider s stochastic systems $L^{[k]}$ with evolution time $T^{[k]}$, $k = \overline{1,s}$.

Definition 7.1 *The system $L_S^{[0]}$, whose evolution is the concatenation of the evolutions of systems $L^{[k]}$, $k = \overline{1,s}$, is called a sequential composition. If the concatenation is performed in a fixed order, then it is an ordered composition, otherwise it is an unordered composition.*

Let S_s be the set of all permutations of degree s. We consider an arbitrary permutation $\delta \in S_s$. Let us assume that the sequential composition $L_S^{[0]}$ passes from the final state of the component system $L^{[\delta(k)]}$ to the initial state of the next component system $L^{[\delta(k+1)]}$ in $\tau_{\delta(k),\,\delta(k+1)}$ time units. Therefore, for the system $L_S^{[0]}$ the matrix $\Lambda = (\tau_{j,k})_{j,k=\overline{1,s}}$ of transition times between intermediate systems is defined.

If the system $L_S^{[0]}$ represents an ordered sequential composition of the systems $L^{[k]}$, $k=\overline{1,s}$, generated by permutation $\delta \in S_s$, then we denote it by

$$L_S^{[0]} = \sum_{k=1}^{s} L^{[\delta(k)]}[\Lambda].$$

Let $\varphi : S_s \to [0,1]$ be a probability function and let $\delta[\varphi]$ be a discrete random variable with distribution φ. If $L_{S,\varphi}^{[0]}$ is an unordered sequential composition of the systems $L^{[k]}$, $k=\overline{1,s}$, generated by distribution φ of ordering, then we denote it by

$$L_{S,\varphi}^{[0]} = \sum_{k=1}^{s} L^{[\delta[\varphi](k)]}[\Lambda].$$

Consider an ordered composition

$$L = \sum_{k=1}^{s} L^{[\delta(k)]}[\Lambda],$$

where $\delta \in S_s$ and $\Lambda = (\tau_{j,k})_{j,k=\overline{1,s}}$. Let T be the evolution time of the composition L and let $T^{[k]}$ be the evolution time of the stochastic system $L^{[k]}$, $k=\overline{1,s}$. Consider that $T^{[k]} < +\infty$, $k=\overline{1,s}$. For the probabilistic characterization of the evolution time T, the following result is required.

Lemma 7.1 *Consider the independent random variables* $\xi^{(j)}$, $j=\overline{1,n}$. *Denote by* $\nu_{kj}^* = \nu_k(\xi^{(j)})$ *the moment of order* k *of the variable* $\xi^{(j)}$, $j=\overline{1,n}$, $\forall k \in \mathbb{N}$. *Then, the moment of order* k *of the variable* $\xi = \sum_{j=1}^{n} \xi^{(j)}$ *is* $\nu_k(\xi) = \nu_{kn}$, *where*

$$\nu_{k1} = \nu_{k1}^*, \ \nu_{kj} = \sum_{i=0}^{k} C_k^i \nu_{i,j-1} \nu_{k-i,j}^*, \ j=\overline{2,n}, \ \forall k \in \mathbb{N}.$$

Proof. Denote $U_j = \sum_{i=1}^{j} \xi^{(i)}$, $j=\overline{1,n}$. We have

$$U_1 = \xi^{(1)}, \ U_j = U_{j-1} + \xi^{(j)}, \ j=\overline{2,n}.$$

Consider $k \geq 0$. It is easy to observe that

$$\nu_k(U_1) = \nu_k(\xi^{(1)})$$

and, for $j = \overline{2,n}$, we have

$$\nu_k(U_j) = \mathbb{E}((U_{j-1} + \xi^{(j)})^k) =$$

$$= \sum_{i=0}^{k} C_k^i \mathbb{E}(U_{j-1}^i)\mathbb{E}((\xi^{(j)})^{k-i}) = \sum_{i=0}^{k} C_k^i \nu_i(U_{j-1})\nu_{k-i}(\xi^{(j)}).$$

Considering $\nu_{kj}^* = \nu_k(\xi^{(j)})$ and $\nu_{kj} = \nu_k(U_j)$, $j = \overline{1,n}$, we obtain the truth of the statement of this lemma. □

It is easy to observe that

$$T = \sum_{k=1}^{s} T^{[k]} + \kappa_\Lambda(\delta),$$

where

$$\kappa_\Lambda(\delta) = \sum_{k=1}^{s-1} \tau_{\delta(k),\delta(k+1)}.$$

The random variables $T^{[k]}$, $k = \overline{1,s}$, are independent and $\kappa_\Lambda(\delta)$ is a constant. Therefore, we can apply Lemma 7.1 for the independent random variables $\xi^{(k)} = T^{[k]}$, $k = \overline{1,s}$ and $\xi^{(s+1)} = \kappa_\Lambda(\delta)$ and obtain the moments $\nu_k(T)$, $k = \overline{0,r}$. The expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$ and the standard deviation $\sigma(T)$ can be determined by using the relation (4.1).

Let us analyze the unordered sequential composition

$$\overline{L} = \sum_{k=1}^{s} L^{[\delta[\varphi](k)]}[\Lambda]$$

with evolution time $\overline{T}$. We have $\overline{T} = T^* + \kappa_\Lambda(\delta[\varphi])$, where $T^* = \sum_{k=1}^{s} T^{[k]}$. Therefore, the moments $\nu_k(T^*)$, $k = \overline{0,r}$, can be determined by using Lemma 7.1 for the independent variables $\xi^{(k)} = T^{[k]}$, $k = \overline{1,s}$. It is easy to observe that the random variable

T^* does not depend on the ordering generated by the random permutation $\delta[\varphi]$. Further, by applying the Newton formula, we have

$$\nu_k(\overline{T}) = \mathbb{E}(\mathbb{E}((T^* + \kappa_\Lambda(\delta[\varphi]))^k \mid \delta[\varphi])) =$$

$$= \sum_{j=0}^{k} C_k^j \nu_{k-j}(T^*)\nu_j(\kappa_\Lambda(\delta[\varphi])),\ k = \overline{0,r},$$

where

$$\nu_j(\kappa_\Lambda(\delta[\varphi])) = \sum_{\delta \in S_s} \varphi(\delta)(\kappa_\Lambda(\delta))^j,\ j = \overline{0,r}.$$

When there exists $j \in \{1,2,\ldots,s\}$ such that $T^{[j]} = +\infty$, then the expectation and the moments of the evolution time T of the given sequential composition are infinite.

Example 7.1 *Consider two systems L_1 and L_2, with a final sequence of states and independent states, with the same set of states $V = \{v_1, v_2\}$ and the same distribution of the states $p^*(v_1) = 0.3$ and $p^*(v_2) = 0.7$. Let $X_1 = (v_1, v_2)$ be the final sequence of states of L_1 and let $X_2 = (v_2, v_1)$ be the final sequence of states of L_2. Consider the sequential composition $L_S = (L_1 + L_2)[\Lambda]$, where $\Lambda = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find the expectation, the variance, the standard deviation and the moments of order 1 and 2 of the evolution times of the systems L_1, L_2 and L_S.*

Solution. Using Algorithm 6.1, we obtain:

System	π_1	π_2	w_2	t_2	$u_{1,0}$	$v_{1,0}$
L_1	0.3	0.7	0.21	3	0.7	1
L_2	0.7	0.3	0.21	3	0.3	1

Table 7.1 Determination of the generating vector

System	$v_{2,0}$	$v_{2,1}$	$u_{2,0}$	$u_{2,1}$	q_0	q_1
L_1	0	0.7	0.3	-0.21	1	-0.21
L_2	0	0.3	0.7	-0.21	1	-0.21

Table 7.2 Determination of the generating vector (part 2)

The evolution times of the systems L_1 and L_2 have the same distribution

$$a \in Rol[\mathbb{R}][2],\quad q = (1,-0.21) \in G[\mathbb{R}][2](a),\quad I_2^{[a]} = (0,0.21).$$

Next, by applying Algorithm 4.4, we obtain

k	*system*	*evol. time*	$\nu_1(T_k)$	$\nu_2(T_k)$	$\mathbb{E}(T_k)$	$\mathbb{V}(T_k)$	$\sigma(T_k)$
1	L_1	T_1	3.761905	22.54195	3.761905	8.39002	2.89655
2	L_2	T_2	3.761905	22.54195	3.761905	8.39002	2.89655

Table 7.3 Determination of the characteristics of the evolution time

For the sequential composition L_S with evolution time T_S we obtain:

k	ν_{k1}^*	ν_{k2}^*	ν_{k3}^*	ν_{k1}	ν_{k2}	ν_{k3}
0	1	1	1	1	1	1
1	3.761905	3.761905	2	3.761905	7.52381	9.52381
2	22.54195	22.54195	4	22.54195	73.38776	107.483

Table 7.4 Characteristics of evolution time of the composition

Therefore, we have the following probabilistic characteristics of the evolution time T_S of the sequential compositions L_S:

system	*evol. time*	$\nu_1(T_S)$	$\nu_2(T_S)$	$\mathbb{E}(T_S)$	$\mathbb{V}(T_S)$	$\sigma(T_S)$
L_S	T_S	9.52381	107.483	9.52381	16.78004	4.09634

Table 7.5 Sequential compositions – final results

7.2 Parallel compositions of zero-order Markov systems with a final sequence of states

Consider s systems $L^{[k]}$ with a final sequence of states and evolution time $T^{[k]}$, $k=\overline{1,s}$. The following definitions were introduced in [88]:

Definition 7.2 *The stochastic system $L_P^{[0]}$, whose evolution consists of simultaneous evolutions of the systems $L^{[k]}$, $k=\overline{1,s}$, is called a parallel composition of the given systems. A parallel composition is called excludable if the completion of every component system immediately stops the evolutions of all component systems. Similarly, a parallel composition is called nonexcludable if the completion of any component system does not stop the evolution of other component systems.*

If L_m represents an excludable parallel composition of the stochastic systems $L^{[k]}$, $k=\overline{1,s}$, then we denote it by

$$L_m = \bigcap_{k=1}^{s} L^{[k]}.$$

Also, if L_M is a nonexcludable parallel composition of the stochastic systems $L^{[k]}$, $k=\overline{1,s}$, then we denote it by

$$L_M = \bigcup_{k=1}^{s} L^{[k]}.$$

Consider a composition L and a nonnegative integer constant θ. We associate with the composition L the system L_θ^* with the final sequence of states

$$X^* = \{x_1^*, x_2^*, \ldots, x_{\theta+1}^*\}$$

and independent states, where the probabilities of the states satisfy the relations

$$p^*(x_1^*)=1,\ p^*(x_j^*)\neq 0,\ j=\overline{1,\theta+1}.$$

Definition 7.3 *The system* L_θ^* *is called a bounded deterministic system with final critical state of order* $\theta+1$ *and independent states (*θ*-IBDSFCS).*

Definition 7.4 *An excludable parallel composition* $L_{m,\theta}=L\cap L_\theta^*$ *is called a composition with upper-bounded evolution and a nonexcludable parallel composition* $L_{M,\theta}=L\cup L_\theta^*$ *is called a composition with lower-bounded evolution.*

Consider the composition

$$L_{m,\theta}=L\cap L_\theta^*$$

with upper-bounded evolution and the composition

$$L_{M,\theta}=L\cup L_\theta^*$$

with lower-bounded evolution. Let T be the evolution time of the system L, $T_{m,\theta}$ be the evolution time of the composition $L_{m,\theta}$ and $T_{M,\theta}$ be the evolution time of the composition $L_{M,\theta}$. We have

$$T_{m,\theta}=\min\{T,\theta\},\ T_{M,\theta}=\max\{T,\theta\}.$$

Consider the distributions

$$a=rep(T),\ b=rep(T_{m,\theta}),\ c=rep(T_{M,\theta}).$$

We obtain the relations:

$$b_n=a_n,\ n=\overline{0,\theta-1};\ b_\theta=1-\sum_{n=0}^{\theta-1}a_n;\ b_n=0,\ \forall n>\theta;$$

$$c_n=0,\ n=\overline{0,\theta-1};\ c_\theta=\sum_{n=0}^{\theta}a_n;\ c_n=a_n,\ \forall n>\theta.$$

The moments of order $k\geq 1$ of the evolution time $T_{m,\theta}$ and $T_{M,\theta}$ are:

$$\nu_k(T_{m,\theta})=\sum_{n=0}^{\infty}n^k b_n=\sum_{n=0}^{\theta-1}n^k a_n+\theta^k\left(1-\sum_{n=0}^{\theta-1}a_n\right)=\theta^k+\sum_{n=0}^{\theta-1}(n^k-\theta^k)a_n;$$

$$\nu_k(T_{M,\theta}) = \sum_{n=0}^{\infty} n^k c_n = \theta^k \sum_{n=0}^{\theta} a_n + \sum_{n=0}^{\infty} n^k a_n - \sum_{n=0}^{\theta} n^k a_n = \nu_k(T) - \sum_{n=0}^{\theta} (n^k - \theta^k) a_n .$$

In this way, we have obtained efficient formulas for determining the moments of the evolution time of the compositions with bounded evolution.

Further, we consider the excludable parallel composition

$$L_m = \bigcap_{k=1}^{s} L^{[k]}$$

and the nonexcludable parallel composition

$$L_M = \bigcup_{k=1}^{s} L^{[k]},$$

where the system $L^{[k]}$ is not an IBDSFCS, $k = \overline{1,s}$. Let T_m be the evolution time of the excludable composition L_m, T_M be the evolution time of the nonexcludable composition L_M and $T^{[j]}$ be the evolution time of the component system $L^{[j]}$, $j = \overline{1,s}$. Assume that $T^{[k]} < +\infty$, $k = \overline{1,s}$. We observe that

$$T_m = \min_{j=1,s} T^{[j]}, \quad T_M = \max_{j=1,s} T^{[j]}.$$

Consider the distributions

$$a^{(j)} = rep(T^{[j]}), \quad j = \overline{1,s},$$

$$a^* = rep(T_m), \quad A^* = rep(T_M).$$

According to Section 6.2, we have that there exists $M_j \in \mathbb{N}^*$ such that

$$a^{(j)} \in Rol[\mathbb{R}][M_j], \quad j = \overline{1,s}.$$

Let us introduce the notation

$$F_n = P(T_m < n), \quad G_n = P(T_M < n),$$

$$F_n^{(j)} = P(T^{[j]} < n), \quad j = \overline{1,s}, \quad n = \overline{0,\infty}.$$

If we consider the sequences

$$F = (F_n)_{n=0}^{\infty}, \quad G = (G_n)_{n=0}^{\infty},$$

$$F^{(j)} = (F_n^{(j)})_{n=0}^{\infty},\ j = \overline{1,s},$$

we have

$$F_n = P(T_m < n) = 1 - P(T^{[j]} \geq n,\ j = \overline{1,s}) =$$

$$= 1 - \prod_{j=1}^{s} P(T^{[j]} \geq n) = 1 - \prod_{j=1}^{s} (1 - F_n^{(j)}),$$

$$G_n = P(T_M \leq n-1) = P(T^{[j]} \leq n-1,\ j = \overline{1,s}) =$$

$$= \prod_{j=1}^{s} P(T^{[j]} \leq n-1) = \prod_{j=1}^{s} F_n^{(j)},$$

$$a_n^* = F_{n+1} - F_n,\ A_n^* = G_{n+1} - G_n,$$

$$F_n^{(j)} = G_n^{[a^{(j)}]}(1) = \sum_{i=0}^{n-1} a_i^{(j)},\ j = \overline{1,s},\ n = \overline{0,\infty}.$$

Let

$$M^* = \prod_{j=1}^{s} (M_j + 1),\ r^* = \prod_{j=1}^{s} (M_j + 2),\ m^* = r^* + 1.$$

According to Theorem 1.6, we obtain

$$F^{(j)} \in Rol[\mathbb{R}][M_j + 1],\ j = \overline{1,s}.$$

Further, by applying Theorem 2.3, we obtain $G \in Rol[\mathbb{R}][M^*]$. Since $\overline{G} = (G_{n+1})_{n=0}^{\infty}$ is a subsequence of the sequence G, we have $\overline{G} \in Rol[\mathbb{R}][M^*]$ with the same generating vector. From Theorem 1.7, we obtain

$$A^* = \overline{G} - G \in Rol[\mathbb{R}][M^*].$$

By applying Theorem 2.2 and Theorem 2.3, we obtain $F \in Rol[\mathbb{R}][m^*]$. Since the subsequence $\overline{F} = (F_{n+1})_{n=0}^{\infty}$ represents a subsequence of F, we have $\overline{F} \in Rol[\mathbb{R}][m^*]$. Using Theorem 1.7, we obtain

$$a^* = \overline{F} - F \in Rol[\mathbb{R}][m^*].$$

The values

$$a_k^*,\ k=\overline{0,2m^*-1},\ \ A_l^*,\ l=\overline{0,2M^*-1},$$

are obtained from the formulas above. By applying an order minimization method, we can compute the dimensions

$$d^*=dim[\mathbb{R}](a^*),\ D^*=dim[\mathbb{R}](A^*)$$

and the minimal generating vectors

$$q^*\in G[\mathbb{R}][d^*](a^*),\ \ Q^*\in G[\mathbb{R}][D^*](A^*).$$

Further, we can apply Algorithm 4.3 or Algorithm 4.4 to determine the main probabilistic characteristics of the evolution time T_m and T_M.

When there exists $j\in\{1,2,\ldots,s\}$ such that the system $L^{[j]}$ is an IBDSFCS, the main probabilistic characteristics of the evolution time of the parallel compositions L_m and L_M are determined in the following way.

Consider the partition $\{C^{(1)},C^{(2)}\}$ of the set $\{L^{[k]},k=\overline{1,s}\}$, where

$$C^{(1)}=\{L^{[k]},k=\overline{1,s}\mid L^{[k]}\ is\ not\ an\ IBDSFCS\},$$

$$C^{(2)}=\{L^{[k]},k=\overline{1,s}\mid L^{[k]}\ is\ an\ IBDSFCS\}.$$

For each $C\in C^{(2)}$, let C be a $\theta(C)-$IBDSFCS. If

$$\theta_m=\min_{C\in C^{(2)}}\theta(C),\ \theta_M=\max_{C\in C^{(2)}}\theta(C),$$

then we consider the arbitrary IBDSFCS

$$C_m\in C^{(2)},\ C_M\in C^{(2)}$$

which satisfy the relations

$$\theta(C_m)=\theta_m,\ \theta(C_M)=\theta_M.$$

Let

$$\Lambda_m=\bigcap_{C\in C^{(1)}}C,\ \Lambda_M=\bigcup_{C\in C^{(1)}}C.$$

We have

$$L_m = \Lambda_m \cap C_m,\ L_M = \Lambda_M \cup C_M .$$

Therefore, L_m is a composition with upper-bounded evolution and the system L_M is a composition with lower-bounded evolution. The probabilistic characteristics of the evolution time of parallel compositions Λ_m and Λ_M can be determined by using the method described above. A method was also developed in this section for determining the probabilistic characteristics of the evolution time of the compositions L_m and L_M with bounded evolution.

If there exists $j \in \{1,2,\ldots,s\}$ such that $T^{[j]} = +\infty$, then the expectation and the moments of T_M are infinite and the following formula holds:

$$L_m = \bigcap_{k \in \{1,2,\ldots,s\} \setminus \{j\}} L^{[k]}.$$

Example 7.2 *Consider two stochastic systems L_1 and L_2 with a final sequence of states and independent states, with the same set of states $V = \{v_1, v_2\}$, the same distribution of the states*

$$p^*(v_1) = 0.3,\ p^*(v_2) = 0.7 .$$

Let $X_1 = (v_1, v_2)$ be the final sequence of states of the system L_1 and let $X_2 = (v_2, v_1)$ be the final sequence of states of the system L_2. Consider the parallel compositions

$$L_m = L_1 \cap L_2,\ L_M = L_1 \cup L_2 .$$

Determine the expectation, the variance, the standard deviation and the moments of order 1 and 2 of the evolution time of L_m and L_M.

Solution. The probabilistic characteristics of the evolution time of stochastic systems L_1 and L_2 were determined in Example 7.1. Referring to excludable parallel composition L_m with evolution time T_m and nonexcludable parallel composition L_M with evolution time T_M, by applying the methods developed above, we obtain:

n	a_n^*	A_n^*
0	0.000000000000	0.000000000000
1	0.375900000000	0.044100000000
2	0.287700000000	0.132300000000
3	0.164921190000	0.166878810000
4	0.086039520000	0.157560480000
5	0.043275185079	0.130646814921
6	0.021440774985	0.101325225015
7	0.010555671028	0.075686708972
8	0.005182727031	0.055278792969
9	0.002541731548	0.039808888652
10	0.001245909578	0.028407791422
11	0.000610592538	0.020149478220
12	0.000299210682	0.014233582866
13	0.000146617505	0.010026561184
14	0.000071843474	0.007049448569
15	0.000035203491	0.004949721028
16	0.000017249750	0.003472203440
17	0.000008452386	0.002434166655
18	0.000004141671	
19	0.000002029419	
20	0.000000994415	
21	0.000000487264	
22	0.000000238759	
23	0.000000116992	
24	0.000000057326	
25	0.000000028090	
26	0.000000013764	
27	0.000000006744	
28	0.000000003305	
29	0.000000001619	
30	0.000000000793	
31	0.000000000389	
32	0.000000000191	
33	0.000000000093	

Table 7.6 Evolution time of parallel compositions – phase 1

Next, using the rank-based minimization method and Algorithm 4.4, we obtain

system	*dimension*	*minimal generating vector*
L_m	3	(0.79, -0.1659, 0.009261)
L_M	5	(1.79,-1.1659,0.341061,-0.0441,0.00194481)

Table 7.7 Evolution time of parallel compositions – phase 2

system	*evol. time T*	$\nu_1(T)$	$\nu_2(T)$	$\mathbb{E}(T)$	$\mathbb{V}(T)$	$\sigma(T)$
L_m	T_m	2.30025	7.59431	2.30025	2.30316	1.51762
L_M	T_M	5.22356	37.4896	5.22356	10.20402	3.19437

Table 7.8 Evolution time of parallel compositions – final results

The results obtained can be validated using the Monte Carlo method. □

7.3 Parallel compositions of stochastic systems with a final sequence of states

Let $L^{[k]}$ be a stochastic system with a final sequence of states and interdependent states, with evolution time $T^{[k]}$, $k=\overline{1,s}$. The excludable parallel composition $L_m=\bigcap_{k=1}^{s}L^{[k]}$ and nonexcludable parallel composition $L_M=\bigcup_{k=1}^{s}L^{[k]}$ are defined in the same way as in the case of parallel compositions of stochastic systems with a final sequence of states and independent states ([85]).

Consider that T_m is the evolution time of the excludable composition L_m, T_M is the evolution time of the nonexcludable composition L_M and $T^{[j]}$ is the evolution time of the component system $L^{[j]}$, $j=\overline{1,s}$. Let $T^{[k]}<+\infty$, $k=\overline{1,s}$. We have

$$T_m=\min_{j=1,s}T^{[j]},\ T_M=\max_{j=1,s}T^{[j]}.$$

Let

$$a^{(j)}=rep(T^{[j]}),\ j=\overline{1,s},\ a^*=rep(T_m),\ A^*=rep(T_M).$$

It was shown in Section 6.5 that there exists $M_j\in\mathbb{N}^*$ such that

$$a^{(j)}\in Rol^*[\mathbb{R}][M_j],\ j=\overline{1,s}.$$

Let

$$F_n=P(T_m<n),\ G_n=P(T_M<n),$$

$$F_n^{(j)} = P(T^{[j]} < n),\ j = \overline{1,s},\ n = \overline{0,\infty}.$$

Consider the sequences

$$F = (F_n)_{n=0}^{\infty},\ G = (G_n)_{n=0}^{\infty},$$

$$F^{(j)} = (F_n^{(j)})_{n=0}^{\infty},\ j = \overline{1,s}.$$

We have

$$F_n = P(T_m < n) = 1 - P(T^{[j]} \geq n,\ j = \overline{1,s}) =$$

$$= 1 - \prod_{j=1}^{s} P(T^{[j]} \geq n) = 1 - \prod_{j=1}^{s} (1 - F_n^{(j)}),$$

$$G_n = P(T_M \leq n-1) = P(T^{[j]} \leq n-1,\ j = \overline{1,s}) =$$

$$= \prod_{j=1}^{s} P(T^{[j]} \leq n-1) = \prod_{j=1}^{s} F_n^{(j)},$$

$$a_n^* = F_{n+1} - F_n,\ A_n^* = G_{n+1} - G_n,$$

$$F_n^{(j)} = G_n^{[a^{(j)}]}(1) = \sum_{i=0}^{n-1} a_i^{(j)},\ j = \overline{1,s},\ n = \overline{0,\infty}.$$

Let

$$M^* = \prod_{j=1}^{s} (M_j + 1),\ r^* = \prod_{j=1}^{s} (M_j + 2),\ m^* = r^* + 1.$$

Using Theorem 1.33, we obtain

$$F^{(j)} \in Rol^*[\mathbb{R}][M_j + 1],\ j = \overline{1,s}.$$

Subsequently, by applying Theorem 2.7, we obtain $G \in Rol^*[\mathbb{R}][M^*]$. Since $\overline{G} = (G_{n+1})_{n=0}^{\infty}$ is a subsequence of the sequence G, we have $\overline{G} \in Rol^*[\mathbb{R}][M^*]$ with the same generating vector. Using Theorem 1.31, we obtain

$$A^* = \overline{G} - G \in Rol^*[\mathbb{R}][M^*].$$

By applying Theorem 2.6 and Theorem 2.7, we obtain $F \in Rol^*[\mathbb{R}][m^*]$. Since $\overline{F} = (F_{n+1})_{n=0}^{\infty}$ represents a subsequence of the sequence F, we have $\overline{F} \in Rol^*[\mathbb{R}][m^*]$. Using Theorem 1.31, we obtain

$$a^* = \overline{F} - F \in Rol^*[\mathbb{R}][m^*].$$

The values

$$a_k^*,\ k = \overline{0,2m^*-1},\ A_l^*,\ l = \overline{0,2M^*-1},$$

are obtained according to the formulas above. By using the rank-based order minimization method, we can calculate the dimensions

$$d^* = dim[\mathbb{R}](a^*),\ D^* = dim[\mathbb{R}](A^*)$$

and the minimal generating vectors

$$q^* \in G^*[\mathbb{R}][d^*](a^*),\ Q^* \in G^*[\mathbb{R}][D^*](A^*).$$

Further, we can apply Algorithm 4.4 to determine the main probabilistic characteristics of the evolution time T_m and T_M.

If there exists $j \in \{1,2,\ldots,s\}$ such that $T^{[j]} = +\infty$, then the expectation and the moments of the evolution time T_M of the nonexcludable parallel composition L_M are infinite and

$$L_m = \bigcap_{k \in \{1,2,\ldots,s\} \setminus \{j\}} L^{[k]}.$$

Example 7.3 *Consider two stochastic systems with a final sequence of states and interdependent states L_1 and L_2 with the same set of states $V = \{v_1, v_2\}$, the same initial distribution of the states*

$$p^*(v_1) = 0.3,\ p^*(v_2) = 0.7$$

and the same transition probabilities

$$p(v_1, v_1) = p(v_2, v_1) = 0.3,$$

$$p(v_1, v_2) = p(v_2, v_2) = 0.7.$$

Let $X_1 = (v_1, v_2)$ *be the final sequence of states of the system* L_1 *and let* $X_2 = (v_2, v_1)$ *be the final sequence of states of the system* L_2. *Consider the compositions*

$$L_S = (L_1 + L_2)[\Lambda],$$

$$L_m = L_1 \cap L_2, \quad L_M = L_1 \cup L_2,$$

where

$$\Lambda = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Determine the expectation, the variance, the standard deviation and the moments of order 1 *and* 2 *of the evolution times of the systems* L_1, L_2, L_S, L_m *and* L_M.

Solution. Let T_k be the evolution time of the system L_k and

$$a^{(k)} = rep(T_k), \ k \in \{1, 2, S, m, M\}.$$

By applying the method described in Section 6.5, we obtain

$$a = a^{(1)} = a^{(2)} \in Rol^*[\mathbb{R}][4],$$

$$q^* = (1, -0.21, 0, 0) \in G^*[\mathbb{R}][4](a),$$

$$I_4^{[a]} = (0, 0.21, 0.21, 0.1659).$$

Therefore, we have the recurrence

$$a_n = a_{n-1} - 0.21a_{n-2}, \ \forall n \geq 4.$$

Next, using the rank-based order minimization method, we obtain

$$a \in Rol^*[\mathbb{R}][2], \ \ q = (1, -0.21) \in G^*[\mathbb{R}][2](a), \ \ I_2^{[a]} = (0, 0.21).$$

Using the method developed and theoretically justified in this section, we have

$$a^{(m)} \in Rol^*[\mathbb{R}][3]$$

with generating vector

$$q^{(m)} = (0.79, -0.1659, 0.0093) \in G^*[\mathbb{R}][3](a^{(m)})$$

and initial state

$$I_3^{[a^{(m)}]} = (0, 0.3759, 0.2877).$$

Also, we obtain

$$a^{(M)} \in Rol^*[\mathbb{R}][5],$$

$$q^{(M)} = (1.79, -1.1659, 0.3411, -0.0441, 0.0019) \in G^*[\mathbb{R}][5](a^{(M)}),$$

$$I_5^{[a^{(M)}]} = (0, 0.0441, 0.1323, 0.1669, 0.1576).$$

By applying Algorithm 4.4 for the sequences a, $a^{(m)}$ and $a^{(M)}$ and the method from Section 7.1 for random variables $T^{(1)}$, $T^{(2)}$ and $T^{(S)}$, we obtain the following final results:

k	*system*	*evol. time*	$\nu_1(T_k)$	$\nu_2(T_k)$	$\mathbb{E}(T_k)$	$\mathbb{V}(T_k)$	$\sigma(T_k)$
1	L_1	T_1	3.7619	22.5420	3.7619	8.3901	2.8966
2	L_2	T_2	3.7619	22.5420	3.7619	8.3901	2.8966
S	L_S	T_S	9.5238	107.483	9.5238	16.7802	4.0964
m	L_m	T_m	2.3003	7.5943	2.3003	2.3029	1.5175
M	L_M	T_M	5.2236	37.4896	5.2236	10.2036	3.1943

Table 7.9 The evolution time of parallel compositions with interdependent states

All these results can be verified using the Monte Carlo method. Also, we can compare these results with the solutions provided by Example 7.1 and Example 7.2, since Example 7.3 represents their union. □

7.4 Geometric and signomial programming

The geometric programming was introduced in 1967 by Duffin ([42], [43]), Peterson ([43], [132]) and Zener ([43]). Wilde and Beightler ([157]) in 1967 and Zener ([161]) in 1971 made several contributions related to multiple extensions and sensitivity analysis. A geometric program is an optimization problem described by an

objective function and constraints of a special form. An excellent guide for geometric programming was provided in [25].

The first numerical methods, based on solving a sequence of linear programs, were elaborated by Avriel ([7]), Duffin ([42]), Rajpogal and Bricker ([138]). Nesterov and Nemirovsky ([124]) described the first interior-point method for solving geometric programs in 1994 and they proved its polynomial complexity. Recent numerical methods were presented by Andersen and Ye ([4]), Boyd and Vandenberghe ([25], [26]), Kortanek ([75]).

The definition of a geometric program requires definitions of monomials and posynomials.

Definition 7.5 *In the context of geometric programming, a monomial is defined as a function* $f:\mathbb{R}^s \to \mathbb{R}$ *of the form*

$$f(x_1, x_2, \ldots, x_s) = c x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s},$$

where $c > 0$ *and* $\alpha_i \in \mathbb{R}$, $i = \overline{1,s}$.

Definition 7.6 *A linear combination of monomials*

$$f(x_1, x_2, \ldots, x_s) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$$

with $c_k > 0$, $k = \overline{1,K}$ *and* $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1,s}$, $k = \overline{1,K}$, *is called a posynomial.*

The set of posynomials is closed under addition, multiplication and multiplication by positive scalars. The notions of a posynomial and a polynomial are not equivalent because

- the exponents of a polynomial are nonnegative integers, whereas those of a posynomial may be arbitrary real numbers;
- the coefficients of a polynomial are arbitrary real numbers, whereas those of a posynomial are positive real numbers.

Using the above definitions, we may define a geometric program as follows.

Definition 7.7 *A geometric program represents an optimization problem of the form*

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1,r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1,m} \\ x_l > 0, & l = \overline{1,s}, \end{cases}$$

where the functions $f_i(x_1, \ldots, x_s)$, $i = \overline{0,r}$, *are posynomials and the functions* $g_j(x_1, \ldots, x_s)$, $j = \overline{1,m}$, *are monomials.*

To solve a geometric program efficiently, it must be transformed into a convex optimization problem. Efficient methods for solving the general convex optimization problems were developed in [26]. The transformation is based on a logarithmic change of variables $y_l = \ln x_l$, $l = \overline{1,s}$ and the logarithmic transformation of the objective function and the constraints. The resulting convex optimization problem has the form:

$$\ln f_0(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) \to \min,$$

$$\begin{cases} \ln f_i(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) \le 0, & i = \overline{1,r} \\ \ln g_j(e^{y_1}, e^{y_2}, \ldots, e^{y_s}) = 0, & j = \overline{1,m}. \end{cases}$$

Unlike the original problem, the resulting problem may appear more complicated. However, it is convex and can be solved efficiently using standard interior-point methods ([25], [26]). Interior-point methods for solving geometric programs are robust, require neither an initial point nor any additional parameters, and always converge to the globally optimal solution.

Definition 7.8 *A function* $f : \mathbb{R}^s \to \mathbb{R}$ *of the form*

$$f(x_1, x_2, \ldots, x_s) = c x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$$

with $c \in \mathbb{R}$ *and* $\alpha_i \in \mathbb{R}$, $i = \overline{1,s}$, *is called a signomial monomial.*

Definition 7.9 *A signomial is defined as a linear combination of signomial monomials, i.e. it is a function*

$$f(x_1, x_2, \ldots, x_s) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$$

with $c_k \in \mathbb{R}$ *and* $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1,s}$, $k = \overline{1,K}$.

It is easy to observe that the signomial

$$f(x_1, x_2, \ldots, x_s) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$$

represents a difference of two posynomials, since

$$f(x_1, x_2, \ldots, x_s) = \sum_{c_k > 0} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}} - \sum_{c_k < 0} (-c_k) x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}} .$$

Also, the set of signomials is closed under addition, subtraction, multiplication, and scalar multiplication. Therefore, we can define a geometric program in the following way.

Definition 7.10 *A signomial program is defined as an optimization problem:*

$$f_0(x_1, x_2, \ldots, x_s) \to \min ,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1,r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1,m} \\ x_l > 0, & l = \overline{1,s} , \end{cases}$$

where $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0,r}$ *and* $g_j(x_1, x_2, \ldots, x_s)$, $j = \overline{1,m}$, *are signomials.*

Therefore, a signomial has the same form as a posynomial, but the coefficients are also allowed to be negative. However, there exists a significant difference between a geometric program and a signomial program. The global optimal solution of the geometric program can always be determined, but only a local optimal solution of the signomial program can be computed efficiently.

In several particular cases, the signomial programs can be handled using geometric programming. It was shown in [25] that the geometric programs with posynomial equality constraints represent such a particular case, i.e. they can be solved using the geometric programs.

Definition 7.11 *A signomial program of the form:*

$$f_0(x_1, x_2, \ldots, x_s) \to \min ,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1,r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1,m} \\ h_k(x_1, x_2, \ldots, x_s) = 1, & k = \overline{1,n} \\ x_l > 0, & l = \overline{1,s}, \end{cases}$$

where $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0,r}$ *and* $h_k(x_1, x_2, \ldots, x_s)$, $k = \overline{1,n}$, *are posynomials and* $g_j(x_1, x_2, \ldots, x_s)$, $j = \overline{1,m}$, *are monomials, is called a geometric program with posynomial equality constraints.*

Assume that, for each posynomial equality constraint

$$h_k(x_1, x_2, \ldots, x_s),\ k = \overline{1,n},$$

we can find a different variable $x_{l(\mathrm{k})}$ with the following properties:

- The variable $x_{l(\mathrm{k})}$ does not appear in any of the monomial equality constraint functions;
- The posynomial $h_k(x_1, x_2, \ldots, x_s)$ is strictly
  - increasing in $x_{l(\mathrm{k})}$, in which case we denote $\lambda(x_{l(k)}) = -1$ or
  - decreasing in $x_{l(\mathrm{k})}$, in which case we denote $\lambda(x_{l(k)}) = 1$;
- The functions $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0,r}$, are all
  - monotone decreasing in $x_{l(\mathrm{k})}$, if $\lambda(x_{l(k)}) = -1$;
  - monotone increasing in $x_{l(\mathrm{k})}$, if $\lambda(x_{l(k)}) = 1$.

Initially, we form the relaxed geometric program

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1,r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1,m} \\ h_k(x_1, x_2, \ldots, x_s) \le 1, & k = \overline{1,n} \\ x_l > 0, & l = \overline{1,s} \end{cases}$$

If f^* is the optimal value of the relaxed problem, then the optimal solutions of the problem

$$\prod_{k=1}^{n}(x_{l(k)})^{\lambda(x_{l(k)})} \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1,r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1,m} \\ h_k(x_1, x_2, \ldots, x_s) \le 1, & k = \overline{1,n} \\ f_0(x_1, x_2, \ldots, x_s) \le f^* & \\ x_l > 0, & l = \overline{1,s} \end{cases}$$

are also optimal solutions for the original problem.

7.5 Optimization of zero-order Markov processes with a final sequence of states

Let L be a stochastic system with the set of states V, where $|V| = N \le \infty$. The state of the system at each discrete time $t = 0,1,2,\ldots$ is $v(t) \in V$. The transition time of the system from the state u to the state v at each discrete time t is one.

A probability function $p: V \to [0,1]$ is defined on the set V, where $p(v)$ represents the probability that L starts its evolution from the state $v \in V$. Also, the probability of the system transition from the state $u \in V$ to the state $v \in V$ at time t is $p(v) \in [0,1]$.

The stochastic system L completes its evolution when it consecutively passes through the final sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$. The evolution time T of the system L is the amount of time needed to reach the final sequence of states.

The system L represents a stochastic system with the final sequence of states X, independent states and the distribution of the states p. We have already studied the moments of the evolution time. In particular, we obtained the expectation of the evolution time.

Further, we consider that the distribution p is not fixed. Therefore, we have the stochastic system $L(p)$ with the final sequence of states X and the distribution of the states p, for each p in the set

$$\Pi = \{y = (y_1, y_2, \ldots, y_N) \in [0,1]^N \mid \sum_{j=1}^{N} y_j = 1\}.$$

Our aim is to determine the optimal distribution $p^* \in \Pi$, which minimizes the expectation of the evolution time $T(p)$ of the stochastic system $L(p)$, i.e.

$$T(p^*) = \min_{p \in \Pi} T(p).$$

Next, we present an approach for solving this problem efficiently. We prove that the optimization of the evolution time of the systems with a final sequence of states and independent states can be reduced to a geometric program ([93]). We will consider the notation from Algorithm 6.2.

Lemma 7.2 *The sequence* $(t(s))_{s=1}^{m}$ *is nondecreasing and*

$$t(1) = 2,\ 2 \le t(s) \le s+1,\ s = \overline{1,m}.$$

Proof. The relations $t(1) = 2$ and

$$2 \le t(s) \le s+1,\ s = \overline{1,m},$$

are obtained from the definition of the coefficients $t(s)$, $s = \overline{1,m}$. Also, we have

$$(x_{t(s)}, x_{t(s)+1}, \ldots, x_s) = (x_1, x_2, \ldots, x_{s+1-t(s)}),$$

$$(x_{t(s+1)}, x_{t(s+1)+1}, \ldots, x_{s+1}) = (x_1, x_2, \ldots, x_{s+2-t(s+1)}),$$

for each s which satisfies the double inequality $1 \le s < m$. This implies that

$$(x_{t(s+1)}, x_{t(s+1)+1}, \ldots, x_s) = (x_1, x_2, \ldots, x_{s+1-t(s+1)})$$

for $1 \le s < m$. Since the value $t = t(s)$ is the smallest natural number which satisfies

$$(x_t, x_{t+1}, \ldots, x_s) = (x_1, x_2, \ldots, x_{s+1-t}),$$

we have $t(s) \le t(s+1)$, for each s, $1 \le s < m$. Therefore, the sequence $(t(s))_{s=1}^{m}$ is nondecreasing. □

Lemma 7.3 *If* $t(s)=2$, *then* $w_s=\pi_1^s$.

Proof. Let $t(s)=2$. Then, by definition, we have $x_1=x_2=\ldots=x_s$, which implies the equality $\pi_1=\pi_2=\ldots=\pi_s$. Finally, we obtain $w_s=\pi_1^s$. □

Lemma 7.4 *If* $t(s)=2$ *and* $k\geq 1$, *then* $z_{sk}=\pi_1^k(1-\pi_1)$.

Proof. Consider $t(s)=2$ and $k\geq 2$. Then

$$z_{s,k}=w_{t(s)-1}z_{s-t(s)+1,k-t(s)+1}=\pi_1 z_{s-1,k-1}.$$

By applying Lemma 7.2, we have

$$2\leq t(s-1)\leq t(s)=2,$$

which implies $t(s-1)=2$, i.e., if $k-1\geq 2$, then,

$$z_{s,k}=\pi_1 z_{s-1,k-1}=\pi_1^2 z_{s-2,k-2}.$$

Repeating it $(k-1)$ times, we obtain

$$z_{s,k}=\pi_1^{k-1}z_{s-(k-1),1}.$$

Since $t(s-(k-1))=2$, we have

$$z_{s,k}=\pi_1^{k-1}z_{s-(k-1),1}=\pi_1^k(1-\pi_1)$$

and the statement of this lemma is proved. The proof is also valid for $k=1$. □

Lemma 7.5 *The relation* $w_s=w_{t(s)-1}w_{s+1-t(s)}$, $s=\overline{1,m}$, *holds.*

Proof. From the definition of the overlapping levels $t(s)$, $s=\overline{1,m}$, it follows that

$$(x_1,x_2,\ldots,x_{s+1-t(s)})=(x_{t(s)},x_{t(s)+1},\ldots,x_s),$$

which implies

$$(\pi_1,\pi_2,\ldots,\pi_{s+1-t(s)})=(\pi_{t(s)},\pi_{t(s)+1},\ldots,\pi_s)$$

for each natural number s in the interval $[1,m]$. We obtain

$$w_{t(s)-1}=\prod_{j=1}^{t(s)-1}\pi_j=\frac{w_s}{\prod\limits_{j=t(s)}^{s}\pi_j}=\frac{w_s}{\prod\limits_{j=1}^{s+1-t(s)}\pi_j}=\frac{w_s}{w_{s+1-t(s)}},\ s=\overline{1,m},$$

which represents the statement of this lemma. □

Lemma 7.6 *For each* $k = \overline{0, m-1}$ *there exist positive integers* m^* *and* k^* *which satisfy the relations*

$$\begin{cases} z_{mk} = \dfrac{w_m}{w_{m^*}} z_{m^* k^*} \\ m^* - k^* = m - k \\ 0 \le k^* < t(m^*). \end{cases}.$$

Proof. This lemma is obtained from Theorem 6.1 and Lemma 7.5. By applying Theorem 6.1, we obtain that there exists an index l such that

$$z_{mk} = w_{t(m)-1} w_{t(m-t(m)+1)-1} \cdot \ldots \cdot w_{t(l)-1} z_{m^* k^*}.$$

Using Lemma 7.5, we have

$$z_{mk} = \frac{w_m}{w_{m+1-t(m)}} \cdot \frac{w_{m+1-t(m)}}{w_{(m-t(m)+1)+1-t(m-t(m)+1)}} \cdot \ldots \cdot \frac{w_l}{w_{m^*}} z_{m^* k^*} = \frac{w_m}{w_{m^*}} z_{m^* k^*}.$$

Also, from Theorem 6.1 we can easily note that the difference between indices does not change at each step of the recurrence, i.e. $m^* - k^* = m - k$. The condition $0 \le k^* < t(m^*)$ represents the stopping rule of the recurrence from Theorem 6.1. □

Theorem 7.1 *The expression* $\mathbb{E}(T(p)) + 1$ *represents a posynomial in the variables* $\pi_1, \pi_2, \ldots, \pi_m$.

Proof. From Theorem 6.1 and Theorem 6.2, we have

$$z_{sk} = \begin{cases} 0, & 0 \le k \le t(s) - 3 \\ -w_{t(s)-1}, & k = t(s) - 2 \\ w_{t(s)-1}(1 - \pi_1), & t(s - t(s) + 1) = 2 \text{ and } k = t(s) - 1 \\ w_{t(s)-1}, & t(s - t(s) + 1) \ge 3 \text{ and } k = t(s) - 1 \\ w_{t(s)-1} z_{s-t(s)+1, k-t(s)+1}, & t(s) \le k \le s - 1. \end{cases}.$$

Consider $z_{mk} = \frac{w_m}{w_{m^*}} z_{m^* k^*}$, where the indices m^* and k^* were defined in Lemma 7.6. The following scenarios are possible:

- *Case* $k^* \le t(m^*) - 3$.

In this case we have $z_{mk} = \frac{w_m}{w_{m^*}} z_{m^* k^*} = 0$.

- *Case* $k^* = t(m^*) - 2$, $t(m^*) \le m^*$ *and* $t(m^* - t(m^*) + 1) \ge 3$.

In this case we have $k^* + 1 = t(m^*) - 1$, which implies

$$z_{m^* k^*} = -w_{t(m^*)-1}, \; z_{m^*, k^*+1} = w_{t(m^*)-1}.$$

Therefore,

$$f_{mk} \overset{def}{=} (k+1) z_{mk} + (k+2) z_{m,k+1} = \frac{w_m}{w_{m^*}} ((k+1) z_{m^* k^*} + (k+2) z_{m^*, k^*+1}) =$$

$$= \frac{w_m}{w_{m^*}} ((k+1)(-w_{t(m^*)-1}) + (k+2) w_{t(m^*)-1}) = \frac{w_m}{w_{m^*}} w_{t(m^*)-1} = \frac{w_m}{w_{m^*+1-t(m^*)}}.$$

It is easy to note that this expression is a posynomial in the variables $\pi_1, \pi_2, \ldots, \pi_m$.

- *Case* $k^* = t(m^*) - 2$, $t(m^*) \le m^*$ *and* $t(m^* - t(m^*) + 1) = 2$.

In this case we have $k^* + 1 = t(m^*) - 1$, which implies

$$z_{m^* k^*} = -w_{t(m^*)-1}, \; z_{m^*, k^*+1} = w_{t(m^*)-1}(1 - \pi_1).$$

Consider $r \ge 2$. Using Lemma 7.4, we obtain:

$$z_{m^*, k^*+r} = w_{t(m^*)-1} z_{m^* - t(m^*)+1, k^*+r-t(m^*)+1} =$$

$$= w_{t(m^*)-1} z_{m^* - t(m^*)+1, r-1} = w_{t(m^*)-1} \pi_1^{r-1} (1 - \pi_1).$$

Further, by applying Lemma 7.3, Lemma 7.5 and Lemma 7.6, we have the relation

$$g_{mk} \overset{def}{=} \sum_{j=k}^{m-1} (j+1) z_{mj} = \sum_{j=k}^{m-1} (j+1) z_{m,k+(j-k)} = \frac{w_m}{w_{m^*}} \sum_{j=k}^{m-1} (j+1) z_{m^*, k^*+(j-k)} =$$

$$= \frac{w_m}{w_{m^*}} \left[(k+1)(-w_{t(m^*)-1}) + \sum_{j=k+1}^{m-1} (j+1) w_{t(m^*)-1} \pi_1^{j-k-1} (1-\pi_1) \right] =$$

$$= \frac{w_m}{w_{m^*+1-t(m^*)}} \left[-(k+1) + \sum_{j=0}^{m-k-2} (j+k+2) \pi_1^j (1-\pi_1) \right] =$$

$$= \frac{w_m}{\pi_1^{m^*-t(m^*)+1}} (1 - m\pi_1^{m-k-1}) = \frac{w_m}{\pi_1^{(m^*-k^*)+(k^*-t(m^*)+2)-1}} (1 - m\pi_1^{m-k-1}) =$$

$$= \frac{w_m}{\pi_1^{m-k-1}} (1 - m\pi_1^{m-k-1}) = \frac{w_m}{\pi_1^{m-k-1}} - mw_m,$$

which implies the formula

$$mw_m + g_{mk} = \frac{w_m}{\pi_1^{m-k-1}}.$$

In this way, we obtain that the expression $mw_m + g_{mk}$ represents a posynomial in the variables $\pi_1, \pi_2, \ldots, \pi_m$.

- *Case* $k^* = m^* - 1$ *and* $t(m^*) = m^* + 1$.

In this case we have

$$z_{mk} = z_{m,m-1} = \frac{w_m}{w_{m^*}} z_{m^*,m^*-1} = \frac{w_m}{w_{m^*}} (-w_{t(m^*)-1}) = \frac{w_m}{w_{m^*}} (-w_{m^*}) = -w_m,$$

which implies the relation

$$mw_m + mz_{m,k} = mw_m - mw_m = 0.$$

- *Case* $k^* = t(m^*) - 1$.

Since $k^* - 1 = t(m^*) - 2$, this case is part of the previous scenarios, which have already been studied.

Consequently, we obtain that the expression $mw_m + \sum_{k=0}^{m-1} (k+1) z_{mk}$ represents a posynomial in the variables $\pi_1, \pi_2, \ldots, \pi_m$. By applying Theorem 6.2, we have

$$\mathbb{E}(T(p))+1=\frac{1+\left[mw_m+\sum_{k=0}^{m-1}(k+1)z_{mk}\right]}{w_m}.$$

From the properties of the posynomials, we obtain that the expression $\mathbb{E}(T(p))+1$ represents a posynomial in the variables $\pi_1, \pi_2, \ldots, \pi_m$. □

Theorem 7.1 is very important. It shows how the initial problem can be reduced to a geometric program

$$\mathbb{E}(T(p))+1 \to \min,$$

$$\begin{cases} \sum_{x\in Y} p(x) \le 1 \\ p(x) > 0, \forall x \in Y \end{cases},$$

where $Y=\{x_1, x_2, \ldots, x_m\}$. If $\pi^*=(\pi^*(x))_{x\in Y}$ is the optimal solution of this geometric program, then $p^*=(p^*(x))_{x\in V}$ is the optimal solution of the initial problem, where

$$\begin{cases} p^*(x)=\pi^*(x), & x\in Y \\ p^*(x)=0, & x\in V\setminus Y \end{cases}.$$

We considered the constraint $\sum_{x\in Y} p(x) \le 1$ instead of $\sum_{x\in Y} p(x)=1$ since

- The expression $\sum_{x\in Y} p(x)$ is a posynomial and it is not a monomial in the variables $p(x)$, $x\in Y$;
- These constraints yield the same optimal solution, since the relations $p^*(x)=0$, $\forall x\in V\setminus Y$ and $\sum_{x\in V} p^*(x)=1$ hold.

We can mention that the relation $p^*(x)=0$, $\forall x\in V\setminus Y$, holds in the initial problem. Indeed, if we had $p^*(x^*)>0$ for at least one state $x^*\in V\setminus Y$, then the states belonging to the final sequence of states would have a lower probability of occurrence and, consequently, the evolution time would be longer and the solution would not be optimal.

In what follows, several particular cases of the problem are analyzed, and explicit optimal solutions are obtained. The following theorems hold.

Theorem 7.2 *If* $t(m)=2$*, then the optimal solution is* $p^*=(p^*(x))_{x\in V}$*, where*

$$p^*(x_1)=1,\ p^*(y)=0,\ \forall y\in V\setminus\{x_1\}$$

and the minimal expectation value of the evolution time is

$$\mathbb{E}(T(p^*))=m-1.$$

Proof. Let $t(m)=2$. We have $x_1=x_2=\ldots=x_m$ and the stochastic system $L(p)$ is a stochastic system with the final critical state x_1. We obtain $Y=\{x_1,x_2,\ldots,x_m\}=\{x_1\}$. Therefore, the optimal solution is $p^*=(p^*(x))_{x\in V}$, where

$$p^*(x_1)=1,\ p^*(y)=0,\ \forall y\in V\setminus\{x_1\}.$$

It is easy to observe that this implies the relation $\mathbb{E}(T(p^*))=m-1$. □

Theorem 7.3 *If* $t(m)=m+1$*, then the components* $p^*(y)$*,* $y\in V$*, of the optimal solution* p^* *are directly proportional to the multiplicities* $m(y)$*,* $y\in V$*, of the related states in the final sequence of states* X *and the minimal expectation value of the evolution time is*

$$\mathbb{E}(T(p^*))=-1+\prod_{y\in Y}\left(\frac{m}{m(y)}\right)^{m(y)}.$$

Proof. Let $t(m)=m+1$. We have

$$z_{mk}=\begin{cases}0, & 0\le k\le m-2\\ -w_m, & k=m-1\end{cases}.$$

From Theorem 6.2, the formula for the expectation of the evolution time is

$$\mathbb{E}(T(p))=-1+w_m^{-1}.$$

Therefore, we have the optimization problem

$$\mathbb{E}(T(p))+1=w_m^{-1}\to\min$$

$$\sum_{x\in Y}p(x)=1,\ p(x)>0,\ \forall x\in Y,$$

which is equivalent to the problem

$$w_m \to \max$$

$$\sum_{x \in Y} p(x) = 1,\ p(x) > 0,\ \forall x \in Y.$$

For every state $x \in Y$, we denote by $m(x)$ its multiplicity in the final sequence of states X. We obtain the optimization problem

$$f((p(x))_{x \in Y}) = \prod_{x \in Y} (p(x))^{m(x)} \to \max,$$

$$\sum_{x \in Y} p(x) = 1,\ p(x) > 0,\ \forall x \in Y.$$

Next, we apply the classical optimization method, which uses the partial derivatives of the objective function. We choose a state $y^* \in Y$ and denote by Y^* the set $Y \setminus \{y^*\}$. We have

$$p(y^*) = 1 - \sum_{x \in Y^*} p(x),$$

which implies

$$f((p(x))_{x \in Y^*}) = (p(y^*))^{m(y^*)} \prod_{x \in Y^*} (p(x))^{m(x)}.$$

For each $y \in Y^*$ we obtain

$$\frac{\partial f}{\partial p(y)} = \prod_{x \in Y^* \setminus \{y\}} (p(x))^{m(x)} . \frac{\partial h(y, y^*)}{\partial p(y)},$$

where

$$h(y, y^*) = (p(y))^{m(y)} (p(y^*))^{m(y^*)}.$$

We have

$$\frac{\partial h(y, y^*)}{\partial p(y)} = m(y)(p(y))^{m(y)-1} (p(y^*))^{m(y^*)} -$$

$$-m(y^*)(p(y))^{m(y)} (p(y^*))^{m(y^*)-1} =$$

$$= (p(y))^{m(y)-1}(p(y^*))^{m(y^*)-1} \cdot (m(y)p(y^*) - m(y^*)p(y)).$$

Therefore, the system

$$\frac{\partial f}{\partial p(y)} = 0, \ \forall y \in Y^*,$$

is equivalent to

$$m(y)p(y^*) - m(y^*)p(y) = 0, \ \forall y \in Y^*,$$

which, in turn, implies the relation

$$p(y) = \frac{m(y)}{m(y^*)} p(y^*), \ \forall y \in Y^*.$$

We obtain the formula

$$1 = \sum_{y \in Y} \frac{m(y)}{m(y^*)} p(y^*) = \frac{p(y^*)}{m(y^*)} \sum_{y \in Y} m(y) = \frac{mp(y^*)}{m(y^*)},$$

from which we have

$$p(y^*) = \frac{m(y^*)}{m}.$$

Next, we obtain the components of the optimal solution:

$$p^*(y) = p(y) = \frac{m(y)}{m}, \forall y \in Y.$$

Finally, we obtain

$$\mathbb{E}(T(p^*)) = -1 + w_m^{-1} = -1 + \prod_{y \in Y} \left(\frac{m}{m(y)} \right)^{m(y)}$$ □

Example 7.4 *We consider a stochastic system L with the final sequence of states*

$$X = (x_1, x_1, x_2, x_2, x_1, x_1)$$

and independent states. Determine the optimal distribution of the states, which minimizes the expectation of the evolution time of the stochastic system L.

Solution. By applying Theorem 6.2, we have

$$\mathbb{E}(T(p)) = -1 + p_1^{-2} + p_1^{-1} + p_1^{-4} p_2^{-2}.$$

The following geometric program is obtained:

$$p_1^{-2} + p_1^{-1} + p_1^{-4} p_2^{-2} \to \min,$$

subject to

$$\begin{cases} p_1 + p_2 \le 1 \\ p_1 > 0,\ p_2 > 0 \end{cases}.$$

Next, we solve this geometric program and we obtain the optimal distribution

$$p^* = (0.673694, 0.326306)$$

and the minimal evolution time

$$\mathbb{E}(T(p^*)) = 48.2808.$$ □

Example 7.5 *We consider the stochastic system L with the final sequence of states*

$$X = (x_1, x_2, x_3, x_1, x_2, x_3)$$

and independent states. Determine the optimal distribution of the states, which minimizes the expectation of the evolution time of the stochastic system L.

Solution. By applying Theorem 6.2, we have

$$\mathbb{E}(T(p)) = -1 + (p_1 p_2 p_3)^{-1} + (p_1 p_2 p_3)^{-2} \to \min,$$

subject to

$$\begin{cases} p_1 + p_2 + p_3 \le 1, \\ p_1 > 0,\ p_2 > 0, p_3 > 0 \end{cases}.$$

We solve this optimization problem by using the geometric programming method and we obtain the optimal distribution

$$p^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

and the minimal evolution time $\mathbb{E}(T(p^*)) = 755$. □

7.6 Optimization of stochastic systems with a final sequence of states

Consider the discrete stochastic system L with the set of states V, such that $|V| = \omega < \infty$. The state of the system at each discrete time $t \in \mathbb{N}$ is $v(t) \in V$.

Let $p^*(v)$ be the probability that $v(0) = v$, $\forall v \in V$ and let $p(u,v)$ be the transition probability from the state $u \in V$ to the state $v \in V$. We consider the fixed states $x_1, x_2, \ldots, x_m \in V$. The evolution of the system completes when it passes consecutively through the states $x_1, x_2, \ldots, x_m$. The stopping time T represents the evolution time of the system L.

In what follows, we consider that the distributions p and p^* are not fixed. Therefore, we have the Markov process $L(p^*, p)$ with the final sequence of states $X = (x_1, x_2, \ldots, x_m)$, the initial distribution of the states p^* and the transition matrix p, for each pair of parameters p and p^*. The problem is to determine the optimal distribution $p^* = \overline{p}^*$ and the optimal transition matrix $p = \overline{p}$, which minimize the expectation of the evolution time $T(p^*, p)$ of the system $L(p^*, p)$ ([87], [92]).

Theorem 7.4 *The optimal distribution is* $\overline{p}^*$, *where*

$$\overline{p}^*(x_1) = 1, \ \overline{p}^*(x) = 0, \ \forall x \in V \setminus \{x_1\}.$$

Proof. For the system to complete its evolution, it is necessary that it pass successively through the states $x_1, x_2, \ldots, x_m$. Therefore, in order for the evolution time to be minimal, the state x_1 must be reached as quickly as possible. Thus, the system must start from x_1, i.e. $\overline{p}^*(x_1) = 1$. Since

$$\sum_{x \in V} \overline{p}^*(x) = 1 \text{ and } \overline{p}^*(x) \geq 0, \ \forall x \in V,$$

it follows that $\overline{p}^*(x) = 0, \ \forall x \in V \setminus \{x_1\}$. □

Theorem 7.5 *Consider the sets*

$$\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\},$$

$$\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}.$$

The optimal transition matrix $\overline{p}$ *satisfies the properties:*

1. $\overline{p}(x, y) = 0$, *if* $(x, y) \in V^2 \setminus \overline{Y}$, $x \in \overline{X}$ *and* $y \in V \setminus \{x_1\}$;
2. $\overline{p}(x, x_1) = 1$, *if* $(x, x_1) \in \overline{Y}$ *and* $(x, z) \in V^2 \setminus \overline{Y}$, $\forall z \in V \setminus \{x_1\}$;
3. $\overline{p}(x, x_1) = 1$ *and* $\overline{p}(x, y) = 0$, $\forall y \in V \setminus \{x_1\}$, *if* $x \in V \setminus \overline{X}$;
4. $\overline{p}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$;
5. $\sum_{(x,y) \in \overline{Y} \cup \{(x, x_1)\}} \overline{p}(x, y) = 1$, $\forall x \in \overline{X}$.

Proof. Let $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$ be the set of states from which an optimal transition can be performed and let $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ be the set of optimal transitions.

1. If $x \in \overline{X}$, then $\exists y_1, y_2, \ldots, y_k \in \overline{X} \cup \{x_m\}$, such that $(x, y_j) \in \overline{Y}$, $j = \overline{1, k}$, where $k \geq 1$. To guarantee that one of these transitions is performed whenever the system is in state $x \in \overline{X}$ or that the system returns to the initial state x_1 whenever necessary, we require that there be no other transition $(x, y) \in V^2 \setminus \overline{Y}$ such that $y \in V \setminus \{x_1\}$, i.e. $\overline{p}(x, y) = 0$.
2. If $(x, x_1) \in \overline{Y}$ and $(x, z) \in V^2 \setminus \overline{Y}$, $\forall z \in V \setminus \{x_1\}$, then $x \in \overline{X}$ and the transition (x, x_1) is the unique possible transition in the set $\overline{Y}$. To guarantee that this transition is performed whenever the system is in state $x \in \overline{X}$ we impose that the relation $\overline{p}(x, x_1) = 1$ holds.
3. For the system to complete its evolution, it is necessary that it pass successively through the states $x_1, x_2, \ldots, x_m$. Therefore, in order for the evolution time to be

minimal, the state x_1 must be reached as quickly as possible. Thus, if the system is in state $x \in V \setminus \overline{X}$, we guarantee that the next state x_1 is reached by imposing the constraint $\overline{p}(x, x_1) = 1$. Since $\sum_{y \in V} \overline{p}(x, y) = 1$ and $\overline{p}(x, y) \geq 0$, $\forall x, y \in V$, it follows that $\overline{p}(x, y) = 0$, $\forall y \in V \setminus \{x_1\}$.

4. We require the condition $\overline{p}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$, since otherwise we would have $\overline{p}(x, z) = 0$ for at least one $(x, z) \in \overline{Y}$, i.e. (x, z) would never occur and the evolution time would be infinite.
5. The relation $\sum_{(x,y) \in \overline{Y} \cup \{(x, x_1)\}} \overline{p}(x, y) = 1$, $\forall x \in \overline{X}$, is obtained from the formula $\sum_{y \in V} \overline{p}(x, y) = 1$, $\forall x \in \overline{X}$ and the Property 1 of this theorem. □

Theorem 7.6 *If* $\delta_{ij}(p) \neq 0$, $i, j = \overline{1,2}$, *then the optimal transition matrix can be found by solving the following geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p)) = d_1 d_2^{-1} \to \min \tag{7.1}$$

$$\begin{cases} (a'): & \sum_{(x,y) \in \overline{Y} \cup \{(x, x_1)\}} p(x, y) = 1, \forall x \in \overline{X} \\ (b'): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (c'): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (d'): & d_{1,1}^{-1} \delta_{1,1}(p) = 1 \\ (e'): & d_{1,2}^{-1} \delta_{1,2}(p) = 1 \\ (f'): & d_{2,1}^{-1} \delta_{2,1}(p) = 1 \\ (g'): & d_{2,2}^{-1} \delta_{2,2}(p) = 1 \\ (h'): & d_i > 0, i = \overline{1,2} \\ (i'): & d_{i,j} > 0, i, j = \overline{1,2} \\ (j'): & p(x, y) > 0, \forall (x, y) \in \overline{Y}. \end{cases} \tag{7.2}$$

and

$$\mathbb{E}(T(p)) = d_1 d_2^{-1} \to \min \tag{7.3}$$

$$\begin{cases} (a''): & \sum\limits_{(x,y)\in \overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X} \\ (b''): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (c''): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (d''): & d_{1,1}^{-1} \delta_{1,2}(p) = 1 \\ (e''): & d_{1,2}^{-1} \delta_{1,1}(p) = 1 \\ (f''): & d_{2,1}^{-1} \delta_{2,2}(p) = 1 \\ (g''): & d_{2,2}^{-1} \delta_{2,1}(p) = 1 \\ (h''): & d_i > 0,\ i = \overline{1,2} \\ (i''): & d_{i,j} > 0,\ i,j = \overline{1,2} \\ (j''): & p(x,y) > 0,\ \forall (x,y) \in \overline{Y}. \end{cases} \tag{7.4}$$

taking into account the properties described in Theorem 7.4 and Theorem 7.5, where $\delta_{i,j}(p)$, $i,j = \overline{1,2}$, *are the posynomials from the decomposition*

$$\mathbb{E}(T(p)) = (\delta_{1,1}(p) - \delta_{1,2}(p))(\delta_{2,1}(p) - \delta_{2,2}(p))^{-1} \tag{7.5}$$

which follow from Algorithm 6.4. The signomial programs (7.1)-(7.2) and (7.3)-(7.4) can be reduced to geometric programs by using the method described in Section 7.4. *If* $\overline{p}^1$ *is the optimal solution of the problem (7.1)-(7.2) and* $\overline{p}^2$ *is the optimal solution of the problem (7.3)-(7.4), then the optimal transition matrix is* $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ *for which* $\mathbb{E}(T(\overline{p}))$ *is minimal. If there exists at least one posynomial* $\delta_{i^*,j^*}(p) \equiv 0$**,** *then, in (7.2) and (7.4), the corresponding posynomial equality constraint is eliminated, and the corresponding substitution* $d_{i^*,j^*} = 0$ *is made in (7.2), while the substitution* $d_{i^*,3-j^*} = 0$ *is made in (7.4).*

Proof. From Theorem 7.4 and the theoretical justification provided for Algorithm 6.4, it follows that the components of the generating vector $q(p)$ of the distribution $a = rep(T(p))$ of the evolution time $T(p)$ are signomials in the variables $p(x,y)$,

$x, y \in V$. Since $\mathbb{E}(T(p)) = \left.\dfrac{\partial G^{[a]}(z)}{\partial z}\right|_{z=1}$, by applying Theorem 1.29, we obtain that $\mathbb{E}(T(p))$ can be expressed as the ratio of two signomials. Since every signomial can be written as the difference of two posynomials, we obtain relation (7.5), where $\delta_{i,j}(p)$ is a posynomial, $i, j = \overline{1,2}$.

Denoting

$$d_{i,1} = \max\{\delta_{i,1}(p), \delta_{i,2}(p)\},\ d_{i,2} = \min\{\delta_{i,1}(p), \delta_{i,2}(p)\},$$

$$d_i = d_{i,1} - d_{i,2} > 0,\ i = \overline{1,2},$$

we obtain the relations

$$0 < \mathbb{E}(T(p)) = d_1 d_2^{-1},$$

$$d_{i,1}^{-1} d_i + d_{i,1}^{-1} d_{i,2} = 1,\ i = \overline{1,2},$$

$$d_{i,j}^{-1}\delta_{i,j}(p) = 1 \text{ or } d_{i,j}^{-1}\delta_{i,3-j}(p) = 1,\ i, j = \overline{1,2}.$$

The formulas

$$\sum_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X},$$

$$p(x,y) > 0,\ \forall (x,y) \in \overline{Y},$$

follow from Theorem 7.5.

Moreover, by applying Theorem 7.5, we can eliminate the variables $p(x,y)$, for which $p(x,y) = 0$ or $p(x,y) = 1$, performing the corresponding substitutions. Thus, we obtain the signomial programs (7.1)-(7.2) and (7.3)-(7.4). Since all the constraints $a' - g'$ and $a'' - g''$ are posynomial equality constraints, these signomial programs are geometric programs with posynomial equality constraints.

In what follows, we illustrate how these geometric programs with posynomial equality constraints can be reduced to geometric programs by using the method described in Section 7.4. We consider only the problem (7.1)-(7.2) since the theoretical justification for the problem (7.3)-(7.4) can be justified similarly.

Therefore, if the posynomials $\delta_{i,j}(p)$, $i,j=\overline{1,2}$, are not monomials, we can select the variable $d_{1,1}$ for the constraint b', the variable d_2 for c', the variable $d_{1,2}$ for e', the variable $d_{2,1}$ for f', the variable $d_{2,2}$ for g', an arbitrary variable $p(x^*,y^*)$ appearing in the posynomial $\delta_{1,1}(p)$ for the constraint d' and, for each $x\in\overline{X}$, an arbitrary variable $p(x,y^*(x))\neq p(x^*,y^*)$ appearing in the corresponding posynomial in a', for the constraints a'. These selected variables satisfy the properties described in Section 7.4, hence, the problem (7.1)-(7.2) can be treated using geometric programming.

If the posynomial $\delta_{1,2}(p)$, $\delta_{2,1}(p)$ or $\delta_{2,2}(p)$ is a monomial, then the corresponding constraint e', f' or g' is simply eliminated, and the corresponding substitution $d_{i^*,j^*}=\delta_{i^*,j^*}(p)$ is performed in the signomial program (7.1)-(7.2). The variables selected for the remaining constraints remain unchanged. Therefore, the problem (7.1)-(7.2) can be treated using geometric programming.

If the posynomial $\delta_{1,1}(p)$ is a nonconstant monomial, then the corresponding constraint d' is simply eliminated, and the corresponding substitution $d_{1,1}=\delta_{1,1}(p)$ is performed in the signomial program (7.1)-(7.2). The variables selected for the constraints a', c', e', f' and g' remain unchanged. Additionally, the variable $p(x^*,y^*)$, which appears in the posynomial $\delta_{1,1}(p)$, is selected for the constraint b'. These selected variables satisfy the properties described in Section 7.4, hence, the problem (7.1)-(7.2) can be treated using geometric programming.

If the posynomials $\delta_{1,1}(p)$ and $\delta_{1,2}(p)$ are both constant, then d_1 is also constant. In this case, the constraints b', d' and e' are simply eliminated. The variables selected for the remaining constraints remain unchanged. Therefore, the problem (7.1)-(7.2) can be treated using geometric programming.

If the posynomial $\delta_{1,1}(p)$ is constant and $\delta_{1,2}(p)$ is nonconstant, then the constraint d' is eliminated, and the substitution $d_{1,1} = \delta_{1,1}(p)$ is performed in the signomial program (7.1)-(7.2). We can select the variable $d_{1,2}$ for the constraint b', an arbitrary variable $p(x^{**}, y^{**})$ appearing in the posynomial $\delta_{1,2}(p)$ for the constraint e' and, for each $x \in \overline{X}$, an arbitrary variable $p(x, y^{**}(x)) \neq p(x^{**}, y^{**})$, appearing in the corresponding posynomial in a', for the constraints a'. The variables selected for the remaining constraints remain unchanged. These selected variables satisfy the properties described in Section 7.4, hence, the problem (7.1)-(7.2) can be treated using geometric programming.

Therefore, we have considered all possible cases. In conclusion, the problems (7.1)-(7.2) and (7.3)-(7.4) can be reduced to geometric programs. □

8 STOCHASTIC SYSTEMS WITH MULTIPLE FINAL SEQUENCES OF STATES

In this chapter we study the stochastic systems with multiple final sequences of states. We analyze both scenarios: the simplified case when the states of the system are independent and the case when the states of the system are interdependent. We prove that the distribution of the evolution time is a homogeneous linear recurrence.

Further, by using the Aho-Corasick and Knuth-Morris-Pratt string-searching algorithms, the order of the recurrence is reduced and enhanced methods are obtained. These approaches enable us to extend the methods and prove that the distribution of the total alert time is also a homogeneous linear recurrence. Moreover, the same strategy can be applied to study the duration of the games defined on stochastic systems with multiple final sequences of states.

Therefore, the obtained results allow us to apply the previously developed algorithms for finding the main probabilistic characteristics of the evolution time, total alert time and game duration. The complexity of the algorithms is polynomial.

8.1 Zero-order Markov processes with multiple final sequences of states

Let L be a discrete stochastic system with a finite set of states V, $|V|=\omega$. At every discrete time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The stochastic system L starts its evolution from the state v with probability $p^*(v)$, for all $v \in V$, where $\sum_{v \in V} p^*(v) = 1$.

Also, the transition from one state u to another state v is performed according to the same probability $p^*(v)$ that depends only on the destination state v, for every $u \in V$ and $v \in V$. Additionally, we assume that r different sequences of states

$$X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1,r},$$

are given and the stochastic system stops its evolution as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in the given order for an arbitrary $\ell \in \{1,2,\ldots,r\}$. The time T when the system stops is the evolution time of the stochastic system L with given final sequences of states

$$X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}.$$

The stochastic system L represents a zero-order Markov process with final sequences of states $X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}$, i.e. a stochastic system with multiple final sequences of states and independent states ([100]). The particular case $r = 1$ was studied in Section 6.2 and it was proved that, in this case, the distribution of the evolution time is a homogeneous linear recurrence sequence.

Next, we consider the generalization of this problem for any $r \geq 1$. This generalized problem is somewhat different from the parallel compositions, studied in Section 7.2, because the dynamics of the systems are mixed and the systems are interdependent.

Our goal is to analyze the evolution time T of the stochastic system L. We prove that the distribution of the evolution time T is a homogeneous linear recurrence. Also, we develop a polynomial algorithm to determine the initial state and the generating vector of this recurrence. Having the generating vector and the initial state of the recurrence, we can use Algorithm 4.4 to determine the main probabilistic characteristics of the evolution time.

In what follows, we will determine the distribution law of the evolution time T. Initially, for $k = \overline{1,m}$ and $\ell = \overline{1,r}$, we consider the notation

$$\begin{gathered} X_k^{(\ell)} = \{x_k^{(\ell)}\},\ \pi_k^{(\ell)} = p^*(x_k^{(\ell)}),\ w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell)}, \\ Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\}, \end{gathered} \tag{8.1}$$

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the evolution time T, i.e

$$a_n = \mathbb{P}(T = n),\ n = \overline{0,\infty}.$$

Since $T \geq m-1$, we have

$$a_n = 0,\ n = \overline{0,m-2}.$$

If $T = m-1$, then $\exists \ell \in \{1,2,\ldots,r\}$ such that

$$v(j) = x_{j+1}^{(\ell)},\ j = \overline{0,m-1},$$

which implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \sum_{\ell=1}^{r}\prod_{j=1}^{m} p^*(x_j^{(\ell)}) =$$

$$= \sum_{\ell=1}^{r}\left(\pi_1^{(\ell)}\pi_2^{(\ell)}\ldots\pi_m^{(\ell)}\right) = \sum_{\ell=1}^{r}\left(\pi_1^{(\ell)}w_m^{(\ell)}\right). \tag{8.2}$$

We consider $\forall n \in \mathbb{Z}$. Let $S(V) = \{A \,|\subseteq V\}$. Denote by $P_{\Phi}^{(\ell)}(n)$ the probability that $T = n$ and $v(j) \in \Phi_j$, $j = \overline{0,t-1}$, for all $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{1,r}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0,m}$, $\ell = \overline{1,r}$:

$$\begin{aligned} \beta_k^{(\ell)}(n) &= P_{(X_1^{(\ell)},X_2^{(\ell)},\ldots,X_k^{(\ell)})}(n), \\ \gamma_k^{(\ell)}(n) &= P_{(X_2^{(\ell)},X_3^{(\ell)},\ldots,X_k^{(\ell)})}(n). \end{aligned} \tag{8.3}$$

For $\forall n \geq m$, we have

$$\beta_k^{(\ell)}(n) = P_{(X_1^{(\ell)},X_2^{(\ell)},\ldots,X_k^{(\ell)})}(n) =$$

$$= \pi_1^{(\ell)}P_{(X_2^{(\ell)},\ldots,X_k^{(\ell)})}(n-1) - \pi_1^{(\ell)}\sum_{j=1}^{r}u_{j,k}^{(\ell)}P_{(X_2^{(j)},\ldots,X_m^{(j)})}(n-1) =$$

$$= \pi_1^{(\ell)}\left(\gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r}u_{j,k}^{(\ell)}\gamma_m^{(j)}(n-1)\right), k = \overline{1,m},\ \ell = \overline{1,r}, \tag{8.4}$$

where

$$u_{j,k}^{(\ell)} = \begin{cases} 1, & Y_k^{(j)} = Y_k^{(\ell)} \\ 0, & Y_k^{(j)} \neq Y_k^{(\ell)} \end{cases}. \tag{8.5}$$

In particular, for $k = 0$, we have

$$\beta_0^{(\ell)}(n) = P(n) = P(n-1) - \sum_{j=1}^{r} \pi_1^{(j)} P_{(X_2^{(j)},\ldots,X_m^{(j)})}(n-1) =$$

$$= \gamma_0^{(\ell)}(n-1) - \sum_{j=1}^{r} u_{j,0}^{(\ell)} \gamma_m^{(j)}(n-1), \ell = \overline{1,r}, \tag{8.6}$$

where

$$u_{j,0}^{(\ell)} = \pi_1^{(j)}. \tag{8.7}$$

We consider the sets

$$T_s^{(\ell)} = \{s+1\} \cup \{t \in \{2,3,\ldots,s\} \mid (x_t^{(\ell)}, x_{t+1}^{(\ell)}, \ldots, x_s^{(\ell)}) \in Y_{s+1-t}\},$$

for each $s = \overline{1,m}$ and $\ell = \overline{1,r}$. The minimum of each set is denoted by

$$t_s^{(\ell)} = \min_{k \in T_s^{(\ell)}} k, s = \overline{1,m}, \ell = \overline{1,r}. \tag{8.8}$$

It represents the smallest index in the sequence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_s^{(\ell)})$ from which, when a final sequence of states $X^{(\tau_s^{(\ell)})} \in X$ is superimposed on itself, the overlapping components coincide. Here by $\tau_s^{(\ell)}$ we denote the smallest index in the set $\{1,2,\ldots,r\}$ that satisfies the given condition.

Next, for $s = \overline{1,m}$ and $\ell = \overline{1,r}$, we obtain

$$\gamma_s^{(\ell)}(n) = P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_s^{(\ell)})}(n) =$$

$$= \pi_2^{(\ell)} \pi_3^{(\ell)} \ldots \pi_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_{t_s^{(\ell)}}, X_{t_s^{(\ell)}+1}, \ldots, X_s)}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} P_{(X_1^{(\tau_s^{(\ell)})}, X_2^{(\tau_s^{(\ell)})}, \ldots, X_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})})}(n - t_s^{(\ell)} + 2) =$$

$$= w_{t_s^{(\ell)}-1}^{(\ell)} \beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n - t_s^{(\ell)} + 2). \tag{8.9}$$

In particular, for $s = 0$, we have

$$\gamma_0^{(\ell)}(n) = a_n = \gamma_1^{(\ell)}(n) = w_1^{(\ell)} \beta_0^{(\tau_1^{(\ell)})}(n) = \beta_0^{(\ell)}(n),$$

which implies

$$\beta_0^{(\ell)}(n)=\gamma_0^{(\ell)}(n-1)-\sum_{j=1}^{r}u_{j,0}^{(\ell)}\gamma_m^{(j)}(n-1)=$$

$$=\beta_0^{(\ell)}(n-1)-\sum_{j=1}^{r}u_{j,0}^{(\ell)}w_{t_m^{(j)}-1}^{(j)}\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1) \tag{8.10}$$

and, for $k=\overline{1,m}$,

$$\beta_k^{(\ell)}(n)=\pi_1^{(\ell)}\left(\gamma_k^{(\ell)}(n-1)-\sum_{j=1}^{r}u_{j,k}^{(\ell)}\gamma_m^{(j)}(n-1)\right)=$$

$$=\pi_1^{(\ell)}\left(w_{t_k^{(\ell)}-1}^{(\ell)}\beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n-t_k^{(\ell)}+1)-\sum_{j=1}^{r}u_{j,k}^{(\ell)}w_{t_m^{(j)}-1}^{(j)}\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1)\right). \tag{8.11}$$

Since

$$2\le t_s^{(\ell)}\le s+1\le m+1,\ s=\overline{1,m},\ \ell=\overline{1,r},$$

there exist real coefficients $v_{jks\ell}^{(i)}$, $k,j,s=\overline{0,m-1}$, $i,\ell=\overline{1,r}$, such that

$$\beta_k^{(\ell)}(n)=\sum_{i=1}^{r}\sum_{j=0}^{m-1}\sum_{s=0}^{m-1}v_{jks\ell}^{(i)}\ \beta_s^{(i)}(n-1-j),\ k=\overline{0,m-1},\ \ell=\overline{1,r},\ \forall n\ge m.$$

Therefore, we have

$$\beta_k(n)=\sum_{j=0}^{m-1}\sum_{s=0}^{m-1}V_{jks}\ \beta_s(n-1-j),\ k=\overline{0,m-1},\ \forall n\ge m,$$

where $V_{jks}=(v_{jks\ell}^{(i)})_{\ell,\,i=\overline{1,r}}$, $\beta_k(n)=(\beta_k^{(\ell)}(n))_{\ell=\overline{1,r}}$, $k,j,s=\overline{0,\,m-1}$. This relation can be written in the form

$$\beta(n)=\sum_{j=0}^{m-1}V_j\ \beta(n-1-j),\ \forall n\ge m,$$

where $V_j=(V_{jks})_{k,s=\overline{0,m-1}}$ and $\beta(n)=((\beta_k(n))_{k=0}^{m-1})^T$, $j=\overline{0,m-1}$, $\forall n\in\mathbb{Z}$. We obtain

$$\beta=(\beta(n))_{n=0}^{\infty}\in Rol^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m],$$

$$V=(V_j)_{j=0}^{m-1}\in G^*[\mathcal{M}_m(\mathcal{M}_r(\mathbb{R}))][m](\beta),$$

which implies $\beta\in Rol^*[\mathbb{R}][m^2r]$, i.e.

$$(\beta_k^{(\ell)}(n))_{n=0}^{\infty}\in Rol^*[\mathbb{R}][m^2r],\ k=\overline{0,m-1},\ \ell=\overline{1,r},$$

with the same generating vector. Since

$$a_n = \beta_0^{(1)}(n), \forall n \geq 0, \tag{8.12}$$

we have

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2 r].$$

Next, we will use only the relation $a \in Rol^*[\mathbb{C}][m^2 r]$, the minimal generating vector being determined by using the rank-based order minimization method. The minimal generating vector $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$ is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{8.13}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}}, \forall n \in \mathbb{N}^* \tag{8.14}$$

and R is the rank of the matrix $A_{m^2 r}^{[a]}$.

In order to apply this minimization method, we need to have only the values a_k, $k = \overline{0, 2m^2 r - 1}$. These values can be determined using the recurrences (8.10) and (8.11) and the relations (8.1), (8.2), (8.5), (8.8) and (8.12).

In this way, we have theoretically justified the following algorithm for determining the main probabilistic characteristics of the evolution time T: the distribution $(\mathbb{P}(T = n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the standard deviation $\sigma(T)$ and the moment $\nu_k(T)$ of order k, $k = 1, 2, \ldots$.

Algorithm 8.1

Input: $X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m$, $\pi_k^{(\ell)}$, $k = \overline{1, m}$, $\ell = \overline{1, r}$;

Output: $\mathbb{E}(T), \mathbb{V}(T), \sigma(T), \nu_k(T), k = \overline{1, t}, t \geq 2$.

1. *Determine the values* a_k, $k = \overline{0, 2m^2 r - 1}$, *using the recurrences (8.10) and (8.11) and the relations (8.1), (8.2), (8.5), (8.8) and (8.12);*

2. *Find the minimal generating vector*

$$q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$$

by solving the system (8.13), taking into account the relation (8.14);

3. *Consider the distribution*

$$a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$$

of the evolution time T as a homogeneous linear recurrence with the initial state $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ and the minimal generating vector $q = (q_k)_{k=0}^{R-1}$, determined at the steps 1 and 2;

4. *Determine the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the standard deviation $\sigma(T)$ and the moment $\nu_k(T)$ of order k, $k = \overline{1,t}$, of the evolution time T, by using Algorithm 4.4.*

8.2 Stochastic systems with multiple final sequences of states

The zero-order Markov processes with multiple final sequences of states were presented in Section 8.1. The homogeneous linear recurrence property of the distribution was used to determine the main probabilistic characteristics of the evolution time by applying Algorithm 4.4.

In what follows, a generalization of these Markov processes is considered. In the same manner, stochastic systems with multiple final sequences of states ([89]) are defined. The difference is that the rows of the transition probability matrix are not all identical. This means that the transition probability from a source state to a destination state depends on both states, rather than solely on the destination state.

Therefore, we consider a discrete stochastic system L with a finite set of states $V = \{v_1, v_2, \ldots, v_\omega\}$, $|V| = \omega$. At each discrete time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The system L begins its evolution in the state v with probability $p^*(v)$, for each $v \in V$, where $\sum_{v \in V} p^*(v) = 1$.

At this point, we introduce the transition probability matrix of the Markov process L. A transition from a state u to another state v occurs with probability $p(u,v)$, which depends on both the source state u and the destination state v, for each $u \in V$ and $v \in V$. Additionally, suppose that r distinct sequences of states

$$X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1,r}$$

are given. The stochastic system terminates its evolution as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in the prescribed order, for some $\ell \in \{1,2,\ldots,r\}$. The time instant T, at which the system terminates, is called the evolution time of the stochastic system L with final sequences of states

$$X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\}.$$

Our objective is to analyze the evolution time T of the stochastic system L. We will prove that the distribution of the evolution time T satisfies a homogeneous linear recurrence and we will determine the initial state and the generating vector of this recurrence. Given the generating vector and the initial state, we can determine the main probabilistic characteristics of the evolution time by using Algorithm 4.4.

Consider $m \geq 2$ and let us introduce the following notation

$$\begin{aligned} &X_k^{(\ell)} = \{x_k^{(\ell)}\},\ \pi_k^{(\ell)} = p^*(x_k^{(\ell)}),\ \pi_{ik}^{(\ell)} = p(x_i^{(\ell)}, x_k^{(\ell)}),\ w_k^{(\ell)} = \prod_{j=3}^{k} \pi_{j-1,j}^{(\ell)}, \\ &Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\}, \end{aligned} \tag{8.15}$$

for $i,k = \overline{1,m}$ and $\ell = \overline{1,r}$.

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the evolution time T, i.e.

$$a_n = \mathbb{P}(T = n),\ n = \overline{0,\infty}.$$

Since $T \geq m-1$, we have

$$a_n = 0,\ n = \overline{0,m-2}.$$

If $T = m-1$, then $\exists \ell \in \{1,2,\ldots,r\}$ such that

$$v(j) = x_{j+1}^{(\ell)},\ j = \overline{0,m-1},$$

which implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \sum_{\ell=1}^{r} \left(p^*(x_1^{(\ell)}) \prod_{j=2}^{m} p(x_{j-1}^{(\ell)}, x_j^{(\ell)}) \right) =$$

$$= \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_{1,2}^{(\ell)} \pi_{2,3}^{(\ell)} \ldots \pi_{m-1,m}^{(\ell)} \right) = \sum_{\ell=1}^{r} \left(\pi_1^{(\ell)} \pi_{1,2}^{(\ell)} w_m^{(\ell)} \right). \tag{8.16}$$

We consider $\forall n \in \mathbb{Z}$ and $S(V) = \{A \mid A \subseteq V\}$. Let $P_\Phi(n)$ be the probability that $T = n$ and $v(j) \in \Phi_j$, $j = \overline{0, t-1}$, assuming that the initial state of the system is known, for each $\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t$, $t \in \mathbb{N}$ and $\ell = \overline{1, r}$. We consider several functions for $k = \overline{1, m}$, $\ell = \overline{1, r}$:

$$\begin{aligned} \beta_k^{(\ell)}(n) &= P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n), \\ \gamma_k^{(\ell)}(n) &= P_{(X_2^{(\ell)}, X_3^{(\ell)}, \ldots, X_k^{(\ell)})}(n). \end{aligned} \tag{8.17}$$

These functions $\beta_k^{(\ell)}(n)$ are extended to $\ell = \overline{r+1, r+\omega}$, in the following way:

$$\beta_k^{(r+i)}(n) = P_{(\{v_i\})}(n),\ i = \overline{1, \omega},\ k = \overline{1, m}, \tag{8.18}$$

where $v_1, v_2, \ldots, v_\omega$, as defined above, are all states of the stochastic system L in a predefined order. If, for each $x \in V$, we denote by $\iota(x)$ the index which satisfies the equality $v_{\iota(x)} = x$, then the relation (8.18) becomes

$$P_{(\{x\})}(n) = P_{(\{v_{\iota(x)}\})}(n) = \beta_1^{(r+\iota(x))}(n), \forall x \in V. \tag{8.19}$$

On the other hand, for each $x \in \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}$, there exists at least one index $\ell(x) \in \{1, 2, \ldots, r\}$ such that $x = x_1^{(\ell(x))}$. Therefore, in this case, we also have

$$\beta_k^{(r+\iota(x))}(n) = P_{(\{x\})}(n) = \beta_1^{(\ell(x))}(n),\ k = \overline{1, m}. \tag{8.20}$$

In contrast, for $x \notin \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}$, we can write the following relations:

$$\beta_k^{(r+\iota(x))}(n) = P_{(\{x\})}(n) = \sum_{y \in V} p(x, y) P_{(\{y\})}(n-1) =$$

$$= \sum_{y \in V} p(x, y) \beta_1^{(r+\iota(y))}(n-1),\ k = \overline{1, m}. \tag{8.21}$$

We first analyze the functions $\beta_k^{(\ell)}(n)$ for $\forall n \geq m$, $k = \overline{1,m}$, $\ell = \overline{1,r}$. For each $k > 1$ we have

$$\beta_k^{(\ell)}(n) = P_{(X_1^{(\ell)}, X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n) =$$

$$= \pi_{1,2}^{(\ell)} P_{(X_2^{(\ell)}, \ldots, X_k^{(\ell)})}(n-1) - \sum_{j:1 \leq j \leq r,\, Y_k^{(j)} = Y_k^{(\ell)}} \pi_{1,2}^{(j)} P_{(X_2^{(j)}, \ldots, X_m^{(j)})}(n-1) =$$

$$= \pi_{1,2}^{(\ell)} \gamma_k^{(\ell)}(n-1) - \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s | Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right). \tag{8.22}$$

Similarly, for $k = 1$, the following formula holds:

$$\beta_1^{(\ell)}(n) = P_{(X_1^{(\ell)})}(n) =$$

$$= \sum_{y \in V} p(x_1^{(\ell)}, y) P_{(\{y\})}(n-1) - \sum_{j:1 \leq j \leq r,\, Y_1^{(j)} = Y_1^{(\ell)}} \pi_{1,2}^{(j)} P_{(X_2^{(j)}, \ldots, X_m^{(j)})}(n-1) =$$

$$= \sum_{y \in V} p(x_1^{(\ell)}, y) \beta_1^{(r+\iota(y))}(n-1) - \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s | Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right). \tag{8.23}$$

By combining (8.22) − (8.23), we obtain

$$\beta_k^{(\ell)}(n) = \pi_{1,2}^{(\ell)} \cdot I_{\mathbb{N} \setminus \{1\}}(k) \cdot \gamma_k^{(\ell)}(n-1) +$$

$$+ \sum_{y \in V} \left(p(x_1^{(\ell)}, y) \cdot I_{\{1\}}(k) \cdot \beta_1^{(r+\iota(y))}(n-1) \right) -$$

$$- \sum_{j=1}^{r} \left(\pi_{1,2}^{(j)} \cdot I_{\{s | Y_k^{(s)} = Y_k^{(\ell)}\}}(j) \cdot \gamma_m^{(j)}(n-1) \right), \tag{8.24}$$

for $\forall n \geq m$, $k = \overline{1,m}$, $\ell = \overline{1,r}$

We consider the sets

$$T_s^{(\ell)} = \{s+1\} \cup \{t \in \{2,3,\ldots,s\} \mid (x_t^{(\ell)}, x_{t+1}^{(\ell)}, \ldots, x_s^{(\ell)}) \in Y_{s+1-t}\},$$

for each $s = \overline{1,m}$ and $\ell = \overline{1,r}$. The minimum of each set is denoted by

$$t_s^{(\ell)} = \min_{k \in T_s^{(\ell)}} k,\ s = \overline{1,m}, \ell = \overline{1,r}. \tag{8.25}$$

It represents the smallest index in the sequence $(x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_s^{(\ell)})$, from which, when a final sequence of states $X^{(\tau_s^{(\ell)})} \in X$ is superimposed on itself, the overlapping com-

ponents coincide. Here by $\tau_s^{(\ell)}$ we denote the smallest index in the set $\{1,2,\ldots,r\}$ that satisfies the given condition.

Next, we analyze the expression $\gamma_s^{(\ell)}(n-1)$ for $s=\overline{1,m}$ and $\ell=\overline{1,r}$. For $s=1$ we have

$$\gamma_1^{(\ell)}(n-1)=P_{()}(n-1)=\sum_{x\in V}P_{(\{x\})}(n-1)=\sum_{x\in V}\beta_1^{(r+\iota(x))}(n-1). \tag{8.26}$$

For $s=\overline{2,m}$, there are two cases. When $t_s^{(\ell)}\le s$ we obtain

$$\gamma_s^{(\ell)}(n-1)=P_{(X_2^{(\ell)},X_3^{(\ell)},\ldots,X_s^{(\ell)})}(n-1)=$$

$$=\pi_{2,3}^{(\ell)}\pi_{3,4}^{(\ell)}\ldots\pi_{t_s^{(\ell)}-1,t_s^{(\ell)}}^{(\ell)}P_{(X_{t_s^{(\ell)}},X_{t_s^{(\ell)}+1},\ldots,X_s)}(n-t_s^{(\ell)}+1)=$$

$$=w_{t_s^{(\ell)}}^{(\ell)}P_{(X_1^{(\tau_s^{(\ell)})},X_2^{(\tau_s^{(\ell)})},\ldots,X_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})})}(n-t_s^{(\ell)}+1)=w_{t_s^{(\ell)}}^{(\ell)}\beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n-t_s^{(\ell)}+1). \tag{8.27}$$

and for $t_s^{(\ell)}=s+1$ we have

$$\gamma_s^{(\ell)}(n-1)=P_{(X_2^{(\ell)},X_3^{(\ell)},\ldots,X_s^{(\ell)})}(n-1)=$$

$$=\pi_{2,3}^{(\ell)}\pi_{3,4}^{(\ell)}\ldots\pi_{s-1,s}^{(\ell)}P_{(x_s^{(\ell)})}(n-s+1)=$$

$$=w_s^{(\ell)}\beta_1^{(r+\iota(x_s^{(\ell)}))}(n-s+1). \tag{8.28}$$

By combining (8.26) − (8.28), we obtain

$$\gamma_s^{(\ell)}(n-1)=\sum_{x\in V}I_{\{1\}}(s)\cdot\beta_1^{(r+\iota(x))}(n-1)+$$

$$+w_{t_s^{(\ell)}}^{(\ell)}\cdot I_{\{k|2\le t_k^{(\ell)}\le k\}}(s)\cdot\beta_{s+1-t_s^{(\ell)}}^{(\tau_s^{(\ell)})}(n-t_s^{(\ell)}+1)+$$

$$+w_s^{(\ell)}\cdot I_{\{k|3\le t_k^{(\ell)}=k+1\}}(s)\cdot\beta_1^{(r+\iota(x_s^{(\ell)}))}(n-s+1), \tag{8.29}$$

for $\forall n\ge m$, $s=\overline{1,m}$, $\ell=\overline{1,r}$.

Since

$$I_{\{1\}}(m)=0,\ \forall m\ge 2,\ I_{\mathbb{N}\setminus\{1\}}(k)\cdot I_{\{1\}}(k)=0,$$

by substituting the relation (8.15) in (8.24), we obtain the following recurrence for $k=\overline{1,m}$, $\ell=\overline{1,r}$ and $\forall n\in\mathbb{Z}$:

$$\beta_k^{(\ell)}(n)=\pi_{1,2}^{(\ell)}\cdot I_{\mathbb{N}\setminus\{1\}}(k)\cdot\gamma_k^{(\ell)}(n-1)+$$

$$+\sum_{y\in V}\Big(p(x_1^{(\ell)},y)\cdot I_{\{1\}}(k)\cdot\beta_1^{(r+\iota(y))}(n-1)\Big)-$$

$$-\sum_{j=1}^{r}\left(\pi_{1,2}^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)}\}}(j)\cdot\gamma_m^{(j)}(n-1)\right)=$$

$$=\sum_{y\in V}\Big(p(x_1^{(\ell)},y)\cdot I_{\{1\}}(k)\cdot\beta_1^{(r+\iota(y))}(n-1)\Big)+$$

$$+\Bigg[\pi_{1,2}^{(\ell)}\cdot I_{\mathbb{N}\setminus\{1\}}(k)\cdot\Bigg(\sum_{y\in V}I_{\{1\}}(k)\cdot\beta_1^{(r+\iota(y))}(n-1)+$$

$$+w_{t_k^{(\ell)}}^{(\ell)}\cdot I_{\{s|2\le t_s^{(\ell)}\le s\}}(k)\cdot\beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n-t_k^{(\ell)}+1)+$$

$$+w_k^{(\ell)}\cdot I_{\{s|3\le t_s^{(\ell)}=s+1\}}(k)\cdot\beta_1^{(r+\iota(x_k^{(\ell)}))}(n-k+1)\Bigg)\Bigg]-$$

$$-\sum_{j=1}^{r}\Bigg[\pi_{1,2}^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)}\}}(j)\cdot\Bigg(\sum_{y\in V}I_{\{1\}}(m)\cdot\beta_1^{(r+\iota(y))}(n-1)+$$

$$+w_{t_m^{(j)}}^{(j)}\cdot I_{\{s|2\le t_s^{(j)}\le s\}}(m)\cdot\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1)+$$

$$+w_m^{(j)}\cdot I_{\{s|3\le t_s^{(j)}=s+1\}}(m)\cdot\beta_1^{(r+\iota(x_m^{(j)}))}(n-m+1)\Bigg)\Bigg]=$$

$$=\sum_{y\in V}\Big(p(x_1^{(\ell)},y)\cdot I_{\{1\}}(k)\Big)\cdot\beta_1^{(r+\iota(y))}(n-1)+$$

$$+\pi_{1,2}^{(\ell)}\cdot w_{t_k^{(\ell)}}^{(\ell)}\cdot I_{\{s|2\le t_s^{(\ell)}\le s\}}(k)\cdot\beta_{k+1-t_k^{(\ell)}}^{(\tau_k^{(\ell)})}(n-t_k^{(\ell)}+1)+$$

$$+\pi_{1,2}^{(\ell)}\cdot w_k^{(\ell)}\cdot I_{\{s|3\le t_s^{(\ell)}=s+1\}}(k)\cdot\beta_1^{(r+\iota(x_k^{(\ell)}))}(n-k+1)-$$

$$-\sum_{j=1}^{r}\left(\pi_{1,2}^{(j)}\cdot w_{t_m^{(j)}}^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)},\,2\le t_m^{(j)}\le m\}}(j)\cdot\beta_{m+1-t_m^{(j)}}^{(\tau_m^{(j)})}(n-t_m^{(j)}+1)\right)-$$

$$-\sum_{j=1}^{r}\left(\pi_{1,2}^{(j)}\cdot w_m^{(j)}\cdot I_{\{s|Y_k^{(s)}=Y_k^{(\ell)},\,3\le t_m^{(j)}=m+1\}}(j)\cdot\beta_1^{(r+\iota(x_m^{(j)}))}(n-m+1)\right). \tag{8.30}$$

From the recurrence relations (8.20), (8.21) and (8.30), there exist real coefficients $v_{jks\ell}^{(i)}$, $j=\overline{0,m-1}$, $k,s=\overline{1,m}$, $i,\ell=\overline{1,r+\omega}$, such that

$$\beta_k^{(\ell)}(n)=\sum_{i=1}^{r+\omega}\sum_{j=0}^{m-1}\sum_{s=1}^{m}v_{jks\ell}^{(i)}\beta_s^{(i)}(n-1-j),\ k=\overline{1,m},\ \ell=\overline{1,r+\omega},\ \forall n\ge m. \tag{8.31}$$

Therefore, we have

$$\beta_k(n)=\sum_{j=0}^{m-1}\sum_{s=1}^{m}V_{jks}\ \beta_s(n-1-j),\ k=\overline{1,m},\ \forall n\ge m,$$

where

$$V_{jks}=(v_{jks\ell}^{(i)})_{\ell,=\overline{1,r+\omega}},\ \beta_k(n)=(\beta_k^{(\ell)}(n))_{\ell=\overline{1,r+\omega}},\ k,s=\overline{1,m},\ j=\overline{0,m-1}.$$

This recurrence can be written in the form

$$\beta(n)=\sum_{j=0}^{m-1}V_j\ \beta(n-1-j),\ \forall n\ge m,$$

where

$$V_j=(V_{jks})_{k,s=\overline{1,m}},\ \beta(n)=((\beta_k(n))_{k=1}^{m})^T,\ j=\overline{0,m-1},\ \forall n\in\mathbb{Z}.$$

From here, we have

$$\beta=(\beta(n))_{n=0}^{\infty}\in Rol^*[\mathcal{M}_m(\mathcal{M}_{r+\omega}(\mathbb{R}))][m],$$

$$V=(V_j)_{j=0}^{m-1}\in G^*[\mathcal{M}_m(\mathcal{M}_{r+\omega}(\mathbb{R}))][m](\beta).$$

Consequently, we obtain

$$\beta\in Rol^*[\mathbb{R}][m^2(r+\omega)],$$

which also implies

$$(\beta_k^{(\ell)}(n))_{n=0}^{\infty}\in Rol^*[\mathbb{R}][m^2(r+\omega)],\ k=\overline{1,m},\ \ell=\overline{1,r+\omega},$$

with the same generating vector. Since

$$a_n=\sum_{x\in V}p^*(x)P_{(x)}(n)=\sum_{x\in V}p^*(x)\beta_1^{(r+\iota(x))}(n),\ \forall n\in\mathbb{Z}, \tag{8.32}$$

we have

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][m^2(r+\omega)].$$

In what follows, we will use only the relation

$$a \in Rol^*[\mathbb{C}][m^2(r+\omega)],$$

the minimal generating vector being determined by using the rank-based order minimization method. Thus, according to this method, the minimal generating vector

$$q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$$

is obtained from the unique solution $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{8.33}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n \in \mathbb{N}^* \tag{8.34}$$

and R is the rank of the matrix $A_{m^2(r+\omega)}^{[a]}$.

To apply this minimization method, the values a_k, $k = \overline{0, 2m^2(r+\omega)-1}$, are required. These values can be found using the recurrences (8.16), (8.20), (8.21), (8.30) and (8.32).

In the case $m = 1$, we have a recurrence formula similar to (8.31). It is easy to observe that the following relations hold for $P_{(\{x\})}(n)$, $\forall x \in V$, $\forall n \in \mathbb{N}$:

$$\beta_1^{(\ell)}(0) = \beta_1^{(r+\iota(x_1^{(\ell)}))}(0) = P_{(X_1^{(\ell)})}(0) = 1,$$

$$\beta_1^{(\ell)}(n) = \beta_1^{(r+\iota(x_1^{(\ell)}))}(n) = P_{(X_1^{(\ell)})}(n) = 0,\ \forall n \geq 1,\ \ell = \overline{1,r},$$

$$\beta_1^{(r+\iota(x))}(0) = P_{(\{x\})}(0) = 0,$$

$$\beta_1^{(r+\iota(x))}(n) = P_{(\{x\})}(n) = \sum_{y \in V} p(x,y) P_{(\{y\})}(n-1) =$$

$$= \sum_{y \in V} p(x,y) \beta_1^{(r+\iota(y))}(n-1),\ \forall n \geq 1,\ \forall x \in V \setminus \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}. \tag{8.35}$$

Finally, these relations can be written in the same form as (8.31):

$$\beta_1^{(\ell)}(n)=\sum_{i=1}^{r+\omega}\sum_{j=0}^{0}\sum_{s=1}^{1}v_{jks\ell}^{(i)}\beta_s^{(i)}(n-1-j),\ \ell=\overline{1,r+\omega},\ \forall n\geq 1.$$

Therefore, the homogeneous linear recurrence for the sequence $a=(a_n)_{n=0}^{\infty}$, proved above, can also be applied to the particular case $m=1$.

In this way, we have theoretically justified the following algorithm for determining the main probabilistic characteristics of the evolution time T: the distribution $(\mathbb{P}(T=n))_{n=0}^{\infty}$, the expectation $\mathbb{E}(T)$, the variance $\mathbb{V}(T)$, the standard deviation $\sigma(T)$ and the moment $\nu_k(T)$ of order k, $k=1,2,\ldots$.

Algorithm 8.2

Input: $X^{(\ell)}=(x_1^{(\ell)},x_2^{(\ell)},\ldots,x_m^{(\ell)})\in V^m$, $\ell=\overline{1,r}$, $p^*(x)$ and $p(x,y)$, $\forall x,y\in V$;

Output: $\mathbb{E}(T),\mathbb{V}(T),\sigma(T),\nu_k(T),k=\overline{1,t},t\geq 2$.

1. *Determine the values* a_k, $k=\overline{0,2m^2(r+\omega)-1}$, *by using the recurrence (8.30) and the relations (8.16), (8.20), (8.21), (8.32) and (8.35);*
2. *Find the minimal generating vector*

$$q=(q_0,q_1,\ldots,q_{R-1})\in G^*[\mathbb{R}][R](a)$$

by solving the system (8.33), taking into account the relation (8.34);

3. *Consider the distribution*

$$a=(a_n)_{n=0}^{\infty}=(\mathbb{P}(T=n))_{n=0}^{\infty}$$

of the evolution time T *as a homogeneous linear recurrence with the initial state* $I_R^{[a]}=(a_n)_{n=0}^{R-1}$ *and the minimal generating vector* $q=(q_k)_{k=0}^{R-1}$, *determined at the steps 1 and 2;*

4. *Determine the expectation* $\mathbb{E}(T)$, *the variance* $\mathbb{V}(T)$, *the standard deviation* $\sigma(T)$ *and the moment* $\nu_k(T)$ *of order* k, $k=\overline{1,t}$, *of the evolution time* T, *by using Algorithm 4.4.*

8.3 State merging for zero-order Markov processes with a final sequence of states

A zero-order Markov process with a final sequence of states was defined as a stochastic system L with a finite set of states V, $|V| = \omega$. For each $v \in V$, the system L starts its evolution from v with probability $p^*(v)$. Also, for every two states $u, v \in V$, the stochastic system L passes from u to v with the same probability $p^*(v)$. The stochastic system L was enriched with a sequence of states $X = (x_1, x_2, \ldots, x_m) \in V^m$ and it terminates its evolution when the states $x_1, x_2, \ldots, x_m$ are consecutively reached.

The evolution time of this system represents the amount of time needed for the system to reach the final sequence of states. So, provided that the system starts its evolution at time $t = 0$ and each transition takes one unit of time, the moment in time T when the system stops is the evolution time of the stochastic system L with the given final sequence of states X (of length m).

The distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T was studied in Section 6.2. It was theoretically established that $a \in Rol[\mathbb{R}][m]$, which means that the sequence a represents a non-degenerate homogeneous linear recurrence of order m over the set $\mathbb{R}$.

Moreover, it was proved that $m = dim[\mathbb{R}](a)$, i.e. the recurrence is $m-$minimal over the set $\mathbb{R}$. This means that there is no place for improvements by trying to reduce the length of the final sequence of states. Even if it had been possible to handle the zero-order Markov process as a standard Markov process and apply optimizations given by Knuth–Morris–Pratt and Aho–Corasick string-searching algorithms (as will be shown in the following sections), this would not have led to better results, because the order of the recurrence had already been minimal.

Anyway, we can slightly improve the computations by applying a state merging strategy. Since all the states are independent, the states not belonging to the final

sequence of states can be merged together into a single macro-state. In this way, all the states from $V \setminus \{x_1, x_2, \ldots, x_m\}$ form a single macro-state ϖ, for which we have

$$p^*(\varpi) = \sum_{v \in V \setminus \{x_1, x_2, \ldots, x_m\}} p^*(v) = 1 - \sum_{v \in \{x_1, x_2, \ldots, x_m\}} p^*(v).$$

The stochastic system L can be handled as a zero-order Markov process with a smaller set of states $\{\varpi, x_1, x_2, \ldots, x_m\}$.

Also, there is an important advantage of this state merging strategy. It also allows us to consider the case when the set of states is infinite. By merging the states not belonging to the final sequence of states, the set of states becomes finite.

8.4 Knuth-Morris-Pratt approach for Markov processes with a final sequence of states

Next, we extend our study to the standard Markov processes with interdependent states and final sequence of states. These stochastic systems were investigated in Section 6.5.

Let L be such a system. Similarly to zero-order Markov processes, the system L starts its evolution from each state $v \in V$ with probability $p^*(v)$ and stops when the final sequence of states X is reached. The difference is that, for every two states $u, v \in V$, the stochastic system L passes from state u to the state v with probability $p(u,v)$ that depends on both states, u and v.

The distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T was studied in Section 6.5. It was theoretically established that $a \in Rol^*[\mathbb{R}][m\omega]$, which means that the sequence a represents a homogeneous linear recurrence of order $m\omega$ over $\mathbb{R}$.

The issue here is that the order $m\omega$ may be non-minimal. We will explore the Knuth–Morris–Pratt algorithm for decreasing this order of recurrence.

As presented in [74], the idea of this string-searching algorithm is to keep track of the longest prefix matched so far while searching for an occurrence of a word within another string. At each position in the string, if the current character is the next

expected character, it is appended to the longest prefix matched so far; otherwise, the prefix is reset or the search falls back to the longest prefix ending with the current character.

We apply this idea to our stochastic system L. We enrich each state $v \in V$ with an additional number k, $0 \le k \le m$, that represents the length of the longest prefix of the final sequence of states matched so far. In other words, the stochastic system L is in the state (v,k) at discrete time t if at the last k moments of time $t-k+1, t-k+2, \ldots, t$ it has been in the states $x_1, x_2, \ldots, x_k$ and this number k is the largest one that satisfies this property. The number k is equal to 0 if no such sequence exists (i.e. the current state is not present in the final sequence of states or it is not critical).

In order to minimize the number of joint states, we consider only the states that are possible. So, for each state $v \ne x_1$, we have the possible state $(v,0)$. The state $(x_1,0)$ is not possible because, in the worst case, the state x_1 acts as fallback indicating the starting of a new matching of the final sequence of states, i.e. the state $(x_1,1)$. Additionally, for each k, $2 \le k \le m$, we have the possible state (x_k,k). In this way, we have built the set of all possible states

$$Z = \bigcup_{k=0}^{m} Z^{(k)},$$

where

$$Z^{(0)} = \{(v,0) \mid v \in V \setminus \{x_1\}\},$$

$$Z^{(k)} = \{(x_k,k)\}, k = \overline{1,m}.$$

This set contains $|Z| = \omega - 1 + m$ states.

We now compute the initial probabilities and transition probability matrix for the new states. It is easy to observe that

$$p^*((v,0)) = p^*(v), \forall v \in V \setminus \{x_1\},$$

$$p^*((x_1,1)) = p^*(x_1),\ p^*((x_k,k)) = 0, k = \overline{2,m}.$$

Regarding transition probabilities, we have the following relations:

$$p((u,0),(v,0)) = p(u,v),\ \forall u,v \in V \setminus \{x_1\},$$

$$p((u,0),(x_1,1)) = p(u,x_1),\ \forall u \in V \setminus \{x_1\},$$

$$p((x_k,k),(v,0)) = p(x_k,v),\ \forall v \in V \setminus \{x_1, x_2, \ldots, x_m\},\ k = \overline{1,m},$$

$$p((x_k,k),(x_{k+1},k+1)) = p(x_k,x_{k+1}),\ k = \overline{1,m-1},$$

$$p((x_k,k),(v,f_k(v))) = p(x_k,v),\ \forall v \in \{x_1, x_2, \ldots, x_m\} \setminus \{x_{k+1}\},\ k = \overline{1,m-1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (x_k,k). For the remaining transitions, the probability is equal to 0.

In order to clarify more accurately what the fallback position means, let us look again at its definition. The position $f_k(v)$ is said to be the fallback position for a transition from the joint state (x_k,k) to the state $v \in \{x_{1,2}, \ldots, x_m\} \setminus \{x_{k+1}\}$ if we have $0 \le f_k(v) \le k$ and $f_k(v)$ is the maximal integer that satisfies the equality

$$(x_1, x_2, \ldots, x_{f_k(v)}) = (x_{k-f_k(v)+2}, x_{k-f_k(v)+3}, \ldots, x_k, v).$$

In other words, when the state v occurs instead of x_{k+1} while being in the joint state (x_k,k), a shortcut happens and the critical occurrence $(x_1,x_2,\ldots,x_k)$ is replaced by the shorter occurrence $(x_1,x_2,\ldots,x_{f_k(v)})$.

Now, having all the transition probabilities well defined, we can easily note that they do not depend on the time when the transition is performed. So, the new stochastic system is also a Markov process. Even if the number of the states was increased, from ω to $\omega_{new} = \omega - 1 + m$, the improvement can be seen in the final sequence of states. Indeed, the new stochastic system has only one final state, i.e. $m_{new} = 1$, because the final sequence of states is $X_{new} = ((x_m,m))$. Instead, the evolution time remains unchanged. As a result, the order of the homogeneous linear recurrence, which describes the distribution of the evolution time, is reduced from $m\omega$ to

$$m_{new} \cdot \omega_{new} = 1 \cdot (\omega - 1 + m) = \omega - 1 + m.$$

8.5 Aho-Corasick approach for Markov processes with final sequences of states

In this section we analyze the complexity reduction for the Markov processes with multiple final sequences of states, introduced and studied in Section 8.2. For such a system, r different sequences of states

$$X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m, \ell = \overline{1,r},$$

are given and the stochastic system stops as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in the given order for an arbitrary $\ell \in \{1,2,\ldots,r\}$.

Let us recall several sets and sequences used in Section 8.2 and additional notation for $k = \overline{1,m}$, $\ell = \overline{1,r}$:

$$X_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(r)}\},\ X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\},$$

$$Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\},$$

$$Z_k^{(\ell)} = \{y \in X_k \mid (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_{k-1}^{(\ell)}, y) \in Y_k\}.$$

From Section 8.2 we know that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T is a homogeneous linear recurrence of order $m^2(r+\omega)$ over the set $\mathbb{R}$. The issue here is that the order $m^2(r+\omega)$ may be non-minimal. Next, we explore the idea of Aho–Corasick algorithm for decreasing this order of the recurrence.

Based on [1], the Aho–Corasick algorithm simultaneously matches multiple words in an input text. This string-searching algorithm tracks the longest prefix matched so far for each word searched within another string. The common prefixes are reused, in order to avoid repetitions and guarantee the uniqueness of the states. At each position in the string, if the current character is the next expected character from a word, it is appended to the longest prefix matched so far for a given word (or another word sharing that prefix with the given word), otherwise the prefix is reset or the search falls back to the longest prefix ending with the current character.

Thus, we apply this idea to our stochastic system L. We enrich each state $v \in V$ with two additional characteristics: the number k, $0 \le k \le m$, which represents

the length of the longest prefix of the final sequences of states matched so far and the corresponding prefix

$$Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),\ \ell \in \{0,1,\ldots,r\}.$$

In other words, the stochastic system L is in the state $(v, Y_k^{(\ell)}, k)$ at time t if at the last k moments of time $t-k+1, t-k+2, \ldots, t$ it has been in the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}$ and this number k is the largest one that satisfies this property among all final sequences of states (i.e. there is no larger k even for another ℓ, $1 \le \ell \le r$). The number k is equal to 0 if no such sequence exists (i.e. the current state is not present in the final sequences of states or it is not critical) and the prefix is considered $Y_0^{(0)} = ()$.

In order to minimize the number of joint states, we consider only the states that are possible and we merge the equivalent states (the joint states that share the same prefix). So, for each state $v \notin X_1 = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(r)}\}$, we have the possible state $(v, Y_0^{(0)}, 0)$. The states of the form $(x_1^{(\ell)}, Y_0^{(0)}, 0)$ are not possible because, in the worst case, the state $x_1^{(\ell)}$ acts as fallback indicating the starting of a new matching of the final sequence of states $X^{(\ell)}$, i.e. the state $(x_1^{(\ell)}, Y_1^{(\ell)}, 1)$. Additionally, for each k and ℓ, $2 \le k \le m$, $1 \le \ell \le r$, we have the possible state $(x_k^{(\ell)}, Y_k^{(\ell)}, k)$. In this way, we have built the set of all possible states

$$Z = \bigcup_{k=0}^{m} Z^{(k)},$$

where

$$Z^{(0)} = \{(v, Y_0^{(0)}, 0) \mid v \in V \setminus X_1\},$$

$$Z^{(k)} = \{(x_k^{(\ell)}, Y_k^{(\ell)}, k) \mid 1 \le \ell \le r\},\ k = \overline{1,m}.$$

This set contains

$$|Z| \le \omega + r(m-1)$$

states (the sign "$\le$" is due to the possible existence of the shared prefixes).

We now compute the initial probabilities and transition probability matrix for the new states. It is easy to observe that

$$p^*((v,Y_0^{(0)},0)) = p^*(v),\ \forall v \in V \setminus X_1,$$

$$p^*((x_1^{(\ell)},Y_1^{(\ell)},1)) = p^*(x_1^{(\ell)}),\ \ell = \overline{1,r},$$

$$p^*((x_k^{(\ell)},Y_k^{(\ell)},k)) = 0,\ k = \overline{2,m},\ \ell = \overline{1,r}.$$

Regarding transition probabilities, we have the following relations:

$$p((u,Y_0^{(0)},0),(v,Y_0^{(0)},0)) = p(u,v),\ \forall u,v \in V \setminus X_1,$$

$$p((u,Y_0^{(0)},0),(x_1^{(\ell)},Y_1^{(\ell)},1)) = p(u,x_1^{(\ell)}),\ \forall u \in V \setminus X_1,\ \ell = \overline{1,r},$$

$$p((x_k^{(\ell)},Y_k^{(\ell)},k),(v,Y_0^{(0)},0)) = p(x_k^{(\ell)},v),\ \forall v \in V \setminus \bigcup_{j=1}^{m} X_j,\ k = \overline{1,m},\ \ell = \overline{1,r},$$

$$p((x_k^{(\ell)},Y_k^{(\ell)},k),(x_{k+1}^{(\ell)},Y_{k+1}^{(\ell)},k+1)) = p(x_k^{(\ell)},x_{k+1}^{(\ell)}),\ k = \overline{1,m-1},\ \ell = \overline{1,r},$$

$$p((x_k^{(\ell)},Y_k^{(\ell)},k),(v,Y_\kappa^{(\ell_\kappa)},\kappa)) = p(x_k^{(\ell)},v),\ \forall v \in \bigcup_{j=1}^{m} X_j \setminus Z_{k+1}^{(\ell)},\ k = \overline{1,m-1},\ \ell = \overline{1,r},$$

where $\kappa = f_k^{(\ell)}(v)$ is the fallback position of the next state v when the current state is $(x_k^{(\ell)},Y_k^{(\ell)},k)$ and $\ell_\kappa \in \{1,2,\ldots,r\}$ is the index of the related final sequence of states for which the fallback happens. For the remaining transitions, the probability is 0.

In this context, the position $f_k^{(\ell)}(v)$ is the fallback position for a transition from the joint state $(x_k^{(\ell)},Y_k^{(\ell)},k)$ to the state $v \in \bigcup_{j=1}^{m} X_j \setminus Z_{k+1}^{(\ell)}$ because $0 \le f_k^{(\ell)}(v) \le k$ and $\kappa = f_k^{(\ell)}(v)$ is the maximal integer that satisfies the equality

$$(x_1^{(\ell_\kappa)},x_2^{(\ell_\kappa)},\ldots,x_\kappa^{(\ell_\kappa)}) = (x_{k-\kappa+2}^{(\ell)},x_{k-\kappa+3}^{(\ell)},\ldots,x_k^{(\ell)},v)$$

for an index $\ell_\kappa \in \{1,2,\ldots,r\}$. This means that, when the state v occurs instead of $x_{k+1}^{(\ell)}$ while being in the joint state $(x_k^{(\ell)},Y_k^{(\ell)},k)$, a shortcut happens and the critical occurrence $(x_1^{(\ell)},x_2^{(\ell)},\ldots,x_k^{(\ell)})$ is replaced by the shorter one $(x_1^{(\ell_\kappa)},x_2^{(\ell_\kappa)},\ldots,x_\kappa^{(\ell_\kappa)})$.

As a result, we can readily observe that the new stochastic system is also a Markov process with multiple final sequences of states. Even if the number of the

states was eventually increased, from ω to $\omega_{new} \le \omega + r(m-1)$, the improvement can be seen in the final sequences of states. The new stochastic system has r final sequences of states of length $m_{new} = 1$, because the final sequences of states are

$$X_{new} = \{((x_m^{(1)}, X^{(1)}, m)), ((x_m^{(2)}, X^{(2)}, m)), \ldots, ((x_m^{(r)}, X^{(r)}, m))\}.$$

The evolution time remains unchanged and the order of the homogeneous linear recurrence that describes its distribution is reduced from $m^2(r+\omega)$ to

$$m_{new}^2 \cdot (r + \omega_{new}) \le 1^2 \cdot (r + \omega + r(m-1)) = \omega + mr.$$

8.6 Enhanced method for zero-order Markov processes with final sequences of states

The zero-order Markov processes with multiple final sequences of states were studied in Section 8.1. They represent the particular case of the Markov processes with multiple final sequences of states, when the states of the system are independent.

From Section 8.1 we know that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the evolution time T is a homogeneous linear recurrence of order $m^2 r$ over $\mathbb{R}$. By applying the state-merging technique, presented in Section 8.3, we can reduce the number of states from ω to

$$\omega^* = \left| \bigcup_{k=1}^{m} X_k \right| + 1 \le mr + 1,$$

where $X_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(r)}\}$. Next, considering the obtained zero-order Markov process as a standard Markov process and applying the enhancement developed in Section 8.5, we obtain a new Markov process with r final sequences of states of length $m_{new} = 1$ and the number of states

$$\omega_{new} \le \omega^* + r(m-1) \le 2mr + 1 - r.$$

The evolution time remains unchanged and the order of the homogeneous linear recurrence that describes its distribution is reduced, finally, from $m^2 r$ to

$$\omega^* + mr \le 2mr + 1.$$

8.7 Total alert time of Markov processes with multiple final sequences of states

According to Section 6.3, a zero-order Markov process with final critical state v^* (of degree m) represents a zero-order Markov process L with the final sequence of states $X=(x_1,x_2,\ldots,x_m)$, where $x_1=x_2=\ldots=x_m=v^*$ and $p=p^*(v^*)$ is the probability of the critical state. The total alert time η was defined as the cumulative time, during which the system remains in its critical state throughout its evolution. Also, the moment of the first alert of order k, $k=\overline{1,m}$, was introduced as the moment in time when, for the first time, the system L consecutively passes k times through its critical state. The number $\gamma=\eta+1$ indicates how many times the critical state is visited.

It is already known from Section 6.3 that the distribution of the evolution time T satisfies the following relations:

$$a=(\mathbb{P}(T=n))_{n=0}^{\infty}\in Rol[\mathbb{R}][m],$$

$$q=((1-p)p^k)_{k=0}^{m-1}\in G[\mathbb{R}][m](a).$$

The expectation and the variance of the evolution time T are:

$$\mathbb{E}(T)=\overline{m}=m-2+\frac{1+\Delta_1(q)}{w_m},$$

$$\mathbb{V}(T)=(m-1)(m-2)+\overline{m}(\overline{m}-2m+3)+\frac{\Delta_1(q)+\Delta_2(q)}{w_m},$$

where

$$w_m=\prod_{k=1}^{m}p^*(x_k)\neq 0,\ \Delta_j(q)=\sum_{k=1}^{m-1}k^jq_k,\ \forall j\in\mathbb{N}^*.$$

To determine the total alert time and the moments of the first alert, we consider the related systems L_k, $k=\overline{1,m}$. The system L_k has the same set of states V, the same critical state v^* and the same probability $p=p^*(v^*)$ of the critical state. The difference is made by the degree of the critical state, which is equal to k for the sto-

chastic system L_k, $k=\overline{1,m}$. It was shown in Section 6.3 that μ_k represents the evolution time of the system L_k, $k=\overline{1,m}$. Also, the total alert time η_k of the system L_k and $\mu_{k-1}+1$ have the same distribution for all $k>1$, where $\eta_1=0$.

Now, let us generalize the criticality and the total alert time definitions for Markov processes with multiple final sequences of states. A state v is considered critical at one moment in time if it is part of one final sequence of states $X^{(\ell)}$ and the system has just consecutively passed through $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)} = v$. It corresponds to a state $(v, Y_k^{(\ell)}, k)$ with $k>0$ in the reduced system (the related Markov process with final sequences of states of length 1, obtained by using the Aho-Corasick algorithm, as described in Section 8.5).

The total alert time η is defined similarly to the cumulative time during which the state is critical. It represents the time spent visiting the set of critical-related states $\bigcup_{k=1}^{m-1} Z^{(k)}$ and its distribution, conditioned by the initial state y, satisfies the following recurrences:

$$P_y(n) = \sum_{z\in Z} p(y,z) P_z(n),\ \forall y \in Z^{(0)},\ \forall n \geq 0,$$

$$P_y(0) = 0,\ P_y(n) = \sum_{z\in Z} p(y,z) P_z(n-1),\ \forall y \in \bigcup_{k=1}^{m-1} Z^{(k)},\ \forall n \geq 1,$$

$$P_y(0) = 1,\ P_y(n) = 0,\ \forall y \in Z^{(m)},\ \forall n \geq 1.$$

Note that the subset of states $Z_\infty^{(0)} \subseteq Z^{(0)}$, from which no critical-related state is reachable, can be ignored, since the probability of time spent visiting it is zero. For the remaining states $Z \setminus Z_\infty^{(0)}$ we have the recurrence

$$P(n) = Q \cdot P(n) + R \cdot P(n-1),\ \forall n \geq 1,$$

where

$$P(n) = (P_y(n))_{y\in Z\setminus Z_\infty^{(0)}},\ \forall n \geq 0,$$

$$Q = (q(y,z))_{y,z\in Z\setminus Z_\infty^{(0)}},\ R = (r(y,z))_{y,z\in Z\setminus Z_\infty^{(0)}},$$

and, for $\forall z \in Z \setminus Z_{\infty}^{(0)}$,

$$q(y,z) = p(y,z), r(y,z) = 0, \forall y \in Z^{(0)} \setminus Z_{\infty}^{(0)},$$

$$q(y,z) = 0, r(y,z) = p(y,z), \forall y \in \bigcup_{k=1}^{m-1} Z^{(k)},$$

$$q(y,z) = 0, r(y,z) = 0, \forall y \in Z^{(m)}.$$

Q is a weakly chained substochastic matrix ([8]) due to the following properties:

- $q(y,z) \in \{p(y,z), 0\} \subseteq [0,1], \forall y, z \in Z \setminus Z_{\infty}^{(0)}$;
- $\sum_{z \in Z \setminus Z_{\infty}^{(0)}} q(y,z) \leq 1, \forall y \in Z \setminus Z_{\infty}^{(0)}$;
- $\sum_{z \in Z \setminus Z_{\infty}^{(0)}} q(y,z) = 0 < 1, \forall y \in \bigcup_{k=1}^{m} Z^{(k)}$;
- for each $y \in Z^{(0)} \setminus Z_{\infty}^{(0)}$ there exists $z \in \bigcup_{k=1}^{m} Z^{(k)}$ such that the state z is reachable from y (in a finite number of steps).

As a consequence, the matrix $I - Q$ is invertible ([8], [143], [125]), which implies

$$P(n) = (I - Q)^{-1} \cdot R \cdot P(n-1), \forall n \geq 1,$$

i.e. $(P(n))_{n=0}^{\infty} \in Rol^*[\mathcal{M}_M(\mathbb{R})][1]$, where M is the size of the square matrix Q.

Next, we have

$$d = (P(n))_{n=0}^{\infty} \in Rol^*[\mathcal{M}_M(\mathbb{R})][1], ((I - Q)^{-1} R) \in G^*[\mathcal{M}_M(\mathbb{R})][1](d) \Rightarrow$$

$$d \in Rol^*[\mathbb{R}][M], H(z) = \mid I - (I - Q)^{-1} R \cdot z \mid \in H^*[\mathbb{R}][M](d) \Rightarrow$$

$$d_y = (P_y(n))_{n=0}^{\infty} \in Rol^*[\mathbb{R}][M], H(z) \in H^*[\mathbb{R}][M](d_y), \forall y \in Z \setminus Z_{\infty}^{(0)} \Rightarrow$$

$$c = (\mathbb{P}(\eta = n))_{n=0}^{\infty} = \sum_{y \in Z \setminus Z_{\infty}^{(0)}} p^*(y) d_y \in Rol^*[\mathbb{R}][M], H(z) \in H^*[\mathbb{R}][M](c) \Rightarrow$$

$$c \in Rol^*[\mathbb{R}][M], H(z) = \mid I - (I - Q)^{-1} R \cdot z \mid \in H^*[\mathbb{R}][M](c).$$

So, the total alert time distribution is a homogeneous linear recurrence over the set $\mathbb{R}$. The order of the recurrence is

$$M = \mid Z \setminus Z_{\infty}^{(0)} \mid \leq \mid Z \mid \leq \omega + r(m-1).$$

9 GAMES ON STOCHASTIC SYSTEMS WITH FINAL SEQUENCES OF STATES

In this chapter, we study the games defined on stochastic systems with a final sequence of states. We investigate both cases: the games defined on zero-order Markov processes with a final sequence of states and the games defined on stochastic systems with a final sequence of states and interdependent states. For these games, we analyze the game duration, the winning probabilities and the optimal cooperative strategies of the players.

Further, by using the Knuth-Morris-Pratt string-searching algorithm, the order of the recurrence is reduced and enhanced methods are obtained. This approach enables us to extend these results for the games defined on stochastic systems with multiple final sequences of states, by using the corresponding Aho-Corasick algorithm.

9.1 Homogeneous linear recurrence games

Let L be a discrete system with the set of states $V \subseteq \mathbb{C}$. At every discrete time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The state $v(t)$ of the system for every t, $0 \le t < m$, is known and, at every discrete time $t \ge m$, the evolution of the system satisfies the relation

$$v(t) = \sum_{k=0}^{m-1} q_k v(t-1-k),$$

where $q_k \in K$, $k = \overline{0, m-1}$, and $K \subseteq \mathbb{C}$. The system L represents a discrete system with homogeneous linear recurrence evolution over K, generated by the generating vector $q = (q_k)_{k=0}^{m-1}$ and the initial state $I_m^{[v]} = (v(t))_{t=0}^{m-1}$.

Consider the following game ([90]). Initially, each player $\mathcal{P}_\ell$ defines his stationary strategy, represented by one generating vector $q^{(\ell)} \in K^{m_\ell}$ and one initial state

$I_{m_\ell}^{[v]}$, $\ell = \overline{0,r-1}$. The game is started by the first player $\mathcal{P}_0$. At every discrete time t the system passes to the state $v(t)$ according to the strategy of the current player $\mathcal{P}_{t \bmod r}$. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the evolution of the system and the game continues in the same way.

Our goal is to prove that the game defined above is a homogeneous linear recurrence system and to provide an efficient approach for determining its generating vector, so that the previously developed methods can be used to characterize it.

The evolution generated by this game is $a = (a_n)_{n=0}^{\infty}$, where

$$a_n = v(n),\ n = \overline{0,\infty}.$$

The players $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{r-1}$ act consecutively on the evolution of the system and the game continues cyclically. In this way, every player $\mathcal{P}_\ell$ directly contributes only to determining the states $a_{nr+\ell}$, $n = \overline{0,\infty}$, for each $\ell \in \{0,1,\ldots,r-1\}$.

Consider the generating function

$$G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$$

of the sequence $a = (a_n)_{n=0}^{\infty}$. The contribution of each player $\mathcal{P}_\ell$ to this generating function is

$$g_\ell(z) = \sum_{n=0}^{\infty} a_{nr+\ell} z^{nr+\ell},\ \ell = \overline{0,r-1}.$$

Based on these contributions, the generating function becomes

$$G^{[a]}(z) = \sum_{\ell=0}^{r-1} g_\ell(z).$$

The function $g_\ell(z)$ can be written in the following way:

$$g_\ell(z) = \sum_{n:\, rn+\ell < m_\ell} a_{rn+\ell} z^{rn+\ell} + \sum_{n:\, rn+\ell \geq m_\ell} z^{rn+\ell} \sum_{k=0}^{m_\ell - 1} q_k^{(\ell)} a_{rn+\ell-k-1} =$$

$$= W_{m_\ell - 1}^{(\ell)}(z) + \sum_{k=0}^{m_\ell - 1} q_k^{(\ell)} z^{k+1} g_{s(\ell,k)}(z),$$

where

$$deg(W_{m_\ell-1}^{(\ell)}(z)) \le m_\ell - 1,$$

$$s(\ell,k) = (\ell - k - 1)\ mod\ r\ ,\ \ell = \overline{0,r-1}.$$

These relations can be rewritten as

$$A(z)(g(z))^T = (W(z))^T,$$

where

$$A(z) = (a_{ij}(z))_{i,j=0}^{r-1},\ \ g(z) = (g_j(z))_{j=0}^{r-1},\ \ W(z) = (W_{m_j-1}^{(j)}(z))_{j=0}^{r-1},$$

$$deg(a_{ij}(z)) \le m_j,\ deg(W_{m_j-1}^{(j)}(z)) \le m_j - 1,\ i,j = \overline{0,r-1}.$$

Using Cramer's rule, we have

$$g_\ell(z) = \frac{A_\ell(z)}{B(z)},$$

where

$$deg(A_\ell(z)) < m,\ deg(B(z)) \le m$$

and

$$m = \sum_{j=0}^{r-1} m_j,\ \ell = \overline{0,r-1}.$$

By computing the generating function

$$G^{[a]}(z) = \sum_{\ell=0}^{r-1} g_\ell(z)$$

and applying Theorem 1.4, we obtain that $a \in Rol^*[\mathbb{C}][m]$. The minimal generating vector can be determined using the rank-based order minimization method.

9.2 Games defined on zero-order Markov processes with a final sequence of states

Let L be a discrete stochastic system with a finite set of states V, $|V| = \omega$. At each discrete time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The stochastic system L

starts its evolution from the state v with probability $p^*(v)$, for each $v \in V$, where the equality $\sum_{v \in V} p^*(v) = 1$ holds.

Also, the transition from one state u to another state v is performed with the same probability $p^*(v)$, which depends only on the destination state v, for each $u \in V$ and $v \in V$. Additionally, a sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m$$

is given and the system stops when the states x_1, x_2, …, x_m are consecutively reached in the given order for the first time. The moment in time T, when the system stops, represents the evolution time of the stochastic system L with the given final sequence of states X.

The stochastic system L, described above, represents a zero-order Markov process with the final sequence of states X, i.e. a stochastic system with a final sequence of states and independent states ([86]). This kind of stochastic system was studied in Section 6.2 and polynomial-time algorithms for determining the main probabilistic characteristics (expectation, variance, standard deviation, moments) of the evolution time were developed.

In what follows, we study a generalization of this problem. Consider the following game. Initially, each player $\mathcal{P}_\ell$ defines their own strategy, represented by a probability distribution $(p^{*(\ell)}(v))_{v \in V}$ over the states of the system, $\ell = \overline{0, r-1}$. The initial probabilities of the system states are determined by the strategy $(p^{*(0)}(v))_{v \in V}$ of the first player $\mathcal{P}_0$. Subsequently, the system transitions successively to the next state according to the strategy of the next player. After the last player $\mathcal{P}_{r-1}$ has acted, the first player $\mathcal{P}_0$ acts again on the evolution of the system, and the game proceeds in this manner until the final sequence of states X is reached for the first time. The winner of the game is the player $\mathcal{P}_{T \bmod r}$ whose strategy is applied last.

Our objective is to study the game duration T, given the strategy $p^{*(\ell)} = (p^{*(\ell)}(v))_{v \in V}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T is a homogeneous linear recurrence and we will develop a polynomial-time algorithm for determining the initial state and the generating vector of this recurrence. Given the generating vector and the initial state, we can determine the main probabilistic characteristics of the game duration by using Algorithm 4.4.

We determine the distribution law of the game duration T. Consider the following notation

$$X_k = \{x_k\},\ \overline{X}_k = V \setminus \{x_k\},$$

$$\pi_k^{(\ell)} = p^{*(\ell)}(x_k),\ w_k^{(\ell)} = \prod_{j=2}^{k} \pi_j^{(\ell \oplus (-1) \oplus j)},$$

for $k = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where

$$c \oplus d = (c + d) \bmod r,\ \forall c, d \in \mathbb{Z}.$$

Let $a = (a_n)_{n=0}^{\infty}$ be the distribution of the game duration T, i.e.

$$a_n = \mathbb{P}(T = n),\ n = \overline{0, \infty}.$$

Since $T \geq m-1$, we have

$$a_n = 0,\ n = \overline{0, m-2}.$$

If $T = m-1$, then

$$v(j) = x_{j+1},\ j = \overline{0, m-1},$$

which implies

$$a_{m-1} = \mathbb{P}(T = m-1) = \pi_1^{(0)} \pi_2^{(1)} \ldots \pi_m^{(m \oplus (-1))} = \pi_1^{(0)} w_m^{(0)}.$$

Consider $\forall n \in \mathbb{Z}$. Also, we consider the set

$$S(V) = \{A \mid A \subseteq V\}.$$

We denote by $P_{\Phi}^{(\ell)}(n)$ the probability that

$$T = n,\ v(j) \in \Phi_j,\ j = \overline{0, t-1}$$

and the player $\mathcal{P}_\ell$ acts first, for each

$$\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t,\ t \in \mathbb{N}$$

and $\ell = \overline{0, r-1}$. We introduce the following functions on $\mathbb{Z}$, $k = \overline{0, m}$, $\ell = \overline{0, r-1}$:

$$\begin{aligned} \alpha_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}(n), \\ \beta_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n), \\ \gamma_k^{(\ell)}(n) &= P^{(\ell)}_{(X_2, X_3, \ldots, X_k)}(n). \end{aligned} \tag{9.1}$$

We have

$$\beta_k^{(\ell)}(n) = P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n) = a_n^{(\ell)} - \sum_{j=1}^{k} \alpha_j^{(\ell)}(n),\ k = \overline{0, m},\ \ell = \overline{0, r-1}, \tag{9.2}$$

where $a_n^{(\ell)} = P_{()}^{(\ell)}(n)$, $\ell = \overline{0, r-1}$.

Consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\},\ s = \overline{1, m}.$$

The minimum of each set is denoted by

$$t_s = \min_{k \in T_s} k,\ s = \overline{1, m}. \tag{9.3}$$

It represents the smallest index in the sequence $(x_1, x_2, \ldots, x_s)$ from which, when $(x_1, x_2, \ldots, x_s)$ is superimposed on itself, the overlapping components coincide.

By using the formula (9.2) for $s = \overline{1, m}$ and $\ell = \overline{0, r-1}$, we obtain

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \ldots, X_s)}(n) =$$

$$= \pi_2^{(\ell)}\, \pi_3^{(\ell \oplus 1)} \ldots \pi_{t_s - 1}^{(\ell \oplus (t_s - 3))} P^{(\ell \oplus (t_s - 2))}_{(X_{t_s}, X_{t_s+1}, \ldots, X_s)}(n - t_s + 2) =$$

$$= w_{t_s - 1}^{(\ell \oplus (-1))} \beta_{s+1-t_s}^{(\ell \oplus (t_s - 2))}(n - t_s + 2) =$$

$$= w_{t_s - 1}^{(\ell \oplus (-1))} \left(a_{n - t_s + 2}^{(\ell \oplus (t_s - 2))} - \sum_{j=1}^{s+1-t_s} \alpha_j^{(\ell \oplus (t_s - 2))}(n - t_s + 2) \right). \tag{9.4}$$

In particular, we obtain the relation

$$\gamma_1^{(\ell)}(n) = a_n^{(\ell)},\ \ell = \overline{0, r-1},\ n = \overline{0, \infty}. \tag{9.5}$$

The values $\alpha_k^{(\ell)}(n)$, $k = \overline{1, m}$, $\ell = \overline{0, r-1}$ are determined in the following way:

$$\alpha_1^{\ell}(n) = P_{(\overline{X}_1)}^{(\ell)}(n) = (1-\pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)}, \tag{9.6}$$

$$\begin{aligned}
\alpha_k^{(\ell)}(n) &= P_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}^{(\ell)}(n) = \pi_1^{(\ell)}\ P_{(X_2, X_3, \ldots, X_{k-1}, \overline{X}_k)}^{(\ell \oplus 1)}(n-1) = \\
&= \pi_1^{(\ell)} \left(P_{(X_2, X_3, \ldots, X_{k-1})}^{(\ell \oplus 1)}(n-1) - P_{(X_2, X_3, \ldots, X_k)}^{(\ell \oplus 1)}(n-1) \right) = \\
&= \pi_1^{(\ell)} \left(\gamma_{k-1}^{(\ell \oplus 1)}(n-1) - \gamma_k^{(\ell \oplus 1)}(n-1) \right),\ k = \overline{2, m}.
\end{aligned} \tag{9.7}$$

Next, we obtain the recurrence:

$$\begin{aligned}
a_n^{(\ell)} &= \sum_{j=1}^{m} \alpha_j^{(\ell)}(n) = (1-\pi_1^{(\ell)}) a_{n-1}^{(\ell \oplus 1)} + \sum_{j=2}^{m} \pi_1^{(\ell)} \left(\gamma_{j-1}^{(\ell \oplus 1)}(n-1) - \gamma_j^{(\ell \oplus 1)}(n-1) \right) = \\
&= (1-\pi_1^{(\ell)})\, a_{n-1}^{(\ell \oplus 1)} + \pi_1^{(\ell)} \left(a_{n-1}^{(\ell \oplus 1)} - \gamma_m^{(\ell \oplus 1)}(n-1) \right) = \\
&= a_{n-1}^{(\ell \oplus 1)} - \pi_1^{(\ell)} \gamma_m^{(\ell \oplus 1)}(n-1),\ \forall n \geq m,\ \ell = \overline{0,\ r-1}.
\end{aligned} \tag{9.8}$$

According to (9.4) − (9.7), by mathematical induction, we can prove that there exist real coefficients

$$u_{jk\ell}^{(i)},\ v_{jk\ell}^{(i)},\ j = \overline{1, m},\ k = \overline{0, j-1},\ \ell = \overline{0, r-1},\ i = \overline{0, r-1},$$

such that

$$\begin{aligned}
\alpha_j^{(\ell)}(n) &= \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u_{jk\ell}^{(i)}\, a_{n-1-k}^{(i)}, \\
\gamma_j^{(\ell)}(n-1) &= \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} v_{jk\ell}^{(i)}\, a_{n-1-k}^{(i)},\ \forall n \in \mathbb{Z}.
\end{aligned} \tag{9.9}$$

From (9.5) and (9.6), for $i = \overline{0, r-1}$ and $\ell = \overline{0, r-1}$, we obtain

$$u_{1,0,\ell}^{(i)} = \begin{cases} 1-\pi_1^{(\ell)}, & i = \ell \oplus 1 \\ 0, & i \neq \ell \oplus 1 \end{cases} \tag{9.10}$$

and

$$v_{1,0,\ell}^{(i)} = \begin{cases} 1, & i = \ell \\ 0, & i \neq \ell \end{cases}. \tag{9.11}$$

By using the representation (9.9), the formula (9.4) takes the form

$$\gamma_s^{(\ell)}(n-1) = w_{t_s-1}^{(\ell \oplus (-1))} \left(a_{(n-1)-t_s+2}^{(\ell \oplus (t_s-2))} - \sum_{j=1}^{s+1-t_s} \sum_{i=0}^{r-1} \sum_{k=0}^{j-1} u_{j,k,\ell \oplus (t_s-2)}^{(i)} a_{n-t_s-k}^{(i)} \right) =$$

$$= w_{t_s-1}^{(\ell \oplus (-1))} \left(a_{(n-1)-(t_s-2)}^{(\ell \oplus (t_s-2))} - \sum_{i=0}^{r-1} \sum_{k=t_s-1}^{s-1} a_{n-1-k}^{(i)} \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,k-t_s+1,\ell \oplus (t_s-2)}^{(i)} \right) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} v_{sk\ell}^{(i)} a_{n-1-k}^{(i)}, \; s = \overline{1,m}, \; \ell = \overline{0,r-1},$$

where

$$v_{s,k,\ell}^{(i)} = \begin{cases} 0, & k \leq t_s - 3 \\ 0, & k = t_s - 2, \; i \neq \ell \oplus (t_s-2) \\ w_{t_s-1}^{(\ell \oplus (-1))}, & k = t_s - 2, \; i = \ell \oplus (t_s-2) \\ -w_{t_s-1}^{(\ell \oplus (-1))} \sum_{j=k-t_s+2}^{s+1-t_s} u_{j,k-t_s+1,\ell \oplus (t_s-2)}^{(i)}, & k \geq t_s - 1 \end{cases} \tag{9.12}$$

$s = \overline{1,m}$, $k = \overline{0,s-1}$, $\ell = \overline{0,r-1}$, $i = \overline{0,r-1}$, and the formula (9.7) becomes

$$\alpha_s^{(\ell)}(n) = \pi_1^{(\ell)} \left(\gamma_{s-1}^{(\ell \oplus 1)}(n-1) - \gamma_s^{(\ell \oplus 1)}(n-1) \right) =$$

$$= \pi_1^{(\ell)} \left(\sum_{i=0}^{r-1} \sum_{k=0}^{s-2} v_{s-1,k,\ell \oplus 1}^{(i)} a_{n-1-k}^{(i)} - \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} v_{s,k,\ell \oplus 1}^{(i)} a_{n-1-k}^{(i)} \right) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} u_{sk\ell}^{(i)} a_{n-1-k}^{(i)}, \; s = \overline{2,m}, \; \ell = \overline{0,r-1},$$

where

$$u_{s,k,\ell}^{(i)} = \begin{cases} \pi_1^{(\ell)} \left(v_{s-1,k,\ell \oplus 1}^{(i)} - v_{s,k,\ell \oplus 1}^{(i)} \right), & k \leq s-2 \\ -\pi_1^{(\ell)} v_{s,k,\ell \oplus 1}^{(i)}, & k = s-1 \end{cases}, \tag{9.13}$$

for $s = \overline{2,m}$, $k = \overline{0,s-1}$, $\ell = \overline{0,r-1}$, $i = \overline{0,r-1}$. The formula (9.8) takes the form

$$a_n^{(\ell)} = a_{n-1}^{(\ell \oplus 1)} - \pi_1^{(\ell)} \gamma_m^{(\ell \oplus 1)}(n-1) = a_{n-1}^{(\ell \oplus 1)} - \pi_1^{(\ell)} \sum_{i=0}^{r-1} \sum_{k=0}^{m-1} v_{m,k,\ell \oplus 1}^{(i)} a_{n-1-k}^{(i)} =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{m-1} q_{ik}^{(\ell)} a_{n-1-k}^{(i)}, \forall n \geq m, \ell = \overline{0, r-1}, \tag{9.14}$$

where

$$q_{ik}^{(\ell)} = \begin{cases} 1 - \pi_1^{(\ell)} v_{m,0,\ell \oplus 1}^{(\ell \oplus 1)}, & i = \ell \oplus 1 \text{ and } k = 0 \\ -\pi_1^{(\ell)} v_{m,k,\ell \oplus 1}^{(i)}, & \text{otherwise} \end{cases}. \tag{9.15}$$

Next, we consider the column vectors

$$A_n = ((a_n^{(\ell)})_{\ell=0}^{r-1})^T, \ n = \overline{0, \infty}.$$

Also, we define the matrices

$$Q^{(k)} = (q_{ik}^{(\ell)})_{\ell, = \overline{0, r-1}}, \ k = \overline{0, m-1}$$

and we consider the sequence

$$Q = (Q^{(k)})_{k=0}^{m-1}.$$

From the relation (9.14), we have the homogeneous linear recurrence

$$A_n = \sum_{k=0}^{m-1} Q^{(k)} A_{n-1-k}, \ \forall n \geq m,$$

i.e.

$$A = (A_n)_{n=0}^{\infty} \in Rol^*[\mathcal{M}_r(\mathbb{R})][m]$$

with the generating vector

$$Q \in G^*[\mathcal{M}_r(\mathbb{R})][m](A).$$

Therefore, the vector sequence A is a homogeneous linear recurrence sequence over the matrix field $\mathcal{M}_r(\mathbb{R})$ with the generating vector Q. Based on [102], we have

$$A \in Rol^*[\mathbb{R}][mr]$$

with characteristic polynomial

$$H(z) = | I_r - zG_m^{[Q]}(z) | \in H^*[\mathbb{R}][mr](A),$$

which implies that

$$a^{(\ell)} = (a_n^{(\ell)})_{n=0}^{\infty} \in Rol^*[\mathbb{R}][mr]$$

with characteristic polynomial

$$H(z) = | I_r - zG_m^{[Q]}(z) | \in H^*[\mathbb{R}][mr](a^{(\ell)}), \ \ell = \overline{0, r-1}.$$

Since the game is started by the player $\mathcal{P}^{(0)}$, the distribution a of the game duration T coincides with $a^{(0)}$, i.e.

$$a = (a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][mr],$$

$$H(z) = |\, I_r - zG_m^{[Q]}(z)\,| \in H^*[\mathbb{R}][mr](a).$$

Next, we will use only the relation $a \in Rol^*[\mathbb{C}][mr]$, the minimal generating vector being determined by using the rank-based order minimization method, which can also be applied to degenerate homogeneous linear recurrences. Thus, according to this method, the minimal generating vector

$$q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$$

is obtained from the unique solution

$$x = (q_{R-1}, q_{R-2}, \ldots, q_0)$$

of the system

$$A_R^{[a]} x^T = (f_R^{[a]})^T, \tag{9.16}$$

where

$$f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1}), {}_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}}, \forall n \in \mathbb{N}^* \tag{9.17}$$

and R is the rank of the matrix $A_{mr}^{[a]}$.

We need only the values a_k, $k = \overline{0, 2mr-1}$. They can be obtained from the formula

$$a_k = a_k^{(0)}, k = \overline{0, 2mr-1}, \tag{9.18}$$

by using the relations (9.4) − (9.8) and the initial conditions

$$a_n = a_n^{(\ell)} = P^{(\ell)}(n) = 0,\ \ell = \overline{0, r-1},\ n = \overline{0, m-2}$$

$$\alpha_k^{(\ell)}(n) = 0, k = \overline{1, m},\ n = \overline{0, m-1},\ \ell = \overline{0, r-1},$$

$$a_{m-1}^{(\ell)} = \pi_1^{(\ell)} w_m^{(\ell)}, \ell = \overline{0, r-1}. \tag{9.19}$$

Therefore, we provided a theoretical foundation for the following algorithm, which determines the main probabilistic characteristics (distribution $(\mathbb{P}(T=n))_{n=0}^{\infty}$, expectation $\mathbb{E}(T)$, variance $\mathbb{V}(T)$, standard deviation $\sigma(T)$ and moments $\nu_k(T)$, $k=1,2,\ldots$) of the game duration T.

Algorithm 9.1

Input: $X=(x_1,x_2,\ldots,x_m)\in V^m$, $\pi_k^{(\ell)}$, $k=\overline{1,m}$, $\ell=\overline{0,r-1}$;

Output: $\mathbb{E}(T),\mathbb{V}(T),\sigma(T),\nu_k(T),k=\overline{1,t},t\geq 2$.

1. *Determine the values* a_k, $k=\overline{0,2mr-1}$, *by using the relations (9.18) and (9.4) – (9.8) and the initial conditions (9.19);*
2. *Find the minimal generating vector* $q=(q_0,q_1,\ldots,q_{R-1})\in G^*[\mathbb{R}][R](a)$ *by solving the system* (9.16)*, considering the relation* (9.17)*;*
3. *Consider the distribution* $a=(a_n)_{n=0}^{\infty}=(\mathbb{P}(T=n))_{n=0}^{\infty}$ *of the game duration* T *as a homogeneous linear recurrence with the initial state* $I_R^{[a]}=(a_n)_{n=0}^{R-1}$ *and the minimal generating vector* $q=(q_k)_{k=0}^{R-1}$*, determined at the steps 1 and 2;*
4. *Determine the expectation* $\mathbb{E}(T)$*, the variance* $\mathbb{V}(T)$*, the standard deviation* $\sigma(T)$ *and the moments* $\nu_k(T)$, $k=\overline{1,t}$, *of the game duration* T*, by using Algorithm 4.4.*

Therefore, stationary games defined on zero-order Markov processes with a final sequence of states have been studied. The duration of these games has been analyzed, and it has been proved that it is a random variable with a homogeneous linear recurrence distribution. Based on these results, the generating function is applied to determine the basic probabilistic characteristics of the game duration. The complexity of Algorithm 9.1 is polynomial. Moreover, this algorithm is also applicable when the state space is infinite.

9.3 Games defined on stochastic systems with a final sequence of states

Let L be a discrete stochastic system with a finite set of states V, $|V|=\omega$. At each discrete time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with probability $p^*(v)$, for each $v \in V$, where $\sum_{v \in V} p^*(v) = 1$.

Also, the transition from one state $u \in V$ to another state $v \in V$ is performed with probability $p(u,v) \in [0,1]$, such that $\sum_{v \in V} p(u,v) = 1$, $\forall u \in V$. Additionally, a sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m$$

is given and the stochastic system stops once the states $x_1, x_2, \ldots, x_m$ are consecutively reached in the given order. The moment in time T, when the system stops, represents the evolution time of the system L with the final sequence of states X.

The stochastic system L, described above, represents a Markov process with the final sequence of states X, i.e. a stochastic system with a final sequence of states and interdependent states. This kind of stochastic system was studied in Section 6.5 and polynomial-time algorithms for determining the main probabilistic characteristics (expectation, variance, standard deviation, moments) of the evolution time were developed.

Next, we consider the following game ([97]). Initially, each player $\mathcal{P}_\ell$ defines their own stationary strategy, represented by a transition matrix

$$(p^{(\ell)}(u,v))_{u,v \in V}, \ \ell = \overline{0, r-1}.$$

The initial probabilities of the states are specified by the distribution $(p^*(v))_{v \in V}$.

The game is initiated by the first player $\mathcal{P}_0$. At each discrete time, the stochastic system successively transitions to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$ has acted, the first player $\mathcal{P}_0$ acts again on

the evolution of the system, and the game proceeds in this manner until the final sequence of states X is reached. The winner of the game is the player $\mathcal{P}_{T \bmod r}$ whose strategy is applied last.

Our objective is to study the game duration T, given the initial state probabilities $p^* = (p^*(v))_{v \in V}$, the stationary strategy $P^{(\ell)} = (p^{(\ell)}(u,v))_{u,v \in V}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, and the final sequence of states X of the stochastic system L. We will prove that the distribution of the game duration T satisfies a homogeneous linear recurrence and we will show how to determine the initial state and the generating vector of this recurrence.

Given the generating vector and the initial state, we can determine the main probabilistic characteristics of the game duration by using Algorithm 4.4. Moreover, based on these results, we will show in the next section how the winning probabilities of the players can be determined.

In what follows, we will determine the probability distribution of the game duration T. We will prove that this distribution satisfies a homogeneous linear recurrence.

Initially, we consider the sets

$$X_j = \{x_j\},\ \overline{X}_j = V \setminus X_j,\ j = \overline{1, m}.$$

We also introduce the notation

$$\pi_j = p^*(x_j),\ \pi_{ij}^{(\ell)} = p^{(\ell)}(x_i, x_j),\ \omega_j^{(\ell)} = \prod_{k=3}^{j} \pi_{k-1,k}^{(\ell \oplus (k-3))},$$

for each $i, j = \overline{1, m}$ and $\ell = \overline{0, r-1}$, where

$$c \oplus d = (c + d) \bmod r,\ \forall c, d \in \mathbb{Z}.$$

If, for each $\ell = \overline{0, r-1}$, there exists an index $j_\ell \in \{2, \ldots, m\}$ such that

$$\pi_{j_\ell - 1, j_\ell}^{(\ell \oplus (j_\ell - 2))} = 0,$$

then the evolution of the stochastic system is not finite; that is, the game duration is infinite. Equivalently, in this case, we have

$$a_n = 0\,,\ \forall n \in \mathbb{N} \text{ and } \mathbb{P}(T = +\infty) = 1.$$

Consequently, the moments of the game duration are also infinite. In what follows, we investigate the remaining cases, in which the game duration is finite.

Consider $\forall n \in \mathbb{Z}$ and the set $S(V) = \{A \,|\, A \subseteq V\}$. We denote by $P_{\Phi}^{(\ell)}(n)$ the probability that

$$T = n\,,\ v(j) \in \Phi_j,\ j = \overline{0, t-1}$$

and the player $\mathcal{P}_\ell$ acts first, provided that the initial state of the system is known, for each

$$\Phi = (\Phi_j)_{j=0}^{t-1} \in (S(V))^t\,,\ t \in \mathbb{N}\,,\ \ell = \overline{0, r-1}.$$

We introduce the following functions on $\mathbb{Z}$, $k = \overline{0, m}$, $\ell = \overline{0, r-1}$:

$$\begin{aligned} \alpha_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_{k-1}, \overline{X}_k)}(n), \\ \beta_k^{(\ell)}(n) &= P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n), \\ \gamma_k^{(\ell)}(n) &= P^{(\ell)}_{(X_2, X_3, \ldots, X_k)}(n). \end{aligned} \tag{9.20}$$

Also, we consider the sets

$$T_s = \{s+1\} \cup \{t \in \{2, 3, \ldots, s\} \,|\, x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\},\ s = \overline{1, m}.$$

The minimum of each set is denoted by

$$t_s = \min_{k \in T_s} k,\ s = \overline{1, m}. \tag{9.21}$$

It represents the smallest index in the sequence $(x_1, x_2, \ldots, x_s)$, from which, when $(x_1, x_2, \ldots, x_s)$ is superimposed on itself, the overlapping components coincide.

Initially, we study the case $m \geq 2$. We have

$$\beta_k^{(\ell)}(n) = P^{(\ell)}_{(X_1, X_2, \ldots, X_k)}(n) = P_n^{(\ell)} - \sum_{j=1}^{k} \alpha_j^{(\ell)}(n),\ k = \overline{0, m},\ \ell = \overline{0, r-1}, \tag{9.22}$$

where

$$P_n^{(\ell)} = P_{()}^{(\ell)}(n)\,,\ \ell = \overline{0, r-1}.$$

Directly from the definition, we obtain

$$\gamma_1^{(\ell)}(n) = P^{(\ell)}(n),\ \ell = \overline{0, r-1}, \forall n \in \mathbb{Z}. \tag{9.23}$$

Let $s \geq 2$. For $t_s \leq s$ and $\ell = \overline{0, r-1}$, we have

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \ldots, X_s)}(n) =$$

$$= \pi_{2,3}^{(\ell)} \pi_{3,4}^{(\ell \oplus 1)} \ldots \pi_{t_s-1, t_s}^{(\ell \oplus (t_s-3))} P^{(\ell \oplus (t_s-2))}_{(X_{t_s}, \ldots X_s)}(n - t_s + 2) =$$

$$= \omega_{t_s}^{(\ell)} P^{(\ell \oplus (t_s-2))}_{(X_1, \ldots, X_{s+1-t_s})}(n - t_s + 2) = \omega_{t_s}^{(\ell)} \beta^{(\ell \oplus (t_s-2))}_{s+1-t_s}(n - t_s + 2) =$$

$$= \omega_{t_s}^{(\ell)} \left(P^{(\ell \oplus (t_s-2))}(n - t_s + 2) - \sum_{j=1}^{s+1-t_s} \alpha_j^{(\ell \oplus (t_s-2))}(n - t_s + 2) \right) \tag{9.24}$$

and, in the case $t_s = s+1$, for $\ell = \overline{0, r-1}$, we obtain

$$\gamma_s^{(\ell)}(n) = P^{(\ell)}_{(X_2, X_3, \ldots, X_s)}(n) =$$

$$= \omega_s^{(\ell)} \sum_{y \in V} p^{(\ell \oplus (s-2))}(x_s, y) P^{(\ell \oplus (s-1))}_{(\{y\})}(n - s + 1) =$$

$$= \sum_{y \in V} \omega_s^{(\ell)} p^{(\ell \oplus (s-2))}(x_s, y) P^{(\ell \oplus (s-1))}_{(\{y\})}(n - t_s + 2). \tag{9.25}$$

Next, we determine the values $\alpha_k^{(\ell)}(n)$, $k = \overline{1, m}$, $\ell = \overline{0, r-1}$. We have

$$\alpha_1^{(\ell)}(n) = P^{(\ell)}_{(\overline{X}_1)}(n) = \sum_{x \in V \setminus \{x_1\}} P^{(\ell)}_{(\{x\})}(n) =$$

$$= \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} P^{(\ell)}_{(\{x\}, \{y\})}(n) = \sum_{x \in V \setminus \{x_1\}} \sum_{y \in V} p^{(\ell)}(x, y) P^{(\ell \oplus 1)}_{\{y\}}(n-1) =$$

$$= \sum_{y \in V} P^{(\ell \oplus 1)}_{\{y\}}(n-1) \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y) = \sum_{y \in V} \psi_1^{(\ell)}(y) P^{(\ell \oplus 1)}_{(\{y\})}(n-1), \tag{9.26}$$

where

$$\psi_1^{(\ell)}(y) = \sum_{x \in V \setminus \{x_1\}} p^{(\ell)}(x, y), \forall y \in V. \tag{9.27}$$

For $k = 2$, we obtain

$$\alpha_2^{(\ell)}(n) = P^{(\ell)}_{(X_1, \overline{X}_2)}(n) = \sum_{y \neq x_2} P^{(\ell)}_{(X_1, \{y\})}(n) = \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P^{(\ell \oplus 1)}_{(\{y\})}(n-1) \tag{9.28}$$

and for $k \geq 3$ we have

$$\alpha_k^{(\ell)}(n) = P^{(\ell)}_{(X_1,X_2,\ldots,X_{k-1},\overline{X}_k)}(n) = \pi_{1,2}^{(\ell)} P^{(\ell\oplus 1)}_{(X_2,X_3,\ldots,X_{k-1},\overline{X}_k)}(n-1) =$$

$$= \pi_{1,2}^{(\ell)} \left(P^{(\ell\oplus 1)}_{(X_2,X_3,\ldots,X_{k-1})}(n-1) - P^{(\ell\oplus 1)}_{(X_2,X_3,\ldots,X_k)}(n-1) \right) =$$

$$= \pi_{1,2}^{(\ell)} \left(\gamma_{k-1}^{(\ell\oplus 1)}(n-1) - \gamma_k^{(\ell\oplus 1)}(n-1) \right). \tag{9.29}$$

From the equality

$$P^{(\ell)}(n) = \sum_{k=1}^{m} \alpha_k^{(\ell)}(n) = \alpha_1^{(\ell)}(n) + \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n), \forall n \geq m, \tag{9.30}$$

by using the relations (9.22), (9.28) and (9.29), we obtain the formula

$$P^{(\ell)}_{X_1}(n) = \beta_1^{(\ell)}(n) = P^{(\ell)}(n) - \alpha_1^{(\ell)}(n) = \alpha_2^{(\ell)}(n) + \sum_{k=3}^{m} \alpha_k^{(\ell)}(n) =$$

$$= \sum_{y\neq x_2} p^{(\ell)}(x_1, y) P^{(\ell\oplus 1)}_{(\{y\})}(n-1) + \sum_{k=3}^{m} \pi_{1,2}^{(\ell)} \left(\gamma_{k-1}^{(\ell\oplus 1)}(n-1) - \gamma_k^{(\ell\oplus 1)}(n-1) \right) =$$

$$= \sum_{y\neq x_2} p^{(\ell)}(x_1, y) P^{(\ell\oplus 1)}_{(\{y\})}(n-1) + \pi_{1,2}^{(\ell)} \left(\gamma_2^{(\ell\oplus 1)}(n-1) - \gamma_m^{(\ell\oplus 1)}(n-1) \right) =$$

$$= \sum_{y\in V} p^{(\ell)}(x_1, y) P^{(\ell\oplus 1)}_{(\{y\})}(n-1) - \pi_{1,2}^{(\ell)}\, \gamma_m^{(\ell\oplus 1)}(n-1),\ \forall n \geq m. \tag{9.31}$$

For $x \neq x_1$, we have

$$P^{(\ell)}_{(\{x\})}(n) = \sum_{y\in V} p^{(\ell)}(x, y) P^{(\ell\oplus 1)}_{(\{y\})}(n-1). \tag{9.32}$$

According to (9.23) − (9.29), by mathematical induction, we can prove that there exist real coefficients

$$u^{(i)}_{jk\ell}(y),\ v^{(i)}_{jk\ell}(y),\ j = \overline{1,m},\ k = \overline{0,j-1},\ y \in V,\ \ell = \overline{0,r-1},\ i = \overline{0,r-1},$$

such that, for each $n \in \mathbb{Z}$, the following relations hold:

$$\begin{cases} \alpha_j^{(\ell)}(n) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y\in V} u^{(i)}_{jk\ell}(y)\, P^{(i)}_{(\{y\})}(n-1-k), \\ \gamma_j^{(\ell)}(n-1) = \sum\limits_{i=0}^{r-1} \sum\limits_{k=0}^{j-1} \sum\limits_{y\in V} v^{(i)}_{jk\ell}(y)\, P^{(i)}_{(\{y\})}(n-1-k). \end{cases} \tag{9.33}$$

For $n < m-1$, these relations are obvious and they hold for all real values $u_{jk\ell}^{(i)}(y)$ and $v_{jk\ell}^{(i)}(y)$. It remains to prove these relations for $n \geq m$, by using the mathematical induction on j.

For $j=1$ we have

$$\alpha_1^{(\ell)}(n) = \sum_{y \in V} \psi_1^{(\ell)}(y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) =$$

$$= \sum_{i=0}^{r-1} \sum_{k=0}^{0} \sum_{y \in V} u_{1k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k) = \sum_{i=0}^{r-1} \sum_{y \in V} u_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where

$$u_{10\ell}^{(i)}(y) = \begin{cases} \psi_1^{(\ell)}(y), & i = \ell \oplus 1 \\ 0, & i \neq \ell \oplus 1 \end{cases} \tag{9.34}$$

and

$$\gamma_1^{(\ell)}(n-1) = P^{(\ell)}(n-1) = \sum_{y \in V} P_{(\{y\})}^{(\ell)}(n-1) = \sum_{i=0}^{r-1} \sum_{y \in V} v_{10\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1),$$

where

$$v_{10\ell}^{(i)}(y) = \begin{cases} 1, & i = \ell \\ 0, & i \neq \ell. \end{cases} \tag{9.35}$$

For $j=2$ we obtain

$$\alpha_2^{(\ell)}(n) = \sum_{y \neq x_2} p^{(\ell)}(x_1, y) P_{(\{y\})}^{(\ell \oplus 1)}(n-1) = \sum_{i=0}^{r-1} \sum_{k=0}^{1} \sum_{y \in V} u_{2k\ell}^{(i)}(y) P_{(\{y\})}^{(i)}(n-1-k),$$

where

$$u_{20\ell}^{(i)}(y) = \begin{cases} 0, & y = x_2 \\ 0, & y \neq x_2,\ i \neq \ell \oplus 1 \\ p^{(\ell)}(x_1, y), & y \neq x_2,\ i = \ell \oplus 1 \end{cases} \tag{9.36}$$

and

$$u_{21\ell}^{(i)}(y) = 0,\ \forall y \in V. \tag{9.37}$$

Also, we have

$$\gamma_2^{(\ell)}(n-1)=P_{(X_2)}^{(\ell)}(n-1)=\sum_{i=0}^{r-1}\sum_{k=0}^{1}\sum_{y\in V}v_{2k\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(n-1-k),$$

where

$$v_{20\ell}^{(i)}(y)=\begin{cases}0, & y\neq x_2\\ 0, & y=x_2,\ i\neq\ell\\ 1, & y=x_2,\ i=\ell\end{cases} \tag{9.38}$$

and

$$v_{21\ell}^{(i)}(y)=0,\ \forall y\in V. \tag{9.39}$$

Therefore, the relations hold for $\forall j\in\{1,2\}$. We suppose that these relations hold for $j=\overline{1,s-1}$, $s\geq 3$, $\forall n<\tau$ and $\forall y\in V$. We have

$$\alpha_s^{(\ell)}(\tau)=\pi_{1,2}^{(\ell)}\left(\gamma_{s-1}^{(\ell\oplus 1)}(\tau-1)-\gamma_s^{(\ell\oplus 1)}(\tau-1)\right)=$$

$$=\pi_{1,2}^{(\ell)}\left(\sum_{i=0}^{r-1}\sum_{k=0}^{s-2}\sum_{y\in V}v_{s-1,k,\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k)-\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V}v_{s,k,\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k)\right)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V}u_{sk\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k),$$

where

$$u_{sk\ell}^{(i)}(y)=\begin{cases}\pi_{1,2}^{(\ell)}(v_{s-1,k,\ell}^{(i)}(y)-v_{sk\ell}^{(i)}(y)), & if\ 0\leq k\leq s-2\\ -\pi_{1,2}^{(\ell)}v_{s,s-1,\ell}^{(i)}(y), & if\ k=s-1.\end{cases} \tag{9.40}$$

For $t_s\leq s$ we obtain

$$\gamma_s^{(\ell)}(\tau-1)=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\oplus(t_s-2))}(\tau-t_s+1)-\sum_{j=1}^{s+1-t_s}\alpha_j^{(\ell\oplus(t_s-2))}(\tau-t_s+1)\right)=$$

$$=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\oplus(t_s-2))}(\tau-t_s+1)-\sum_{j=1}^{s+1-t_s}\sum_{i=0}^{r-1}\sum_{k=0}^{j-1}\sum_{y\in V}u_{j,k,\ell\oplus(t_s-2)}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-t_s-k)\right)=$$

$$=\omega_{t_s}^{(\ell)}\left(\sum_{y\in V}P_{(\{y\})}^{(\ell\oplus(t_s-2))}(\tau-1-(t_s-2))-\sum_{i=0}^{r-1}\sum_{k=t_s-1}^{s-1}\sum_{y\in V}P_{(\{y\})}^{(i)}(\tau-1-k)\sum_{j=k-t_s+2}^{s+1-t_s}u_{j,k-t_s+1,\ell\oplus(t_s-2)}^{(i)}(y)\right)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1}\sum_{y\in V}v_{sk\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k),$$

where

$$v_{sk\ell}^{(i)}(y)=\begin{cases}0, & 0\le k\le t_s-3\\ 0, & k=t_s-2,\ i\ne \ell\oplus(t_s-2)\\ \omega_{t_s}^{(\ell)}, & k=t_s-2,\ i=\ell\oplus(t_s-2)\\ -\omega_{t_s}^{(\ell)}\sum\limits_{j=k-t_s+2}^{s+1-t_s}u_{j,k-t_s+1,\ell\oplus(t_s-2)}^{(i)}(y), & t_s-1\le k\le s-1\end{cases} \tag{9.41}$$

and for $t_s=s+1$ we have

$$\gamma_s^{(\ell)}(\tau-1)=\sum_{y\in V}\omega_s^{(\ell)}p^{(\ell\oplus(s-2))}(x_s,y)P_{(\{y\})}^{(\ell\oplus(s-1))}(\tau-1-(t_s-2))=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{j-1}\sum_{y\in V}v_{sk\ell}^{(i)}(y)P_{(\{y\})}^{(i)}(\tau-1-k),$$

where

$$v_{sk\ell}^{(i)}(y)=\begin{cases}0, & 0\le k\le s-2\\ 0, & k=s-1,\ i\ne \ell\ \oplus\ (s-1)\\ \omega_s^{(\ell)}p^{(\ell\oplus(s-2))}(x_s,y), & k=s-1,\ i=\ell\ \oplus\ (s-1).\end{cases} \tag{9.42}$$

Therefore, we have proved relations (9.33) and obtained formulas (9.34) − (9.42) for determining the coefficients. Substituting relations (9.33) into (9.31) and (9.32), we obtain

$$P_{(X_1)}^{(\ell)}(n)=\sum_{y\in V}p^{(\ell)}(x_1,y)P_{(\{y\})}^{(\ell\oplus1)}(n-1)-\pi_{1,2}^{(\ell)}\gamma_m^{(\ell\oplus1)}(n-1)=$$

$$=\sum_{y\in V}p^{(\ell)}(x_1,y)P_{(\{y\})}^{(\ell\oplus1)}(n-1)-\pi_{1,2}^{(\ell)}\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}v_{m,k,\ell\oplus1}^{(i)}(y)P_{(\{y\})}^{(i)}(n-1-k)=$$

$$=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w_{k,\ell}^{(i)}(x_1,y)P_{(\{y\})}^{(i)}(n-1-k)$$

and, for any $x\ne x_1$, we have

$$P_{(\{x\})}^{(\ell)}(n)=\sum_{y\in V}p^{(\ell)}(x,y)P_{(\{y\})}^{(\ell\oplus1)}(n-1)=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w_{k,\ell}^{(i)}(x,y)P_{(\{y\})}^{(i)}(n-1-k),$$

where

$$w_{k\ell}^{(i)}(x,y)=\begin{cases} p^{(\ell)}(x_1,y)-\pi_{1,2}^{(\ell)}v_{m,0,\ell\oplus 1}^{(\ell\oplus 1)}(y), & x=x_1,\ k=0,\ i=\ell\oplus 1 \\ -\pi_{1,2}^{(\ell)}v_{m,0,\ell\oplus 1}^{(i)}(y), & x=x_1,\ k=0,\ i\neq \ell\oplus 1 \\ -\pi_{1,2}^{(\ell)}v_{m,k,\ell\oplus 1}^{(i)}(y), & x=x_1,\ 1\le k\le m-1 \\ p^{(\ell)}(x,y), & x\neq x_1,\ k=0,\ i=\ell\oplus 1 \\ 0, & x\neq x_1,\ k=0,\ i\neq \ell\oplus 1 \\ 0, & x\neq x_1,\ 1\le k\le m-1 \end{cases} \tag{9.43}$$

Therefore, we obtain the recurrence relation

$$P_{(\{x\})}^{(\ell)}(n)=\sum_{i=0}^{r-1}\sum_{k=0}^{m-1}\sum_{y\in V}w_{k\ell}^{(i)}(x,y)P_{(\{y\})}^{(i)}(n-1-k),\ \forall x\in V,\ \forall n\ge m,\ \ell=\overline{0,r-1}.$$

We have

$$P_{(\{x\})}(n)=\sum_{k=0}^{m-1}\sum_{y\in V}W_k(x,y)P_{(\{y\})}(n-1-k),\ \forall x\in V,\ \forall n\ge m,$$

where

$$W_k(x,y)=(w_{k\ell}^{(i)}(x,y))_{\ell,\,i=\overline{0,r-1}},$$

$$P_{(\{x\})}(n)=(P_{(\{x\})}^{(\ell)}(n))_{\ell=\overline{0,r-1}},\ \forall x,y\in V,\ k=\overline{0,m-1}.$$

This recurrence relation can be written in the form

$$h_n=\sum_{k=0}^{m-1}W_k h_{n-1-k},\ \forall n\ge m,$$

where

$$W_k=(W_k(x,y))_{x,y\in V},\ h_n=((P_{(\{x\})}(n))_{x\in V})^T,\ k=\overline{0,m-1},\ \forall n\in\mathbb{Z}.$$

From this relation, we obtain

$$h=(h_n)_{n=0}^{\infty}\in Rol^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m],$$

$$W=(W_k)_{k=0}^{m-1}\in G^*[\mathcal{M}_\omega(\mathcal{M}_r(\mathbb{R}))][m](h).$$

Using the properties of five-dimensional homogeneous linear recurrences, we have

$$h\in Rol^*[\mathbb{R}][mr\omega],$$

which implies

$$\left(P_{(\{x\})}^{(\ell)}(n)\right)_{n=0}^{\infty}\in Rol^*[\mathbb{R}][mr\omega],\ \forall x\in V,\ \ell=\overline{0,r-1},$$

with the same generating vector. Since

$$a^{(\ell)}(n)=\sum_{x\in V}p^{*}(x)P^{(\ell)}_{(\{x\})}(n),\ \forall n\in\mathbb{N},$$

we have

$$a^{(\ell)}=(a^{(\ell)}(n))_{n=0}^{\infty}\in Rol^{*}[\mathbb{R}][mr\omega],\ \ell=\overline{0,r-1},$$

with the same generating vector. Next, since the game is initiated by player $\mathcal{P}^{(0)}$, the distribution a of the game duration T coincides with $a^{(0)}$, i.e. we have

$$a=(a_n)_{n=0}^{\infty}\in Rol^{*}[\mathbb{R}][mr\omega]$$

with the same generating vector.

In what follows, we will use only the relation

$$a\in Rol^{*}[\mathbb{C}][mr\omega],$$

with the minimal generating vector being determined using the rank-based order minimization method. Thus, according to this method, the minimal generating vector

$$q=(q_0,q_1,\ldots,q_{R-1})\in G^{*}[\mathbb{C}][R](a)$$

is obtained from the unique solution $x=(q_{R-1},q_{R-2},\ldots,q_0)$ of the system

$$A_R^{[a]}x^T=(f_R^{[a]})^T, \tag{9.44}$$

where

$$f_R^{[a]}=(a_R,a_{R+1},\ldots,a_{2R-1}),\ A_n^{[a]}=(a_{i+j})_{i,j=\overline{0,n-1}},\ \forall n\in\mathbb{N}^{*} \tag{9.45}$$

and R is the rank of the matrix $A_{mr\omega}^{[a]}$.

For this purpose, we only need the values a_k, $k=\overline{0,2mr\omega-1}$. These values are determined from the formula

$$a_k=a_k^{(0)},\ k=\overline{0,2mr\omega-1}, \tag{9.46}$$

using relations (9.22) − (9.32) and the initial conditions

$$a_n=a_n^{(\ell)}=P^{(\ell)}(n)=P^{(\ell)}_{(\{x\})}(n)=0,\ \forall x\in V,\ \ell=\overline{0,r-1},\ n=\overline{0,m-2},$$

$$\alpha_k^{(\ell)}(n)=0,\ k=\overline{1,m},\ n=\overline{0,m-1},\ \ell=\overline{0,r-1},$$

$$P^{(\ell)}(m-1)=\pi_{1,2}^{(\ell)}w_m^{(\ell+1)},\ a_{m-1}^{(\ell)}=\pi_1P^{(\ell)}(m-1),\ \ell=\overline{0,r-1},$$

$$P^{(\ell)}_{(\{x_1\})}(m-1) = P^{(\ell)}(m-1),\ \ell = \overline{0,r-1},$$

$$P^{(\ell)}_{(\{x\})}(m-1) = 0,\ \forall x \in V \setminus \{x_1\},\ \ell = \overline{0,r-1}. \tag{9.47}$$

In the case $m = 1$, we have different formulas for determining the values of the conditional probabilities

$$P^{(\ell)}_{(\{x\})}(n),\ \ell = \overline{0,r-1},\ \forall x \in V,\ \forall n \in \mathbb{N}.$$

It is readily seen that these values can be obtained using the following formulas:

$$P^{(\ell)}_{(X_1)}(0) = 1,\ P^{(\ell)}_{(X_1)}(n) = 0,\ \forall n \in \mathbb{N}^*,\ \ell = \overline{0,r-1},$$

$$P^{(\ell)}_{(\{x\})}(0) = 0,\ P^{(\ell)}_{(\{x\})}(n) = \sum_{y \in V} p^{(\ell)}(x,y) P^{(\ell \oplus 1)}_{(\{y\})}(n-1),\ \forall n \in \mathbb{N}^*,\ \forall x \in V \setminus \{x_1\}. \tag{9.48}$$

Therefore, we have theoretically justified the following algorithm for determining the main probabilistic characteristics of the game duration T.

Algorithm 9.2

Input: $X = (x_1, x_2, \ldots, x_m) \in V^m$, π_j, $\pi^{(\ell)}_{i,j}$, $i, j = \overline{1,m}$, $\ell = \overline{0,r-1}$;

Output: $\mathbb{E}(T), \mathbb{V}(T), \sigma(T), \nu_k(T), k = \overline{1,t}, t \geq 2$.

1. *Determine the values* a_k, $k = \overline{0, 2mr\omega - 1}$, *by using the relations (9.46), (9.22)−(9.32) and the initial conditions (9.47)−(9.48);*
2. *Find the minimal generating vector*

   $$q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{R}][R](a)$$

   by solving the system (9.44), taking into account the relation (9.45);
3. *Consider the distribution* $a = (a_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$ *of the game duration* T *as a homogeneous linear recurrence with the initial state* $I_R^{[a]} = (a_n)_{n=0}^{R-1}$ *and the minimal generating vector* $q = (q_k)_{k=0}^{R-1}$, *determined at the steps 1 and 2;*
4. *Determine the expectation* $\mathbb{E}(T)$, *the variance* $\mathbb{V}(T)$, *the standard deviation* $\sigma(T)$ *and the moments* $\nu_k(T)$, $k = \overline{1,t}$, *of the game duration* T, *by using Algorithm 4.4.*

9.4 Winning probabilities and optimal cooperative strategies

Another problem of interest is the determination of the players' winning probabilities. To solve this problem, we consider the game in its generalized form.

Consider a finite game Γ with r players $\mathcal{P}_\ell$, $\ell = \overline{0,r-1}$, where each player applies their own strategy S_ℓ, $0 \le \ell < r$, in a given cyclic order:

$$(S_0, S_1, \ldots, S_{r-1}, S_0, S_1, \ldots).$$

Let T be the game duration. The winner of the game is the player $\mathcal{P}_{T \bmod r}$ whose strategy is applied last.

Assume that the distribution

$$d = (d_n)_{n=0}^{\infty} = (\mathbb{P}(T = n))_{n=0}^{\infty}$$

of the game duration T is a homogeneous linear recurrence, i.e. there exists an integer $m \in \mathbb{N}^*$ and the generating vector $q = (q_k)_{k=0}^{m-1} \in \mathbb{C}^m$, such that

$$d_n = \sum_{k=0}^{m-1} q_k d_{n-1-k}, \ \forall n \ge m.$$

This means that we have

$$d \in Rol^*[\mathbb{C}][m], \ q \in G^*[\mathbb{C}][m](d).$$

We will show how to find the winning probability ω_ℓ of each player $\mathcal{P}_\ell$, $\ell = \overline{0,r-1}$.

If we consider the subsequence $d^{(\ell)} = (d_{rn+\ell})_{n=0}^{\infty}$ of d, we have

$$\omega_\ell = G^{[d^{(\ell)}]}(1), \ \ell = \overline{0,r-1}.$$

We obtain $d^{(\ell)} \in Rol^*[\mathbb{C}][m]$, $\ell = \overline{0,r-1}$, with a common generating vector.

The minimal generating vector of these sequences can be determined using the rank-based order minimization method and the initial states of these sequences can be obtained using the initial state and the generating vector of the distribution d of the game duration. Finally, the winning probability ω_ℓ is obtained by applying the generating function formula

$$G^{[a]}(z)=\frac{G_m^{[a]}(z)-z\sum\limits_{k=0}^{m-1}q_k z^k G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)},$$

for $z:=1$ and $a:=d^{(\ell)}$, $\ell=\overline{0,r-1}$.

In what follows, we assume that the distributions p^* and $p^{(\ell)}$, $\ell=\overline{0,r-1}$, are not fixed. Therefore, we consider the game $\Gamma(p^*,p^{(0)},p^{(1)},\ldots,p^{(r-1)})$ with the final sequence of states X, the initial distribution of the states p^* and the strategies of the players $p^{(\ell)}$, $\ell=\overline{0,r-1}$, for each set of parameters p^* and $p^{(\ell)}$, $\ell=\overline{0,r-1}$. The problem is to determine the optimal distribution $\overline{p}^*=p^*$ and the optimal strategies $\overline{p}^{(\ell)}=p^{(\ell)}$, $\ell=\overline{0,r-1}$, which minimize the expectation of the duration $T(p^*,p^{(0)},p^{(1)},\ldots,p^{(r-1)}))$ of the game $\Gamma(p^*,p^{(0)},p^{(1)},\ldots,p^{(r-1)})$.

As in Section 9.2, the following theorem holds:

Theorem 9.1 *The optimal initial distribution of the states is* $\overline{p}^*$ *with*

$$\overline{p}^*(x_1)=1,\ \overline{p}^*(x)=0,\ \forall x\in V\setminus\{x_1\}.$$

Theorem 9.2 *Consider the set of active final states*

$$\overline{X}=\{x_1,x_2,\ldots,x_{m-1}\},$$

the set of final transitions

$$\overline{Y}=\{(x_1,x_2),(x_2,x_3),\ldots,(x_{m-1},x_m)\}$$

and the set of branching states

$$\overline{Z}=\{y\in\overline{X}\setminus\{x_1\}\mid\exists x\in\overline{X},\exists z\in\overline{X}\cup\{x_m\},z\neq y:(x,y)\in\overline{Y},(x,z)\in\overline{Y}\}.$$

The optimal strategies $\overline{p}^{(\ell)}$, $\ell=\overline{0,r-1}$, *satisfy the following properties:*

1. $\overline{p}^{(\ell)}(x,x_1)=1$, *if* $(x,x_1)\in\overline{Y}$ *and* $(x,z)\notin\overline{Y}$, $\forall z\neq x_1$;
2. $\overline{p}^{(\ell)}(x,x_1)=1$, $\forall x\notin\overline{X}$;
3. $\overline{p}^{(\ell)}(x,x_1)>0$, $\forall x\in\overline{Z}$ *and* $\overline{p}^{(\ell)}(x,x_1)=0$ *if* $(x,x_1)\notin\overline{Y}$, $x\in\overline{X}\setminus\overline{Z}$;

4. $\overline{p}^{(\ell)}(x,y)=0$, *if* $(x,y)\notin\overline{Y}$ *and* $y\neq x_1$;

5. $\overline{p}^{(\ell)}(x,y)>0,\ \forall(x,y)\in\overline{Y}$;

6. $\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}}\overline{p}^{(\ell)}(x,y)=1,\forall x\in\overline{X}$.

Theorem 9.3 *Let* $p=(p^{(0)},p^{(1)},\ldots,p^{(r-1)})$. *If* $\delta_{i,j}(p)\not\equiv 0,\ i,j=\overline{1,2}$, *then the optimal transition matrix can be found by solving the geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p^*,p))=d_1d_2^{-1}\to\min, \tag{9.49}$$

with the constraints

$$\begin{cases}\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}}p^{(\ell)}(x,y)=1,\forall x\in\overline{X},\ \ell=\overline{0,r-1}\\ d_{1,1}^{-1}d_1+d_{1,1}^{-1}d_{1,2}=1,\ d_{2,1}^{-1}d_2+d_{2,1}^{-1}d_{2,2}=1\\ d_{1,1}^{-1}\delta_{1,1}(p)=1,\ d_{1,2}^{-1}\delta_{1,2}(p)=1,\ d_{2,1}^{-1}\delta_{2,1}(p)=1,\ d_{2,2}^{-1}\delta_{2,2}(p)=1\\ d_1,d_2,d_{1,1},d_{1,2},d_{2,1},d_{2,2}>0\\ p^{(\ell)}(x,y)>0,\forall(x,y)\in\overline{Y},\ \ell=\overline{0,r-1}\\ p^{(\ell)}(x,x_1)>0,\ \forall x\in\overline{Z},\ell=\overline{0,r-1}\end{cases} \tag{9.50}$$

and (9.49) with the constraints

$$\begin{cases}\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}}p^{(\ell)}(x,y)=1,\forall x\in\overline{X},\ \ell=\overline{0,r-1}\\ d_{1,1}^{-1}d_1+d_{1,1}^{-1}d_{1,2}=1,\ d_{2,1}^{-1}d_2+d_{2,1}^{-1}d_{2,2}=1\\ d_{1,1}^{-1}\delta_{1,2}(p)=1,\ d_{1,2}^{-1}\delta_{1,1}(p)=1,\ d_{2,1}^{-1}\delta_{2,2}(p)=1,\ d_{2,2}^{-1}\delta_{2,1}(p)=1\\ d_1,d_2,d_{1,1},d_{1,2},d_{2,1},d_{2,2}>0\\ p^{(\ell)}(x,y)>0,\forall(x,y)\in\overline{Y},\ \ell=\overline{0,r-1}\\ p^{(\ell)}(x,x_1)>0,\ \forall x\in\overline{Z},\ell=\overline{0,r-1}\end{cases} \tag{9.51}$$

in accordance with the properties described in Theorem 9.1 and Theorem 9.2, where $\delta_{i,j}(p),\ \ i,j=\overline{1,2}$, *are the posynomials from the decomposition*

$$\mathbb{E}(T(p^*,p))=(\delta_{1,1}(p)-\delta_{1,2}(p))(\delta_{2,1}(p)-\delta_{2,2}(p))^{-1}, \tag{9.52}$$

which follow from Algorithm 9.2. The signomial programs (9.49) − (9.50) and (9.49) − (9.51) can be handled as geometric programs using the method described in [25]. If $\overline{p}^1$ *is the optimal solution of the problem (9.49) − (9.50) and* $\overline{p}^2$ *is the optimal solution of the problem (9.49) − (9.51), then the optimal transition matrix is the element* $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ *for which the value* $\mathbb{E}(T(\overline{p^*}, \overline{p}))$ *is minimal. If there exists at least one* $\delta_{i^*,j^*}(p) \equiv 0$, *then in (9.50) and (9.51) the corresponding posynomial equality constraints are eliminated and the related substitution* $d_{i^*,j^*} = 0$ *is performed in (9.50) and the substitution* $d_{i^*,3-j^*} = 0$ *is performed in (9.51).*

Therefore, Theorem 9.3 shows us how to determine the optimal cooperative strategies of the players using signomial and geometric programming approaches.

9.5 Stochastic games with real transition costs and final sequence of states

Let $L_{\mathbb{R}}$ be a system with a finite set of states $V_{\mathbb{R}}$, $|V_{\mathbb{R}}| = \omega_{\mathbb{R}}$. The system $L_{\mathbb{R}}$ starts its evolution from the state $v \in V_{\mathbb{R}}$ with probability $p^*_{\mathbb{R}}(v)$, $\forall v \in V_{\mathbb{R}}$. Also, at every step t, the system $L_{\mathbb{R}}$ passes from the state $u \in V_{\mathbb{R}}$ to the state $v \in V_{\mathbb{R}}$ with transition probability $p_{\mathbb{R}}(u,v)$ and transition cost $c_{\mathbb{R}}(u,v) \in \mathbb{R}$, which do not depend on t.

We assume that a sequence of states

$$X_{\mathbb{R}} = (x_1, x_2, \ldots, x_m)$$

is given and the system $L_{\mathbb{R}}$ stops as soon as the states $x_1, x_2, \ldots, x_m$ are reached consecutively in the given order. We denote by $C_{\mathbb{R}}$ the evolution outcome of the stochastic system $L_{\mathbb{R}}$, i.e.

$$C_{\mathbb{R}} = \sum_{(u,v) \in E_{\mathbb{R}}} c_{\mathbb{R}}(u,v),$$

where $E_{\mathbb{R}}$ is the set of transitions that compose the evolution of the given system $L_{\mathbb{R}}$.

The system $L_{\mathbb{R}}$ represents a Markov process with a set of states $V_{\mathbb{R}}$, an initial distribution of the states $p^*_{\mathbb{R}} = (p^*_{\mathbb{R}}(v))_{v\in V_{\mathbb{R}}}$ and a transition probability matrix $p_{\mathbb{R}} = (p_{\mathbb{R}}(u,v))_{u,v\in V_{\mathbb{R}}}$, a real transition-cost matrix $c_{\mathbb{R}} = (c_{\mathbb{R}}(u,v))_{u,v\in V_{\mathbb{R}}}$ and a final sequence of states $X_{\mathbb{R}}$.

Next, the following game $\Gamma_{\mathbb{R}}$ is considered ([98]). Initially, each player $\mathcal{P}_\ell$, $\ell = \overline{0,r-1}$, defines his stationary strategy, represented by one transition matrix

$$p^{(\ell)}_{\mathbb{R}} = (p^{(\ell)}_{\mathbb{R}}(u,v))_{u,v\in V_{\mathbb{R}}},\ \ell = \overline{0,r-1}.$$

The initial distribution of the states is $p^*_{\mathbb{R}} = (p^*_{\mathbb{R}}(v))_{v\in V_{\mathbb{R}}}$. The game is started by the first player $\mathcal{P}_0$. At every discrete time, the system $L_{\mathbb{R}}$ passes to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$ has acted, the first player $\mathcal{P}_0$ acts on the evolution of the system and the game continues in this way until the given final sequence of states $X_{\mathbb{R}}$ is reached. The winner of the game is the player who acts last.

Our goal is to study the game outcome $C_{\mathbb{R}}$, knowing

- the initial distribution of the states $p^*_{\mathbb{R}} = (p^*_{\mathbb{R}}(v))_{v\in V_{\mathbb{R}}}$;
- the real cost transition matrix $c_{\mathbb{R}} = (c_{\mathbb{R}}(u,v))_{u,v\in V_{\mathbb{R}}}$;
- the stationary strategy $p^{(\ell)}_{\mathbb{R}} = (p^{(\ell)}_{\mathbb{R}}(u,v))_{u,v\in V_{\mathbb{R}}}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0,r-1}$;
- the final sequence of states $X_{\mathbb{R}}$ of the system $L_{\mathbb{R}}$.

We will show how this problem can be reduced to the unit transition costs using the extended-network method and how to find the winning probabilities of the players.

Reduction to the case of positive real transition costs.

First of all, let us reduce the game network to the case when the transition costs are positive real numbers. This helps us to avoid the operational issues caused by negative and zero numbers.

Let us consider a new game $\Gamma_{\mathbb{R}^+}$, defined on the stochastic system $L_{\mathbb{R}^+}$ with the same final sequence of states $X_{\mathbb{R}^+} = X_{\mathbb{R}}$, finite set of states $V_{\mathbb{R}^+} = V_{\mathbb{R}}$, initial distribution

of the states $p^*_{\mathbb{R}^+} = p^*_{\mathbb{R}}$ and stationary strategy $p^{(\ell)}_{\mathbb{R}^+} = p^{(\ell)}_{\mathbb{R}}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$. The difference between the games $\Gamma_{\mathbb{R}^+}$ and $\Gamma_{\mathbb{R}}$ consists in the way in which the transition costs are defined.

Let us denote by

$$c^{\min}_{\mathbb{R}} = \min_{u,v \in V_{\mathbb{R}}} c_{\mathbb{R}}(u,v)$$

the minimal transition cost in the game $\Gamma_{\mathbb{R}}$. We define the transition costs in the new game $\Gamma_{\mathbb{R}^+}$ in the following way:

$$c_{\mathbb{R}^+}(u,v) = c_{\mathbb{R}}(u,v) - c^{\min}_{\mathbb{R}} + 1 \geq 1, \ \forall u,v \in V_{\mathbb{R}^+}. \tag{9.53}$$

Next, we consider a new game Γ_1, defined on the stochastic system L_1 with the same final sequence of states $X_1 = X_{\mathbb{R}}$, finite set of states $V_1 = V_{\mathbb{R}}$, initial distribution of the states $p^*_1 = p^*_{\mathbb{R}}$ and stationary strategy $p^{(\ell)}_1 = p^{(\ell)}_{\mathbb{R}}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$. The transition costs in the new game Γ_1 are all equal to 1.

It is easy to observe that the following relation holds:

$$C_{\mathbb{R}} = C_{\mathbb{R}^+} + (c^{\min}_{\mathbb{R}} - 1) \cdot C_1, \tag{9.54}$$

where C_1 is the outcome of the game Γ_1, $C_{\mathbb{R}^+}$ is the outcome of the game $\Gamma_{\mathbb{R}^+}$ and $C_{\mathbb{R}}$ is the outcome of the game $\Gamma_{\mathbb{R}}$. Indeed, we have

$$C_{\mathbb{R}^+} = \sum_{(u,v) \in E_{\mathbb{R}^+}} c_{\mathbb{R}^+}(u,v) =$$

$$= \sum_{(u,v) \in E_{\mathbb{R}}} \left(c_{\mathbb{R}}(u,v) - c^{\min}_{\mathbb{R}} + 1 \right) =$$

$$= \sum_{(u,v) \in E_{\mathbb{R}}} c_{\mathbb{R}}(u,v) - \sum_{(u,v) \in E_{\mathbb{R}}} \left(c^{\min}_{\mathbb{R}} - 1 \right) =$$

$$= \sum_{(u,v) \in E_{\mathbb{R}}} c_{\mathbb{R}}(u,v) - (c^{\min}_{\mathbb{R}} - 1) \sum_{(u,v) \in E_{\mathbb{R}}} 1 =$$

$$= C_{\mathbb{R}} - (c^{\min}_{\mathbb{R}} - 1) \cdot C_1.$$

Reduction to the case of positive rational transition costs.

At this step, we reduce the game network to the case in which the transition costs of the related stochastic system are positive rational values. We approximate each real transition cost by a rational number so that the error in the expected game outcome is less than a given value $\varepsilon > 0$.

We consider the game $\Gamma_{\mathbb{Q}^+}$, defined on the stochastic system $L_{\mathbb{Q}^+}$ with the same final sequence of states $X_{\mathbb{Q}^+} = X_{\mathbb{R}^+}$, finite set of states $V_{\mathbb{Q}^+} = V_{\mathbb{R}^+}$, initial distribution of the states $p^*_{\mathbb{Q}^+} = p^*_{\mathbb{R}^+}$ and stationary strategy $p^{(\ell)}_{\mathbb{Q}^+} = p^{(\ell)}_{\mathbb{R}^+}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$. We define the transition costs in the new game $\Gamma_{\mathbb{Q}^+}$ in the following way:

$$c_{\mathbb{Q}^+}(u,v) = c_{\mathbb{R}^+}(u,v) + \varepsilon(u,v),\ \forall u,v \in V_{\mathbb{Q}^+}, \tag{9.55}$$

choosing a nonnegative value $\varepsilon(u,v)$ such that

$$c_{\mathbb{Q}^+}(u,v) \in \mathbb{Q}^+,\ \forall u,v \in V_{\mathbb{Q}^+}$$

and

$$\left| \mathbb{E}\left(C_{\mathbb{Q}^+}\right) - \mathbb{E}\left(C_{\mathbb{R}^+}\right) \right| \le \varepsilon, \tag{9.56}$$

where $\varepsilon > 0$ is the prescribed accuracy.

Consider

$$0 \le \varepsilon(u,v) \le \varepsilon^* < 1,\ \forall u,v \in V_{\mathbb{Q}^+}. \tag{9.57}$$

We obtain

$$C_{\mathbb{Q}^+} = \sum_{(u,v)\in E_{\mathbb{Q}^+}} c_{\mathbb{Q}^+}(u,v) = \sum_{(u,v)\in E_{\mathbb{R}^+}} \left(c_{\mathbb{R}^+}(u,v) + \varepsilon(u,v)\right) =$$

$$= \sum_{(u,v)\in E_{\mathbb{R}^+}} c_{\mathbb{R}^+}(u,v) + \sum_{(u,v)\in E_{\mathbb{R}^+}} \varepsilon(u,v) =$$

$$= \sum_{(u,v)\in E_{\mathbb{R}^+}} c_{\mathbb{R}^+}(u,v) + \varepsilon(u,v) \cdot \sum_{(u,v)\in E_{\mathbb{R}^+}} 1 = C_{\mathbb{R}^+} + \varepsilon(u,v) \cdot C_1,$$

which implies

$$\left|\mathbb{E}\left(C_{\mathbb{Q}^+}\right)-\mathbb{E}\left(C_{\mathbb{R}^+}\right)\right|=\left|\mathbb{E}\left(C_{\mathbb{Q}^+}-C_{\mathbb{R}^+}\right)\right|=$$

$$=\mathbb{E}(\varepsilon(u,v)\cdot C_1)=\varepsilon(u,v)\cdot\mathbb{E}(C_1)\le\varepsilon^*\mathbb{E}(C_1). \tag{9.58}$$

If we have $\varepsilon^*\mathbb{E}(C_1)\le\varepsilon$, then the following inequality also holds:

$$\left|\mathbb{E}\left(C_{\mathbb{Q}^+}\right)-\mathbb{E}\left(C_{\mathbb{R}^+}\right)\right|\le\varepsilon.$$

Consider an arbitrary value

$$\varepsilon^*\in\left(0,\min\left\{1,\varepsilon/\mathbb{E}(C_1)\right\}\right].$$

If the natural number s satisfies the condition $\varepsilon^*>10^{-s}$, then we can choose

$$\varepsilon(u,v)=10^{-s}(1-\{10^s c_{\mathbb{R}^+}(u,v)\}),\ \forall u,v\in V_{\mathbb{Q}^+}. \tag{9.59}$$

and these values satisfy relations (9.55), (9.56) and (9.57), where $\{a\}$ denotes the fractional part of an arbitrary real number a. From (9.55) and (9.59), we obtain

$$c_{\mathbb{Q}^+}(u,v)=10^{-s}(1+\lfloor 10^s c_{\mathbb{R}^+}(u,v)\rfloor),\ \forall u,v\in V_{\mathbb{Q}^+}, \tag{9.60}$$

where $\lfloor a\rfloor$ denotes the integer part of the arbitrary real number a. Therefore, we can consider

$$E\left(C_{\mathbb{R}^+}\right)\approx E\left(C_{\mathbb{Q}^+}\right)$$

with the approximation error ε.

Reduction to the case of positive integer transition costs.

At this step, we reduce the game network to the case in which the transition costs of the related stochastic system are positive integer values. We will do this transformation by changing the measurement unit, i.e. we will multiply each positive rational transition cost

$$c_{\mathbb{Q}^+}(u,v),\ \forall u,v\in V_{\mathbb{Q}^+},$$

by μ, where μ is the least common multiple of the denominators of these rational transition costs.

Consider a new game $\Gamma_{\mathbb{N}^*}$, defined on the stochastic system $L_{\mathbb{N}^*}$ with the same final sequence of states $X_{\mathbb{N}^*} = X_{\mathbb{Q}^+}$, finite set of states $V_{\mathbb{N}^*} = V_{\mathbb{Q}^+}$, initial distribution of the states $p^*_{\mathbb{N}^*} = p^*_{\mathbb{Q}^+}$ and stationary strategy $p^{(\ell)}_{\mathbb{N}^*} = p^{(\ell)}_{\mathbb{Q}^+}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$. We define the transition costs in the new game $\Gamma_{\mathbb{N}^*}$ in the following way:

$$c_{\mathbb{N}^*}(u,v) = \mu \cdot c_{\mathbb{Q}^+}(u,v),\ \forall u,v \in V_{\mathbb{N}^*}. \tag{9.61}$$

In this way, we have

$$c_{\mathbb{N}^*}(u,v) \in \mathbb{N}^*,\ \forall u,v \in V_{\mathbb{N}^*}.$$

We obtain

$$C_{\mathbb{N}^*} = \sum_{(u,v)\in E_{\mathbb{N}^*}} c_{\mathbb{N}^*}(u,v) = \sum_{(u,v)\in E_{\mathbb{N}^*}} \left(\mu \cdot c_{\mathbb{Q}^+}(u,v)\right) =$$

$$= \mu \cdot \sum_{(u,v)\in E_{\mathbb{Q}^+}} c_{\mathbb{Q}^+}(u,v) = \mu \cdot C_{\mathbb{Q}^+}, \tag{9.62}$$

which implies

$$C_{\mathbb{Q}^+} = \frac{1}{\mu} \cdot C_{\mathbb{N}^*}.$$

Players' states synchronization.

The next step prepares the transition costs for application of the extended-network method. The application of the extended-network method must not change the player who acts in state v at each transition (u, v). For this reason, we will ensure that after this transformation, each positive integer transition cost is congruent to 1 modulo r, where r is the number of the players.

Let us consider a new game Γ_f, defined on the stochastic system L_f with the same final sequence of states $X_f = X_{\mathbb{N}^*}$, finite set of states $V_f = V_{\mathbb{N}^*}$, initial distribution of the states $p^*_f = p^*_{\mathbb{N}^*}$ and stationary strategy $p^{(\ell)}_f = p^{(\ell)}_{\mathbb{N}^*}$ of each player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$. We define the transition costs in the new game Γ_f in the following way:

$$c_f(u,v) = r \cdot c_{\mathbb{N}^*}(u,v) + 1,\ \forall u,v \in V_f, \tag{9.63}$$

i.e. we apply the function

$$f : \mathbb{N}^* \to \mathbb{N}^*,\ f(t) = rt + 1,$$

to each positive integer transition cost.

In this way, we will have

$$c_f(u,v) \in \mathbb{N}^*,\ c_f(u,v) \equiv 1 (\text{mod } r),\ \forall u,v \in V_f.$$

Also,

$$C_f = \sum_{(u,v)\in E_f} c_f(u,v) = \sum_{(u,v)\in E_f} \left(r \cdot c_{\mathbb{N}^*}(u,v) + 1 \right) =$$

$$= r \cdot \sum_{(u,v)\in E_{\mathbb{N}^*}} c_{\mathbb{N}^*}(u,v) + C_1 = r \cdot C_{\mathbb{N}^*} + C_1, \tag{9.64}$$

which implies

$$C_{\mathbb{N}^*} = \frac{1}{r} \cdot \left(C_f - C_1 \right).$$

Reduction to the case of unit transition costs.

At this step, we reduce the game network to the case in which the transition costs of the related stochastic system are equal to 1. We do this transformation by applying the extended-network method, i.e. each edge (u,v) with positive integer transition cost $c_f(u,v)$ will be replaced with $c_f(u,v)$ edges with unit transition cost, $\forall u,v \in V_f$. In this way, the total transition cost from an arbitrary state $u \in V_f$ to another state $v \in V_f$ remains unchanged. It is only necessary to ensure that the transition probability from the state u to the state v also remains unchanged after applying the extended-network method.

Let us consider a new game Γ_{ext}, defined on the stochastic system L_{ext} in the following way. For each state $v \in V_f$ we choose the state $u \in V_f$ such that

$$c_f(u,v) = \max_{z \in V_f^-(v)} c_f(z,v),$$

where

$$V_f^-(v)=\left\{z\in V_f \;\middle|\; \sum_{l=0}^{r-1} p_f^{(l)}(z,v)>0\right\}.$$

We split the transition (u,v) into $c_f(u,v)$ unit transitions using the new states

$$z_0(u,v)=u;\ z_1(u,v);\ z_2(u,v);\ \ldots;\ z_{c_f(u,v)-1}(u,v);\ z_{c_f(u,v)}(u,v)=v.$$

If $c_f(u,v)>1$, then the stationary strategies $p_f^{(\ell)}$, $\ell=\overline{0,r-1}$, are updated in the following way:

- new rows and columns, filled with zeros, that correspond to intermediate states $z_j(u,v)$, $j=\overline{1,c_f(u,v)-1}$, are added;
- the entry $p_f^{(\ell)}(z_j(u,v),z_{j+1}(u,v))$ is set to 1, $j=\overline{1,c_f(u,v)-1}$;
- the entry $p_f^{(\ell)}(z,z_{c_f(u,v)-c_f(z,v)+1}(u,v))$ is set to $p_f^{(\ell)}(z,v)$ and, after that, the entry $p_f^{(\ell)}(z,v)$ is set to 0, for each $z\in V_f^-(v)$ that satisfies the inequality $c_f(z,v)>1$.

We consider

$$p_f^*(z_j(u,v))=0,\ j=\overline{1,c_f(u,v)-1},$$

because the game cannot be started from an intermediate state. The final sequence of states becomes

$$\begin{aligned}X_{ext}=\ & (x_1,\ z_1(x_1,x_2),\ z_2(x_1,x_2),\ldots,z_{c_f(x_1,x_2)-1}(x_1,x_2),x_2,\\ & z_1(x_2,x_3),z_2(x_2,x_3),\ldots,z_{c_f(x_2,x_3)-1}(x_2,x_3),x_3,\\ & \ldots,\\ & z_1(x_{m-1},x_m),z_2(x_{m-1},x_m),\ldots,z_{c_f(x_{m-1},x_m)-1}(x_{m-1},x_m),x_m).\end{aligned}$$

Therefore, if we put all these things together, we obtain that the new game Γ_{ext} is defined on the stochastic system L_{ext}, with a finite set of states V_{ext}, a final sequence of states X_{ext}, an initial distribution of the states p_{ext}^* and a stationary strategy $p_{ext}^{(\ell)}$ of each player $\mathcal{P}_\ell$, $\ell=\overline{0,r-1}$, defined in the way described above. Also, each transition cost in

this stochastic system is equal to 1. This implies that the outcome C_{ext} of the game Γ_{ext} is

$$C_{ext} = \sum_{(u,v)\in E_{ext}} c_{ext}(u,v) = \sum_{(u,v)\in E_{ext}} 1 = \left|E_{ext}\right| = \sum_{(u,v)\in E_f} c_f(u,v) = C_f\,, \tag{9.65}$$

which implies

$$C_f = C_{ext}\,.$$

Expectation of the game outcome.

Let us summarize the results obtained above. We have the following formula for calculating the outcome $C_{\mathbb{R}}$ of the game $\Gamma_{\mathbb{R}}$:

$$C_{\mathbb{R}} = C_{\mathbb{R}^+} + (c_{\mathbb{R}}^{\min} - 1)\cdot C_1 \approx C_{\mathbb{Q}^+} + (c_{\mathbb{R}}^{\min} - 1)\cdot C_1 =$$

$$= \frac{1}{\mu}\cdot C_{\mathbb{N}^*} + (c_{\mathbb{R}}^{\min} - 1)\cdot C_1 = \frac{C_f - C_1}{r\mu} + (c_{\mathbb{R}}^{\min} - 1)\cdot C_1 =$$

$$= \frac{C_{ext} - C_1}{r\mu} + (c_{\mathbb{R}}^{\min} - 1)\cdot C_1 = \frac{1}{r\mu}\cdot C_{ext} + \left(c_{\mathbb{R}}^{\min} - 1 - \frac{1}{r\mu}\right)\cdot C_1\,.$$

So, the expectation of the outcome $C_{\mathbb{R}}$ of the game $\Gamma_{\mathbb{R}}$ is

$$\mathbb{E}(C_{\mathbb{R}}) \approx \frac{1}{r\mu}\cdot \mathbb{E}(C_{ext}) + \left(c_{\mathbb{R}}^{\min} - 1 - \frac{1}{r\mu}\right)\cdot \mathbb{E}(C_1)\,. \tag{9.66}$$

The values $\mathbb{E}(C_{ext})$ and $\mathbb{E}(C_1)$ can be determined according to Section 9.3, because both games Γ_{ext} and Γ_1 are defined on stochastic systems with unit transition costs (which, in this particular case, can be considered unit transition times).

Winning probabilities of the players.

From the definition of the winning probabilities of the players, it is easy to observe that they depend only on the network structure and the transition probabilities of the stochastic system on which the game is defined. Indeed, if we consider that each transition time in the game $\Gamma_{\mathbb{R}}$ is equal to 1, and $T_{\mathbb{R}}$ is the duration of the game

$\Gamma_{\mathbb{R}}$, then the winning probability $w_{\mathbb{R}}^{(\ell)}$ of the player $\mathcal{P}_\ell$, $\ell = \overline{0, r-1}$, is given by the following relation:

$$w_{\mathbb{R}}^{(\ell)} = P(T_{\mathbb{R}} \equiv \ell \pmod r) = \sum_{k=0}^{\infty} P(T_{\mathbb{R}} = k + \ell)\,,\ \ell = \overline{0, r-1}\,. \tag{9.67}$$

Therefore, the winning probabilities of the players do not depend on the transition costs. The winning probabilities of the players in the game $\Gamma_{\mathbb{R}}$ are the same as the winning probabilities of the players in the game Γ_1, which can be found according to Section 9.4.

9.6 Knuth-Morris-Pratt approach for Markov games with a final sequence of states

Let L be a Markov process with the final sequence of states X, as defined in Section 6.5, with initial distribution of states $(p^*(v))_{v\in V}$. A game defined on the stochastic system L was considered in Section 9.3.

According to the game rules, each player $\mathcal{P}_\ell$ has their own strategy, which is a transition matrix $(p^{(\ell)}(u,v))_{u,v\in V}$, $\ell = \overline{0, r-1}$. The game is started by the first player $\mathcal{P}_0$. At every discrete time, the stochastic system passes consecutively to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{r-1}$ has acted, the first player $\mathcal{P}_0$ acts on the evolution of the system and the game continues in this way until the given final sequence of states X is reached. The winner of the game is the player $\mathcal{P}_{T \bmod r}$ whose strategy is applied last.

It is known from Section 9.3 that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence of order $mr\omega$ over the set $\mathbb{R}$.

Let us apply the optimization provided by the Knuth–Morris–Pratt string-searching algorithm. Each state $v \in V$ is enriched with a number k, $0 \le k \le m$, that represents the length of the longest prefix of the final sequence of states X matched so far. The set of states becomes

$$Z = \bigcup_{k=0}^{m} Z^{(k)},$$

where

$$Z^{(0)} = \{(v,0) \mid v \in V \setminus \{x_1\}\},$$

$$Z^{(k)} = \{(x_k,k)\}, k = \overline{1,m}.$$

This set contains

$$|Z| = \omega - 1 + m$$

states.

The initial probabilities and the strategies of the players, $\ell = \overline{0,r-1}$, are:

$$p^*((v,0)) = p^*(v), \forall v \in V \setminus \{x_1\},$$

$$p^*((x_1,1)) = p^*(x_1),\ p^*((x_k,k)) = 0,\ k = \overline{2,m},$$

$$p^{(\ell)}((u,0),(v,0)) = p^{(\ell)}(u,v), \forall u,v \in V \setminus \{x_1\},$$

$$p^{(\ell)}((u,0),(x_1,1)) = p^{(\ell)}(u,x_1), \forall u \in V \setminus \{x_1\},$$

$$p^{(\ell)}((x_k,k),(v,0)) = p^{(\ell)}(x_k,v), \forall v \in V \setminus \{x_1, x_2, \ldots, x_m\},\ k = \overline{1,m},$$

$$p^{(\ell)}((x_k,k),(x_{k+1},k+1)) = p^{(\ell)}(x_k,x_{k+1}),\ k = \overline{1,m-1},$$

$$p^{(\ell)}((x_k,k),(v,f_k(v))) = p^{(\ell)}(x_k,v), \forall v \in \{x_1, x_2, \ldots, x_m\} \setminus \{x_{k+1}\},\ k = \overline{1,m-1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (x_k,k). For the remaining transitions, the probability is equal to 0.

So, the initial game is transformed into an equivalent game with the final sequence of states $X_{new} = ((x_m,m))$ of length $m_{new} = 1$. The number of the players r remains the same and the number of states of the new game becomes $\omega_{new} = \omega - 1 + m$. The game duration remains unchanged and its distribution becomes a homogeneous linear recurrence of smaller order

$$m_{new} \cdot r \cdot \omega_{new} = 1 \cdot r \cdot (\omega - 1 + m) = r(\omega - 1 + m).$$

9.7 State merging for zero-order Markov games with a final sequence of states

The games defined on zero-order Markov processes with a final sequence of states were studied in Section 9.2 and represent a particular case of the games from Section 9.6. It is known that the distribution $a = (\mathbb{P}(T = n))_{n=0}^{\infty}$ of the game duration T is a homogeneous linear recurrence of order mr over $\mathbb{R}$. By applying the state-merging technique, we can reduce the number of states from ω to

$$\omega^* = |\{x_1, x_2, \ldots, x_m\}| + 1 \le m + 1.$$

We can consider this zero-order game as a general game with a final sequence of states and apply the enhancement from Section 9.6, obtaining a new game with a final sequence of states of length $m_{new} = 1$ and the number of states

$$\omega_{new} = \omega^* - 1 + m \le 2m,$$

but this transformation does not give us any improvement in this case. Indeed, the game duration remains unchanged and the order of the homogeneous linear recurrence, which describes its distribution, becomes

$$r(\omega^* - 1 + m) \ge r(2 - 1 + m) > mr.$$

So, only the state-merging technique gives us a small improvement for the games defined on zero-order Markov processes with a final sequence of states. This state-merging strategy also allows us to consider the games defined on zero-order Markov processes with an infinite set of states. By merging the states that do not belong to the final sequence of states, the entire set of states becomes finite.

9.8 Extended-network method for stochastic games with a final sequence of states

Consider the game described in Section 9.3, defined on the stochastic system L with the set of states $V = \{v_1, v_2, \ldots, v_\omega\}$ and the final sequence of states

$$X = (x_1, x_2, \ldots, x_m) \in V^m.$$

Initially, each player $\mathcal{P}_s$ defines their own strategy, represented by a transition matrix

$$(p^{(s)}(u,v))_{u,v\in V},\ s=\overline{0,R-1}.$$

The initial probabilities of the states are $(p^*(v))_{v\in V}$.

The game is initiated by the first player $\mathcal{P}_0$. At each discrete time, the stochastic system successively transitions to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{R-1}$ has acted, the first player $\mathcal{P}_0$ acts again on the evolution of the system, and the game continues until the final sequence of states X is reached for the first time. The winner of the game is the player $\mathcal{P}_{T\ mod\ R}$ whose strategy is applied last.

In what follows, we present the extended-network method as an alternative to the approach from Section 9.3. We consider a copy of each state $v\in V$ for each player $\mathcal{P}_s$, $s=\overline{0,R-1}$. Therefore, the set of states becomes

$$U=V\times\{0,1,\ldots,R-1\}.$$

The state is (v,s) at a given discrete time t if the state is v and $t\ mod\ R=s$. The final sequence of states X corresponds to the multiple sequences of states

$$X^{(s)}=((x_1,s),(x_2,s\oplus 1),\ldots,(x_m,s\oplus(m-1)))\in U^m,\ s=\overline{0,R-1},$$

where

$$j\oplus k=(j+k)\ mod\ R,\ \forall j,k\in\mathbb{N}.$$

For $\forall v\in V$, the new initial probabilities are

$$p^*((v,0))=p^*(v),$$

$$p^*((v,s))=0,\ s=\overline{1,R-1}.$$

Also, for $s=\overline{0,R-1}$ and $\forall u,v\in V$, the new transition probabilities are

$$p((u,s),(v,s\oplus 1))=p^{(s)}(u,v),$$

$$p((u,s),(v,k))=0,\ k\neq s\oplus 1.$$

Therefore, the game is converted into a stochastic system with ωR states and R final sequences of states

$$X^{(s)},\ s=\overline{0,R-1}.$$

According to Section 8.5, the distribution of the game duration T is a homogeneous linear recurrence of order $R(\omega+m)$. The probabilistic analysis of T can be performed by using Algorithm 4.4 and the winning probabilities of the players can be obtained based on Section 9.4.

9.9 Extended-network method for games with multiple final sequences of states

Let L be a Markov process with the set of states $V=\{v_1,v_2,\ldots,v_\omega\}$ and the final sequences of states

$$X^{(\ell)}=(x_1^{(\ell)},x_2^{(\ell)},\ldots,x_m^{(\ell)})\in V^m,\ \ell=\overline{1,r}.$$

As in Section 9.3, we consider the following game. Initially, each player $\mathcal{P}_s$ defines their own stationary strategy, represented by a transition matrix

$$(p^{(s)}(u,v))_{u,v\in V},\ s=\overline{0,R-1}.$$

The initial probabilities of the states are specified by the distribution $(p^*(v))_{v\in V}$.

The game is initiated by the first player $\mathcal{P}_0$. At each discrete time, the stochastic system successively transitions to the next state according to the strategy of the current player. After the last player $\mathcal{P}_{R-1}$ has acted, the first player $\mathcal{P}_0$ acts again on the evolution of the system, and the game proceeds in this manner until one of the final sequences of states $X^{(\ell)}$, $\ell=\overline{1,r}$, is reached for the first time. The winner of the game is the player $\mathcal{P}_{T \bmod R}$ whose strategy is applied last.

Our objective is to study the game duration T, given the initial state probabilities $(p^*(v))_{v\in V}$, the stationary strategy

$$P^{(s)}=(p^{(s)}(u,v))_{u,v\in V}$$

of each player $\mathcal{P}_s$, $s=\overline{0,R-1}$, and the final sequences of states $X^{(\ell)}$, $\ell=\overline{1,r}$ of the stochastic system L.

Next, the extended-network method is applied. We consider a copy of each state $v \in V$ for each player $\mathcal{P}_s$, $s = \overline{0, R-1}$. Therefore, the set of states becomes

$$U = V \times \{0,1,\ldots,R-1\}.$$

The state is (v,s) at a given discrete time t if the state is v and $t \; mod \; R = s$. For each $\ell \in \{1,2,\ldots,r\}$, the final sequence of states $X^{(\ell)}$ corresponds to the multiple sequences of states

$$X^{(\ell,s)} = ((x_1^{(\ell)}, s), (x_2^{(\ell)}, s \oplus 1), \ldots, (x_m^{(\ell)}, s \oplus (m-1))) \in U^m, \; s = \overline{0, R-1},$$

where

$$j \oplus k = (j+k) \; mod \; R, \; \forall j,k \in \mathbb{N}.$$

For $\forall v \in V$, the new initial probabilities are

$$p^*((v,0)) = p^*(v),$$

$$p^*((v,s)) = 0, \; s = \overline{1, R-1}.$$

Also, for $s = \overline{0, R-1}$ and $\forall u, v \in V$, the new transition probabilities are

$$p((u,s),(v,s \oplus 1)) = p^{(s)}(u,v),$$

$$p((u,s),(v,k)) = 0, \; k \neq s \oplus 1.$$

Therefore, the game is converted into a stochastic system with ωR states and rR final sequences of states

$$X^{(\ell,s)}, \; \ell = \overline{1,r}, \; s = \overline{0, R-1}.$$

According to Section 8.5, the distribution of the game duration T is a homogeneous linear recurrence of order $R(\omega + mr)$. The probabilistic analysis of T can be performed by using Algorithm 4.4 and the winning probabilities of the players can be obtained based on Section 9.4.

10 HIDDEN AND NESTED MODELS WITH FINAL SEQUENCES OF STATES

In this chapter, we study the hidden Markov models with final sequences of observable states. We analyze the evolution time and the total alert time of these models and we prove that their distributions are homogeneous linear recurrences.

Further, by using Knuth-Morris-Pratt and Aho-Corasick string-searching algorithms, we extend the obtained results for the hidden Markov models with hierarchical emissions. The hierarchical Markov models with final sequences of states are also studied.

10.1 Hidden models with final sequence of observable states

Consider a hidden Markov model (HMM), represented by an underlying Markov process L with finite sets of hidden states $H = \{h_1, h_2, \ldots, h_\varpi\}$ and observable states $V = \{v_1, v_2, \ldots, v_\omega\}$. The system L starts from each state $h \in H$ with initial probability $p^*(h)$. At each discrete time $t = 0,1,2,\ldots$, the stochastic system L passes from a state $g \in H$ to the next state $h \in H$ with probability $p(g,h)$ and emits each observable state $v \in V$ with probability $e(h,v)$. Additionally, a final sequence of observable states

$$U = (u_1, u_2, \ldots, u_m) \in V^m$$

is given. The amount of time T needed to observe U represents the evolution time.

To reduce the complexity of determining the evolution time of hidden Markov models, a joint Markov process J over the set of states $H \times V$ is considered. The process J is in the state $(h,v) \in H \times V$ at time t if the hidden state of L is $h \in H$ and the observed state is $v \in V$ at the same time t. The final sequence of observable

states $U = (u_1, u_2, \ldots, u_m) \in V^m$ is mapped to a set of multiple final sequences of states of the form

$$((x_1, u_1), (x_2, u_2), \ldots (x_m, u_m))$$

for each tuple $(x_1, x_2, \ldots, x_m) \in H^m$.

So, J is a Markov process with multiple final sequences of states. We have

- The number of the states: $\omega_J = \omega \cdot \varpi$;
- The number of final sequences of states: $r_J = \varpi^m$;
- The length of the final sequences of states: $m_J = m$;
- The order of the recurrence: $m_J^2 \cdot (r_J + \omega_J) = m^2 \cdot (\varpi^m + \omega \cdot \varpi)$.

At this stage, the parameters have exponential size. We will explore the Knuth–Morris–Pratt algorithm for decreasing this order of recurrence.

In what follows, we enrich each state $(h, v) \in H \times V$ with an additional number k, $0 \le k \le m$, that represents the length of the longest prefix of the final sequence of states observed so far. The stochastic system J is in the state (h, v, k) at time t if at the last k moments of time $t-k+1, t-k+2, \ldots, t$ the states $u_1, u_2, \ldots, u_k$ have been observed and this number k is the largest one that satisfies this property. The number k is considered equal to 0 if no such sequence exists. The set of all possible states is

$$Z = \bigcup_{k=0}^{m} Z^{(k)},$$

where

$$Z^{(0)} = \{(h, v, 0) \mid h \in H, v \in V \setminus \{u_1\}\},$$

$$Z^{(k)} = \{(h, u_k, k) \mid h \in H\}, k = \overline{1, m}.$$

This set contains

$$| Z | = \varpi \cdot (\omega - 1 + m)$$

states.

For each $g, h \in H$, the initial and transition probabilities of the new states are:

$$p^*((h,v,0)) = p^*(h)\cdot e(h,v),\ \forall v \in V \setminus \{u_1\},$$

$$p^*((h,u_1,1)) = p^*(h)\cdot e(h,u_1),\ p^*((h,u_k,k)) = 0,\ k = \overline{2,m}.$$

$$p((g,u,0),(h,v,0)) = p(g,h)\cdot e(h,v),\ \forall u,v \in V \setminus \{u_1\},$$

$$p((g,u,0),(h,u_1,1)) = p(g,h)\cdot e(h,u_1),\ \forall u \in V \setminus \{u_1\},$$

$$p((g,u_k,k),(h,v,0)) = p(g,h)\cdot e(h,v),\ \forall v \in V \setminus \{u_1,\ldots,u_m\},\ k = \overline{1,m},$$

$$p((g,u_k,k),(h,u_{k+1},k+1)) = p(g,h)\cdot e(h,u_{k+1}),\ k = \overline{1,m-1},$$

$$p((g,u_k,k),(h,v,f_k(v))) = p(g,h)\cdot e(h,v),\ \forall v \in \{u_1,\ldots,u_m\} \setminus \{u_{k+1}\},\ k = \overline{1,m-1},$$

where $f_k(v)$ is the fallback position of the next state v when the current joint state is (g,u_k,k). For the remaining transitions, the probability is equal to 0.

Thus, the parameters of the stochastic system J have polynomial size:

- The number of states: $\omega_{new} = \varpi\cdot(\omega - 1 + m)$;
- The final sequences of states: $\{((h,u_m,m)) \mid h \in H\}$;
- The number of final sequences of states: $r_{new} = \varpi$;
- The length of the final sequence of states: $m_{new} = 1$.

The distribution of the evolution time T is a homogeneous linear recurrence of order

$$m_{new}^2\cdot(r_{new} + \omega_{new}) = \varpi\cdot(\omega + m).$$

10.2 Hidden models with multiple final sequences of observable states

Now, let us consider a similar HMM with multiple final sequences of observable states ([99])

$$U^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_m^{(\ell)}) \in V^m,\ \ell = \overline{1,r}.$$

The underlying system stops as soon as the states $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_m^{(\ell)}$ are observed for an arbitrary $\ell \in \{1,2,\ldots,r\}$. The evolution time T represents the amount of time needed to observe $U^{(\ell)}$.

As in Section 10.1, the joint Markov process J over $H \times V$ is considered. The process J is in the state $(h,v) \in H \times V$ at time t if the hidden state of L is $h \in H$ and the observed state is $v \in V$. Each final sequence of observable states

$$U^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_m^{(\ell)}) \in V^m, \ \ell \in \{1,2,\ldots,r\},$$

is mapped to a subset of final sequences of states of the form

$$((x_1, u_1^{(\ell)}), (x_2, u_2^{(\ell)}), \ldots (x_m, u_m^{(\ell)}))$$

for each tuple $(x_1, x_2, \ldots, x_m) \in H^m$.

So, J is a Markov process with multiple final sequences of states. We have

- The number of the states: $\omega_J = \omega \cdot \varpi$;
- The number of final sequences of states: $r_J = r \cdot \varpi^m$;
- The length of the final sequences of states: $m_J = m$;
- The order of the recurrence: $m_J^2 \cdot (r_J + \omega_J) = m^2 \cdot (r \cdot \varpi^m + \omega \cdot \varpi)$.

The parameters are also exponential in this case. We will explore the Aho-Corasick algorithm for decreasing this order of recurrence.

We enrich each state $(h,v) \in H \times V$ with two additional characteristics: the number k, $0 \le k \le m$, which represents the length of the longest prefix of the final sequence of states observed so far, and the corresponding prefix

$$Y_k^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)}), \ \ell \in \{0,1,\ldots,r\}.$$

The HMM is in the state $(h, v, Y_k^{(\ell)}, k)$ if at the last k moments of time the states $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)} = v$ have been observed and k is the largest one with this property among all $U^{(\ell)}$, $\ell = \overline{1,r}$. We have $k = 0$ and $Y_0^{(0)} = ()$ if no such sequence exists. The set of all possible states is

$$Z = \bigcup_{k=0}^{m} Z^{(k)},$$

where

$$Z^{(0)} = \{(h, v, Y_0^{(0)}, 0) \,|\, h \in H, v \in V \setminus U_1\},$$

$$Z^{(k)} = \{(h, u_k^{(\ell)}, Y_k^{(\ell)}, k) \mid h \in H, 1 \le \ell \le r\}, k = \overline{1,m}.$$

This set contains

$$|Z| \le \varpi \cdot [\omega + r(m-1)]$$

states.

As before, we use the following notation for $k = \overline{1,m}$, $\ell = \overline{1,r}$:

$$U_k = \{u_k^{(1)}, u_k^{(2)}, \ldots, u_k^{(r)}\},\ U = \{U^{(1)}, U^{(2)}, \ldots, U^{(r)}\},$$

$$Y_k^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)}),\ Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\},$$

$$Z_k^{(\ell)} = \{y \in U_k \mid (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{k-1}^{(\ell)}, y) \in Y_k\}.$$

For each $h \in H$, the initial probabilities for the new states are:

$$p^*((h, v, Y_0^{(0)}, 0)) = p^*(h) \cdot e(h, v),\ \forall v \in V \setminus U_1,$$

$$p^*((h, u_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p^*(h) \cdot e(h, u_1^{(\ell)}),\ \ell = \overline{1,r},$$

$$p^*((h, u_k^{(\ell)}, Y_k^{(\ell)}, k)) = 0,\ k = \overline{2,m},\ \ell = \overline{1,r}.$$

Also, for each $g, h \in H$ and $\ell = \overline{1,r}$, the transition probabilities become:

$$p((g, u, Y_0^{(0)}, 0), (h, v, Y_0^{(0)}, 0)) = p(g, h) \cdot e(h, v),\ \forall u, v \in V \setminus U_1,$$

$$p((g, u, Y_0^{(0)}, 0), (h, u_1^{(\ell)}, Y_1^{(\ell)}, 1)) = p(g, h) \cdot e(h, u_1^{(\ell)}),\ \forall u \in V \setminus U_1,$$

$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, v, Y_0^{(0)}, 0)) = p(g, h) \cdot e(h, v), \forall v \in V \setminus \bigcup_{j=1}^{m} U_j, k = \overline{1,m},$$

$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, u_{k+1}^{(\ell)}, Y_{k+1}^{(\ell)}, k+1)) = p(g, h) \cdot e(h, u_{k+1}^{(\ell)}),\ k = \overline{1, m-1},$$

$$p((g, u_k^{(\ell)}, Y_k^{(\ell)}, k), (h, v, Y_\kappa^{(\ell_\kappa)}, \kappa)) = p(g, h) \cdot e(h, v),\ \forall v \in \bigcup_{j=1}^{m} U_j \setminus Z_{k+1}^{(\ell)},\ k = \overline{1, m-1},$$

where $\kappa = f_k^{(\ell)}(v)$ is the fallback position of the next state v when the current state is $(g, u_k^{(\ell)}, Y_k^{(\ell)}, k)$ and $\ell_\kappa \in \{1, 2, \ldots, r\}$ is the index of the final sequence of states for which the fallback happens. For the remaining transitions, the probability is 0.

So, for the stochastic system J we have the following parameters:

- The number of states: $\omega_{new} \le \varpi \cdot [\omega + r(m-1)]$;

- The final sequences of states: $\{((h,u_m^{(\ell)},U^{(\ell)},m))\,|\in H,\,\ell=\overline{1,r}\}$;
- The number of final sequences of states: $r_{new}=r\cdot\varpi$;
- The length of the final sequences of states: $m_{new}=1$.

The distribution of the evolution time T is a homogeneous linear recurrence of order

$$m_{new}^2\cdot(r_{new}+\omega_{new})\le\varpi\cdot(\omega+mr).$$

Now, let us perform the criticality assessment for the hidden Markov models with final sequences of observable states. An observable state v is considered to be critical at one moment in time if it is part of one final sequence of states $U^{(\ell)}$ and the states $u_1^{(\ell)},u_2^{(\ell)},\ldots,u_k^{(\ell)}=v$ have just been observed. It corresponds to an augmented state $(h,v,Y_k^{(\ell)},k)$ with $k>0$ in the reduced stochastic system J, which represents a Markov process with final sequences of states.

The total alert time η is defined similarly to that in Section 8.7, as the cumulative time during which critical states are observed. It represents the occupation time of the set of critical-related states $\bigcup_{k=1}^{m-1}Z^{(k)}$. Its distribution, conditioned by the initial state $y\in Z$, satisfies the same recurrences from Section 8.7. The subset of states $Z_\infty^{(0)}\subseteq Z^{(0)}$, from which no critical-related state is reachable, can be ignored and, for the remaining states $Z\setminus Z_\infty^{(0)}$, the following recurrence holds:

$$P(n)=Q\cdot P(n)+R\cdot P(n-1),\ \forall n\ge 1.$$

Since Q is a weakly chained substochastic matrix, $I-Q$ is invertible and

$$P(n)=(I-Q)^{-1}\cdot R\cdot P(n-1),\ \forall n\ge 1.$$

Thus, the distribution of the total alert time η is a homogeneous linear recurrence of order M over $\mathbb{R}$, with characteristic polynomial $H(z)$, where

$$H(z)=\left|I-(I-Q)^{-1}R\cdot z\right|,$$

$$M\le|Z|\le\varpi\cdot[\omega+r(m-1)].$$

10.3 Hidden models with hierarchical emissions and final sequences of states

A hidden Markov model with hierarchical emissions is a multi-layered stochastic process with hidden and observable states. The first layer of the model is represented by the evolution of the underlying Markov process, which produces a series of hidden states. At each moment in time, every hidden state from the first layer generates another state on the second layer according to specific emission probabilities. A similar emission process occurs for each intermediate hidden layer and every state in the current layer generates another state on the next layer. The last layer of the model is considered observable. The evolution of the hidden Markov model with hierarchical emissions is terminated when, for the first time, an arbitrary sequence from a set of predefined final sequences of states is observed.

Let L be a hidden Markov model with hierarchical emissions, organized into s layers, as defined above. The set of all possible hidden states from each layer j is

$$H^{(j)} = \left\{ h_1^{(j)}, h_2^{(j)}, \ldots, h_{\varpi^{(j)}}^{(j)} \right\}, \; j = \overline{0, s-1}$$

and the set of all possible observable states from the last layer s is

$$V = H^{(s)} = \left\{ h_1^{(s)}, h_2^{(s)}, \ldots, h_{\varpi^{(s)}}^{(s)} \right\}.$$

The evolution of the model is driven by the underlying Markov process from the first layer (the layer with number $j = 0$). It starts from a hidden state $h \in H = H^{(0)}$ with probability $p^*(h)$. Also, the transition from one hidden state $g \in H$ to another hidden state $h \in H$ is performed with probability $p(g,h)$ at every discrete time.

At each discrete time t, every hidden state $h \in H^{(j)}$ from the layer j, $0 \le j \le s-1$, emits a state $v \in H^{(j+1)}$ on the next layer $j+1$ with probability $e^{(j)}(h,v)$. The evolution of the hidden Markov model is terminated when one of the final sequences of observable states

$$U^{(\ell)} = (u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_m^{(\ell)}) \in V^m, \; \ell = \overline{1, r},$$

is reached, i.e. the states $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_m^{(\ell)}$ are consecutively observed for an arbitrary $\ell \in \{1, 2, \ldots, r\}$.

As in Section 10.2, an observable state v is considered to be critical at one moment in time t if it is part of one final sequence of observable states $U^{(\ell)}$ and the states $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_k^{(\ell)} = v$ have just been observed. The evolution time T represents the amount of time needed to observe $U^{(\ell)}$ and the total alert time η is the cumulative time during which a critical state is observed.

Now, let us study the evolution time T and the total alert time η of the hidden Markov models with hierarchical emissions and final sequences of observable states $U^{(\ell)}$, $\ell = \overline{1,r}$. We collapse the hierarchical model into a hidden Markov model with final sequences of observable states.

Therefore, we iteratively exclude the intermediate layer j, $j = \overline{1,s-1}$, using the following formula for the emission probabilities:

$$e^{(0)}(h,v) = \sum_{u \in H^{(j)}} \left(e^{(0)}(h,u) \cdot e^{(j)}(u,v) \right), \forall h \in H, \forall v \in H^{(j+1)}.$$

We obtain a hidden Markov model with the set of hidden states H and the set of observable states V.

The initial and transition probabilities remain unchanged. Also, the final sequences of observable states remain the same: $U^{(\ell)}$, $\ell = \overline{1,r}$. The emission probabilities are

$$e(h,v) = e^{(0)}(h,v), \forall h \in H, \forall v \in V.$$

In conclusion, the distributions of the evolution time T and total alert time η are homogeneous linear recurrences over $\mathbb{R}$. They can be studied according to Section 10.2 and Algorithm 4.4.

10.4 Nested models with a final sequence of states for each process

We consider the nested systems with a final sequence of states for each inner process and the outer process. The system L acts as the outer process for the nested model. For each outer state $h \in V$, we define an inner process $L^{(h)}$ with a finite set of states $V^{(h)}$, where $\left|V^{(h)}\right| = \omega^{(h)}$. Also, we consider the final sequence of states

$$X^{(h)} = \left(x_1^{(h)}, x_2^{(h)}, \ldots, x_{m^{(h)}}^{(h)}\right) \in \left(V^{(h)}\right)^{m^{(h)}}.$$

The initial probabilities of the states of the process $L^{(h)}$ are $p^*(v)$, $\forall v \in V^{(h)}$ and the transition probabilities are $p(u,v)$, $\forall u,v \in V^{(h)}$.

The evolution of the nested model is defined as follows. Initially, one outer state $g \in V$ is activated according to the initial probability $p^*(g)$. Consequently, the evolution of the entire nested stochastic model begins by randomly selecting one of the outer states according to the initial probability distribution. At this stage, no inner process is active, and the selected outer state determines which subsystem will be executed first. Thus, the initial probability distribution of the outer process completely specifies the starting configuration of the nested model.

The activation of the outer state $g \in V$ triggers the evolution of the corresponding inner process $L^{(g)}$, which represents a Markov process with its own final sequence of states $X^{(g)}$ and its own initial probabilities $p^*(u)$ and transition probabilities $p(u,v)$, $\forall u,v \in V^{(g)}$. Once an outer state becomes active, the corresponding inner process evolves autonomously according to its initial and transition probabilities. During this period, the evolution of the outer process is temporarily suspended, while the behavior of the entire nested model is completely determined by the evolution of the currently active inner process.

The active inner process continues until its own final sequence of states $X^{(g)}$ is reached. As in the case of classical stochastic systems with final sequences of states, the inner process terminates only after the prescribed sequence of states has

been observed for the first time. The duration of this evolution is therefore a random variable determined by the probabilistic structure of the corresponding inner process. Throughout this interval, no transition between the outer states takes place, since the control remains inside the currently active inner process.

Once the final sequence of inner states $X^{(g)}$ is reached, the control is instantly moved back to the parent outer state g, and the next outer state $h \in V$ is activated according to the transition probability $p(g,h)$ of the outer process. Consequently, the completion of an inner process immediately initiates a transition in the outer process, which selects the next inner process to be executed. The newly activated outer state determines the corresponding inner process, and the same mechanism is repeated. In this way, the evolution of the nested model alternates between transitions of the outer process and complete executions of the corresponding inner processes. The overall stochastic behavior of the model is therefore determined jointly by the transition probabilities of the outer process and by the probabilistic evolution of all associated inner processes.

The evolution continues until the final sequence of outer states X is reached, including the evolution of the inner system corresponding to the last outer state. Thus, the stopping condition of the entire nested model is defined exclusively at the level of the outer process, while every transition of the outer process is itself generated by the completion of an inner stochastic process. Consequently, the evolution of the model depends simultaneously on the probabilistic behavior of the outer process and on the evolution times of all activated inner processes.

Each transition between the inner states and each activation of the outer states is performed in one unit of time. Therefore, both types of events contribute uniformly to the total evolution time of the model. More precisely, every transition inside an inner process increases the elapsed time by one unit, while each transition of the outer process resulting from the activation of a new inner process also requires one unit of time. Consequently, the total evolution time is obtained as the cumulative duration of

all transitions performed during the evolution of the inner processes together with all activations of the outer states until the terminating outer sequence is completed.

Our aim is to study the evolution time T of the nested model. To achieve this objective, we convert the model into a Markov process with a final sequence of states. The proposed transformation preserves the evolution time of the original nested model while replacing its hierarchical structure with an equivalent single-level stochastic system.

In what follows, we apply the Knuth-Morris-Pratt algorithm. We enrich each pair of outer and inner states (h,v), where $h \in V$ and $v \in V^{(h)}$, with the following additional characteristics:

- the number j, $0 \le j \le m$, that represents the length of the longest prefix of the final sequence of outer states matched so far;
- the number k, $0 \le k \le m^{(h)}$, that represents the length of the longest prefix of the final sequence of states of the inner process $L^{(h)}$ matched so far.

The nested model is in the augmented state $[(h,j),(v,k)]$ if the last j activated outer states are $x_1, x_2, \ldots, x_j = h$ and, at the last k moments of time, it has been in the inner states $x_1^{(h)}, x_2^{(h)}, \ldots, x_k^{(h)} = v$, where j and k are the largest values with this property among final sequences of states of the outer and inner processes. We have $j = 0$, respectively $k = 0$ if no such sequence exists.

We note that (h,j) represents the augmented state of the outer process L and (v,k) is the augmented state of the inner process $L^{(h)}$. Therefore, the augmented state of the nested model represents the joint state composed of the augmented states of the outer process and the corresponding inner process.

The set of states of the outer process L is

$$Z = \bigcup_{j=0}^{m} Z^{(j)},$$

where

$$|Z| = \omega - 1 + m$$

and

$$Z^{(0)} = \{(h,0) \mid h \in V \setminus \{x_1\}\},$$

$$Z^{(j)} = \{(x_j, j)\},\ j = \overline{1,m}.$$

Also, the set of states of the inner process $L^{(h)}$ is

$$Z^{(h)} = \bigcup_{k=0}^{m^{(h)}} Z^{(h,k)},$$

where

$$\left|Z^{(h)}\right| = \omega^{(h)} - 1 + m^{(h)},$$

$$Z^{(h,0)} = \{(v,0) \mid v \in V^{(h)} \setminus \{x_1^{(h)}\}\},$$

$$Z^{(h,k)} = \{(x_k^{(h)}, k)\},\ k = \overline{1,m^{(h)}}.$$

Therefore, the total number of augmented states of the nested model is

$$\omega_{new} = \sum_{(h,j)\in Z} \left|Z^{(h)}\right| = (\omega - 1 + m)(\omega_{\max} - 1 + m_{\max}),$$

where

$$\omega_{\max} = \max_{h\in V} \omega^{(h)},\ m_{\max} = \max_{h\in V} m^{(h)}.$$

The initial probabilities of the new augmented states are

$$p^*([(h,0),(v,0)]) = p^*(h)\, p^*(v),\ \forall h \in V \setminus \{x_1\}, \forall v \in V^{(h)} \setminus \{x_1^{(h)}\},$$

$$p^*([(h,0),(x_1^{(h)},1)]) = p^*(h)\, p^*(x_1^{(h)}),\ \forall h \in V \setminus \{x_1\},$$

$$p^*([(x_1,1),(v,0)]) = p^*(x_1)\, p^*(v),\ \forall v \in V^{(x_1)} \setminus \{x_1^{(x_1)}\},$$

$$p^*\left(\left[\left(x_1,1\right),\left(x_1^{(x_1)},1\right)\right]\right) = p^*\left(x_1\right) p^*\left(x_1^{(x_1)}\right).$$

The remaining states have the initial probability equal to zero.

The transition probabilities from each terminal inner state with non-critical parent outer state to the initial inner state of the next outer state are:

$$p\left(\left[\left(g,0\right),\left(x_{m^{(g)}}^{(g)}, m^{(g)}\right)\right],\left[\left(h,0\right),\left(v,0\right)\right]\right) =$$

$$= p(g,h)\, p^*(v),\ \forall g,h \in V \setminus \{x_1\}, \forall v \in V^{(h)} \setminus \{x_1^{(h)}\},$$

$$p\left(\left[(g,0),\left(x_{m^{(g)}}^{(g)},m^{(g)}\right)\right],\left[(h,0),\left(x_1^{(h)},1\right)\right]\right)=$$

$$=p(g,h)p^*(x_1^{(h)}),\ \forall g,h\in V\setminus\{x_1\},$$

$$p\left(\left[(g,0),\left(x_{m^{(g)}}^{(g)},m^{(g)}\right)\right],\left[(x_1,1),(v,0)\right]\right)=$$

$$=p(g,x_1)p^*(v),\ \forall g\in V\setminus\{x_1\},\forall v\in V^{(x_1)}\setminus\{x_1^{(x_1)}\},$$

$$p\left(\left[(g,0),\left(x_{m^{(g)}}^{(g)},m^{(g)}\right)\right],\left[(x_1,1),\left(x_1^{(x_1)},1\right)\right]\right)=$$

$$=p(g,x_1)p^*\left(x_1^{(x_1)}\right),\ \forall g\in V\setminus\{x_1\}.$$

The transition probabilities from each terminal inner state with critical parent outer state (x_j,j) to the initial inner state of the next non-critical outer state are:

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[(h,0),(v,0)\right]\right)=$$

$$=p(x_j,h)p^*(v),\ \forall h\in V\setminus\{x_1,x_2,\ldots,x_m\},\forall v\in V^{(h)}\setminus\{x_1^{(h)}\},j=\overline{1,m},$$

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[(h,0),\left(x_1^{(h)},1\right)\right]\right)=$$

$$=p(x_j,h)p^*(x_1^{(h)}),\ \forall h\in V\setminus\{x_1,x_2,\ldots,x_m\},j=\overline{1,m}.$$

The transition probabilities from each terminal inner state with critical parent outer state (x_j,j) to the initial inner state of the next critical outer state $(x_{j+1},j+1)$ are:

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[\left(x_{j+1},j+1\right),(v,0)\right]\right)=$$

$$=p(x_j,x_{j+1})p^*(v),\ \forall v\in V^{(x_{j+1})}\setminus\{x_1^{(x_{j+1})}\},j=\overline{1,m},$$

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[\left(x_{j+1},j+1\right),\left(x_1^{(x_{j+1})},1\right)\right]\right)=$$

$$=p(x_j,x_{j+1})p^*(x_1^{(x_{j+1})}),\ j=\overline{1,m}.$$

When the next critical outer state results from a fallback, we have:

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[(h,\iota),(v,0)\right]\right)=$$

$$=p(x_j,h)p^*(v),\ \forall h\in\{x_1,x_2,\ldots,x_m\}\setminus\{x_{j+1}\},\forall v\in V^{(h)}\setminus\{x_1^{(h)}\},j=\overline{1,m-1},$$

$$p\left(\left[\left(x_j,j\right),\left(x_{m^{(x_j)}}^{(x_j)},m^{(x_j)}\right)\right],\left[\left(h,\iota\right),\left(x_1^{(h)},1\right)\right]\right)=$$

$$=p(x_j,h)p^*(x_1^{(h)}),\ \forall h\in\{x_1,x_2,\ldots,x_m\}\setminus\{x_{j+1}\},\ j=\overline{1,m-1},$$

where $\iota=f_j(h)$ is the fallback position of the next outer state h when the current augmented outer state is (x_j,j).

The fallback has already been defined in previous sections. According to its definition, the position $f_j(h)$ represents the fallback position in the transition from one state (x_j,j) to another state $h\in\{x_1,x_2,\ldots,x_m\}\setminus\{x_{j+1}\}$ because $0\le f_j(h)\le j$ and $\iota=f_j(h)$ is the maximal integer that satisfies

$$(x_1,x_2,\ldots,x_t)=(x_{j-\iota+2},x_{j-\iota+3},\ldots,x_j,h).$$

This means that, when h occurs instead of x_{j+1}, while the system is in the state (x_j,j), a shortcut happens and the critical sequence $(x_1,x_2,\ldots,x_j)$ is replaced by the shorter sequence $(x_1,x_2,\ldots,x_t)$.

It remains only to write the formulas for the transitions within the inner processes. Therefore, for $\forall h\in V$ and for $j=\overline{1,m}$, we have:

$$p\left(\left[\left(h,j\right),\left(u,0\right)\right],\left[\left(h,j\right),\left(v,0\right)\right]\right)=$$

$$=p(u,v),\forall u,v\in V^{(h)}\setminus\{x_1^{(h)}\},$$

$$p\left(\left[\left(h,j\right),\left(u,0\right)\right],\left[\left(h,j\right),\left(x_1^{(h)},1\right)\right]\right)=$$

$$=p(u,x_1^{(h)}),\forall u\in V^{(h)}\setminus\{x_1^{(h)}\},$$

$$p\left(\left[\left(h,j\right),\left(x_k^{(h)},k\right)\right],\left[\left(h,j\right),\left(v,0\right)\right]\right)=$$

$$=p(x_k^{(h)},v),\forall v\in V^{(h)}\setminus\{x_1^{(h)},x_2^{(h)},\ldots,x_m^{(h)}\},\ k=\overline{1,m^{(h)}},$$

$$p\left(\left[\left(h,j\right),\left(x_k^{(h)},k\right)\right],\left[\left(h,j\right),\left(x_{k+1}^{(h)},k+1\right)\right]\right)=$$

$$=p(x_k^{(h)},x_{k+1}^{(h)}),\ k=\overline{1,m^{(h)}-1},$$

$$p\left(\left[\left(h,j\right),\left(x_k^{(h)},k\right)\right],\left[\left(h,j\right),\left(v,\kappa\right)\right]\right)=$$

$$= p(x_k^{(h)}, v), \forall v \in \{x_1^{(h)}, x_2^{(h)}, \ldots, x_m^{(h)}\} \setminus \{x_{k+1}^{(h)}\}, k = \overline{1, m^{(h)} - 1},$$

where $\kappa = f_k^{(h)}(v)$ is the fallback position of the next inner state v when the current inner state is $(x_k^{(h)}, k)$. The remaining transitions have the probability equal to zero.

Therefore, we have converted the nested model into a Markov process with a final sequence of states. The number of states is

$$\omega_{new} = (\omega - 1 + m)(\omega_{\max} - 1 + m_{\max}),$$

the final sequence of states is

$$X_{new} = \left(\left[\left(x_m, m\right), \left(x_{m^{(x_m)}}, m^{(x_m)}\right)\right]\right)$$

and its length is $m_{new} = 1$.

It is easy to see that the evolution time does not change during this conversion. As a result, the evolution time of the nested model is equal to the evolution time of the obtained system with a final sequence of states. According to Section 8.4, the evolution time is a homogeneous linear recurrence of order

$$\omega_{new} - 1 + m_{new} = (\omega - 1 + m)(\omega_{\max} - 1 + m_{\max}).$$

Consequently, Algorithm 4.4 can be applied to determine the main probabilistic characteristics of the evolution time T.

In particular, when all inner processes share the same set of states, the same final sequence of states and the same initial and transition probabilities, the evolution times of all inner processes have the same expectation E. The evolution time T of the nested model can be written as the following sum:

$$T = T_L + T^{(0)} + T^{(1)} + \ldots + T^{(T_L)},$$

where T_L is the evolution time of the outer process and $T^{(t)}$, $t = \overline{0, T_L}$, are the evolution times of the inner processes corresponding to the activated outer states at the discrete steps $t = 0, 1, 2, \ldots, T_L$. According to Wald's equation, we have

$$\mathbb{E}(T^{(0)} + T^{(1)} + \ldots + T^{(T_L)}) = (1 + \mathbb{E}(T_L))E$$

Therefore, in this particular case, the expectation of the evolution time T is

$$\mathbb{E}(T) = \mathbb{E}(T_L) + (1 + \mathbb{E}(T_L))E = E + (1 + E) \cdot \mathbb{E}(T_L),$$

where E and $\mathbb{E}(T_L)$ can be computed using the previously developed algorithms.

Moreover, when the outer process has the same set of states, the same final sequence of states and the same initial and transition probabilities as all inner processes, the evolution time of the outer process has the same expectation E as the evolution times of all inner processes. Therefore, in this case, we have

$$\mathbb{E}(T) = E + (1+E)\cdot\mathbb{E}(T_L) = E + (1+E)\cdot E = E^2 + 2E,$$

where E can be computed using the previously developed algorithms.

10.5 Nested models with multiple final sequences of states

Let L be a stochastic system with a finite set of states V, where $|V| = \omega$. At time $t = 0$, the system L starts its evolution from each state $g \in V$ with probability $p^*(g)$. Also, at each discrete time, it passes from one state $g \in V$ to another state $h \in V$ with probability $p(g,h)$.

Additionally, we assume that r sequences of states

$$X^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}) \in V^m,\ \ell = \overline{1,r},$$

are given and the stochastic system stops as soon as the states $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_m^{(\ell)}$ are reached consecutively in the given order, for an arbitrary $\ell \in \{1,2,\ldots,r\}$. The time T_L, when the system stops, represents the evolution time of the system L.

The stochastic system L represents a Markov process with the final sequences of states $X^{(\ell)}$, $\ell = \overline{1,r}$. For such a stochastic system, it was proved in Section 8.2 that the distribution of the evolution time T_L is a homogeneous linear recurrence of order $m^2(r+\omega)$ over the set $\mathbb{R}$. Also, it was further shown in Section 8.5 that the order of the recurrence can be reduced to at most $\omega + mr$ by using the Aho-Corasick string-searching algorithm. Based on this property, Algorithm 4.4 can be applied to determine the main probabilistic characteristics (expectation, variance, standard deviation, moments) of the evolution time T_L.

Further, we consider the nested models with final sequences of states. The system L acts as the outer process for the nested model. For each outer state $h \in V$, we define an inner process $L^{(h)}$ with a finite set of states $V^{(h)}$, where $|V^{(h)}| = \omega^{(h)}$. Also, we consider the final sequences of states

$$X^{(h,\lambda)} = \left(x_1^{(h,\lambda)}, x_2^{(h,\lambda)}, \ldots, x_{m^{(h)}}^{(h,\lambda)}\right) \in (V^{(h)})^{m^{(h)}}, \lambda = \overline{1, r^{(h)}}.$$

The initial probabilities of the states of the process $L^{(h)}$ are $p^*(v)$, $\forall v \in V^{(h)}$ and the transition probabilities are $p(u,v)$, $\forall u, v \in V^{(h)}$.

The evolution of the nested model is defined in the following way. Initially, one outer state $g \in V$ is activated according to the initial probability $p^*(g)$. The activation of the outer state $g \in V$ triggers the evolution of the corresponding inner process $L^{(g)}$, which represents a Markov process with its own final sequences of states $X^{(g,\lambda)}$, $\lambda = \overline{1, r^{(g)}}$, and its own initial probabilities $p^*(u)$ and transition probabilities $p(u,v)$, $\forall u, v \in V^{(g)}$. Once one of the final sequences of inner states $X^{(g,\lambda)}$, $\lambda = \overline{1, r^{(g)}}$, is reached, the control is instantly moved back to the parent outer state g and the next outer state $h \in V$ is activated according to the transition probability $p(g,h)$ of the outer process. The evolution continues until one of the given final sequences of outer states $X^{(\ell)}$, $\ell = \overline{1, r}$, is reached.

Each transition between the inner states and each activation of the outer states is performed in one unit of time. In order to characterize the evolution time T of the nested model we will convert the model into a Markov process with multiple final sequences of states.

We first recall several sets and sequences used and additional notation for $k = \overline{1, m}$, $\ell = \overline{1, r}$, which are valid for the outer system L:

$$X_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(r)}\},$$

$$X = \{X^{(1)}, X^{(2)}, \ldots, X^{(r)}\},$$

$$Y_k^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_k^{(\ell)}),$$

$$Y_k = \{Y_k^{(1)}, Y_k^{(2)}, \ldots, Y_k^{(r)}\},$$

$$Z_k^{(\ell)} = \{y \in X_k \mid (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_{k-1}^{(\ell)}, y) \in Y_k\}.$$

This notation can also be extended to the inner systems $L^{(h)}$, $\forall h \in V$. Therefore, for $k = \overline{1, m^{(h)}}$ and $\lambda = \overline{1, r^{(h)}}$, we have:

$$X_k^{(h)} = \{x_k^{(h,1)}, x_k^{(h,2)}, \ldots, x_k^{(h,r^{(h)})}\},$$

$$X^{(h)} = \{X^{(h,1)}, X^{(h,2)}, \ldots, X^{(h,r^{(h)})}\},$$

$$Y_k^{(h,\lambda)} = (x_1^{(h,\lambda)}, x_2^{(h,\lambda)}, \ldots, x_k^{(h,\lambda)}),$$

$$Y_k^{(h)} = \{Y_k^{(h,1)}, Y_k^{(h,2)}, \ldots, Y_k^{(h,r^{(h)})}\},$$

$$Z_k^{(h,\lambda)} = \{y \in X_k^{(h)} \mid (x_1^{(h,\lambda)}, x_2^{(h,\lambda)}, \ldots, x_{k-1}^{(h,\lambda)}, y) \in Y_k^{(h)}\}.$$

In what follows, we apply the Aho-Corasick algorithm. We enrich each pair of outer and inner states (h, v), where $h \in V$ and $v \in V^{(h)}$, with the following additional characteristics:

- the number j, $0 \le j \le m$, that represents the length of the longest prefix of the final sequence of outer states matched so far;
- the corresponding prefix

$$Y_j^{(\ell)} = (x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_j^{(\ell)}), \ell \in \{0,1,\ldots,r\};$$

- the number k, $0 \le k \le m^{(h)}$, that represents the length of the longest prefix of the final sequence of states of the inner process $L^{(h)}$ matched so far;
- the corresponding prefix

$$Y_k^{(h,\lambda)} = (x_1^{(h,\lambda)}, x_2^{(h,\lambda)}, \ldots, x_k^{(h,\lambda)}), \lambda \in \{0,1,\ldots,r^{(h)}\}.$$

The nested model is in the augmented state $[(h, Y_j^{(\ell)}, j), (v, Y_k^{(h,\lambda)}, k)]$ if the last j activated outer states are $x_1^{(\ell)}, x_2^{(\ell)}, \ldots, x_j^{(\ell)} = h$ and, at the last k moments of time, it has been in the inner states $x_1^{(h,\lambda)}, x_2^{(h,\lambda)}, \ldots, x_k^{(h,\lambda)} = v$, where j and k are the largest integers with this property among all final sequences of states of the outer and inner

processes. We have $j=0$ and $Y_0^{(0)}=()$, respectively $k=0$ and $Y_0^{(h,0)}=()$ if no such sequence exists.

We note that $(h,Y_j^{(\ell)},j)$ represents the augmented state of the outer process L and $(v,Y_k^{(h,\lambda)},k)$ is the augmented state of the inner process $L^{(h)}$. Therefore, the augmented state of the nested model represents the joint state composed of augmented states of the outer and the corresponding inner process.

The set of all possible states of the outer process L is

$$Z=\bigcup_{j=0}^{m} Z^{(j)},$$

where

$$|Z|\le \omega+r(m-1),$$

$$Z^{(0)}=\{(h,Y_0^{(0)},0)\,|\,h\in V\setminus X_1\},$$

$$Z^{(j)}=\{(x_j^{(\ell)},Y_j^{(\ell)},j)\,|\,1\le \ell\le r\},\ j=\overline{1,m}.$$

Also, the set of all possible states of the inner process $L^{(h)}$ is

$$Z^{(h)}=\bigcup_{k=0}^{m^{(h)}} Z^{(h,k)},$$

where

$$|Z^{(h)}|\le \omega^{(h)}+r^{(h)}(m^{(h)}-1),$$

$$Z^{(h,0)}=\{(v,Y_0^{(h,0)},0)\,|\,v\in V^{(h)}\setminus X_1^{(h)}\},$$

$$Z^{(h,k)}=\{(x_k^{(h,\lambda)},Y_k^{(h,\lambda)},k)\,|\,1\le \lambda\le r^{(h)}\},\ k=\overline{1,m^{(h)}}.$$

Therefore, the total number of augmented states of the nested model is

$$\omega_{new}=\sum_{(h,Y_j^{(\ell)},j)\in Z}|Z^{(h)}|\le[\omega+r(m-1)]\cdot[\omega_{max}+r_{max}(m_{max}-1)],$$

where

$$\omega_{max}=\max_{h\in V}\omega^{(h)},\ r_{max}=\max_{h\in V}r^{(h)},\ m_{max}=\max_{h\in V}m^{(h)}.$$

For $\forall h\in V\setminus X_1$, the initial probabilities of the new augmented states are

$$p^*\left(\left[\left(h,Y_0^{(0)},0\right),\left(v,Y_0^{(h,0)},0\right)\right]\right)=$$

$$=p^*(h)p^*(v),\ \forall v\in V^{(h)}\setminus X_1^{(h)},$$

$$p^*\left(\left[\left(h,Y_0^{(0)},0\right),\left(x_1^{(h,\lambda)},Y_1^{(h,\lambda)},1\right)\right]\right)=$$

$$=p^*(h)p^*\left(x_1^{(h,\lambda)}\right),\lambda=\overline{1,r^{(h)}},$$

and for $\ell=\overline{1,r}$, the initial probabilities are

$$p^*\left(\left[\left(x_1^{(\ell)},Y_1^{(\ell)},1\right),\left(v,Y_0^{(x_1^{(\ell)},0)},0\right)\right]\right)=$$

$$=p^*(x_1^{(\ell)})p^*(v),\ \forall v\in V^{(x_1^{(\ell)})}\setminus X_1^{(x_1^{(\ell)})},$$

$$p^*\left(\left[\left(x_1^{(\ell)},Y_1^{(\ell)},1\right),\left(x_1^{(x_1^{(\ell)},\lambda)},Y_1^{(x_1^{(\ell)},\lambda)},1\right)\right]\right)=$$

$$=p^*(x_1^{(\ell)})p^*\left(x_1^{(x_1^{(\ell)},\lambda)}\right),\lambda=\overline{1,r^{(x_1^{(\ell)})}}.$$

The remaining states have the initial probability equal to zero.

The transition probabilities from each terminal inner state with non-critical parent outer state to the initial inner state of the next outer state are:

$$p\left(\left[\left(g,Y_0^{(0)},0\right),\left(x_{m^{(g)}}^{(g,\lambda)},Y_{m^{(g)}}^{(g,\lambda)},m^{(g)}\right)\right],\left[\left(h,Y_0^{(0)},0\right),\left(v,Y_0^{(h,0)},0\right)\right]\right)=$$

$$=p(g,h)p^*(v),\ \forall g,h\in V\setminus X_1,\ \forall v\in V^{(h)}\setminus X_1^{(h)},\ \lambda=\overline{1,r^{(g)}},$$

$$p\left(\left[\left(g,Y_0^{(0)},0\right),\left(x_{m^{(g)}}^{(g,\lambda)},Y_{m^{(g)}}^{(g,\lambda)},m^{(g)}\right)\right],\left[\left(h,Y_0^{(0)},0\right),\left(x_1^{(h,\bar{\lambda})},Y_1^{(h,\bar{\lambda})},1\right)\right]\right)=$$

$$=p(g,h)p^*\left(x_1^{(h,\bar{\lambda})}\right),\ \forall g,h\in V\setminus X_1,\ \lambda=\overline{1,r^{(g)}},\ \bar{\lambda}=\overline{1,r^{(h)}},$$

$$p\left(\left[\left(g,Y_0^{(0)},0\right),\left(x_{m^{(g)}}^{(g,\lambda)},Y_{m^{(g)}}^{(g,\lambda)},m^{(g)}\right)\right],\left[\left(x_1^{(\ell)},Y_1^{(\ell)},1\right),\left(v,Y_0^{(x_1^{(\ell)},0)},0\right)\right]\right)=$$

$$=p\left(g,x_1^{(\ell)}\right)p^*(v),\ \forall g\in V\setminus X_1,\ \ell=\overline{1,r},\ \forall v\in V^{(x_1^{(\ell)})}\setminus X_1^{(x_1^{(\ell)})},\ \lambda=\overline{1,r^{(g)}},$$

$$p\left(\left[\left(g,Y_0^{(0)},0\right),\left(x_{m^{(g)}}^{(g,\lambda)},Y_{m^{(g)}}^{(g,\lambda)},m^{(g)}\right)\right],\left[\left(x_1^{(\ell)},Y_1^{(\ell)},1\right),\left(x_1^{(x_1^{(\ell)},\bar{\lambda})},Y_1^{(x_1^{(\ell)},\bar{\lambda})},1\right)\right]\right)=$$

$$= p\left(g, x_1^{(\ell)}\right) p^*\left(x_1^{(x_1^{(\ell)},\bar{\lambda})}\right), \forall g \in V \setminus X_1, \ell = \overline{1,r}, \lambda = \overline{1,r^{(g)}}, \bar{\lambda} = \overline{1,r^{(x_1^{(\ell)})}}.$$

Also, for $\ell = \overline{1,r}$ and $j = \overline{1,m}$, the transition probabilities from each terminal inner state with critical parent outer state $(x_j^{(\ell)}, Y_j^{(\ell)}, j)$ to the initial inner state of the next non-critical outer state are:

$$p\left(\left[\left(x_j^{(\ell)}, Y_j^{(\ell)}, j\right), \left(x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, m^{(x_j^{(\ell)})}\right)\right], \left[\left(h, Y_0^{(0)}, 0\right), \left(v, Y_0^{(h,0)}, 0\right)\right]\right) =$$

$$= p\left(x_j^{(\ell)}, h\right) p^*(v), \forall h \in V \setminus \bigcup_{i=1}^{m} X_i, \forall v \in V^{(h)} \setminus X_1^{(h)}, \lambda = \overline{1,r^{(x_j^{(\ell)})}},$$

$$p\left(\left[\left(x_j^{(\ell)}, Y_j^{(\ell)}, j\right), \left(x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, m^{(x_j^{(\ell)})}\right)\right], \left[\left(h, Y_0^{(0)}, 0\right), \left(x_1^{(h,\bar{\lambda})}, Y_1^{(h,\bar{\lambda})}, 1\right)\right]\right) =$$

$$= p\left(x_j^{(\ell)}, h\right) p^*\left(x_1^{(h,\bar{\lambda})}\right), \forall h \in V \setminus \bigcup_{i=1}^{m} X_i, \lambda = \overline{1,r^{(x_j^{(\ell)})}}, \bar{\lambda} = \overline{1,r^{(h)}}.$$

Next, for $\ell = \overline{1,r}$ and $j = \overline{1,m-1}$, the transition probabilities from each terminal inner state with critical parent outer state $(x_j^{(\ell)}, Y_j^{(\ell)}, j)$ to the initial inner state of the next critical outer state $(x_{j+1}^{(\ell)}, Y_{j+1}^{(\ell)}, j+1)$ are:

$$p\left(\left[\left(x_j^{(\ell)}, Y_j^{(\ell)}, j\right), \left(x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, m^{(x_j^{(\ell)})}\right)\right], \left[\left(x_{j+1}^{(\ell)}, Y_{j+1}^{(\ell)}, j+1\right), \left(v, Y_0^{(x_{j+1}^{(\ell)},0)}, 0\right)\right]\right) =$$

$$= p\left(x_j^{(\ell)}, x_{j+1}^{(\ell)}\right) p^*(v), \forall v \in V^{(x_{j+1}^{(\ell)})} \setminus X_1^{(x_{j+1}^{(\ell)})}, \lambda = \overline{1,r^{(x_j^{(\ell)})}},$$

$$p\left(\left[(x_j^{(\ell)}, Y_j^{(\ell)}, j), (x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)}, m^{(x_j^{(\ell)})})\right], \left[(x_{j+1}^{(\ell)}, Y_{j+1}^{(\ell)}, j+1), (x_1^{(x_{j+1}^{(\ell)},\bar{\lambda})}, Y_1^{(x_{j+1}^{(\ell)},\bar{\lambda})}, 1)\right]\right) =$$

$$= p\left(x_j^{(\ell)}, x_{j+1}^{(\ell)}\right) p^*\left(x_1^{\left(x_{j+1}^{(\ell)},\bar{\lambda}\right)}\right), \lambda = \overline{1,r^{(x_j^{(\ell)})}}, \bar{\lambda} = \overline{1,r^{(x_{j+1}^{(\ell)})}}.$$

When the next critical outer state results from a fallback, we have, for $\ell = \overline{1,r}$ and $j = \overline{1,m-1}$, the following formulas:

$$p\left(\left[\left(x_j^{(\ell)},Y_j^{(\ell)},j\right),\left(x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)},Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)},m^{(x_j^{(\ell)})}\right)\right],\left[\left(h,Y_\iota^{(\ell_\iota)},\iota\right),\left(v,Y_0^{(x_{j+1}^{(\ell)},0)},0\right)\right]\right)=$$

$$=p\left(x_j^{(\ell)},h\right)p^*(v),\ \forall h\in\bigcup_{i=1}^{m}X_i\setminus Z_{j+1}^{(\ell)},\ \forall v\in V^{(h)}\setminus X_1^{(h)},\ \lambda=\overline{1,r^{(x_j^{(\ell)})}},$$

$$p\left(\left[\left(x_j^{(\ell)},Y_j^{(\ell)},j\right),\left(x_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)},Y_{m^{(x_j^{(\ell)})}}^{(x_j^{(\ell)},\lambda)},m^{(x_j^{(\ell)})}\right)\right],\left[\left(h,Y_\iota^{(\ell_\iota)},\iota\right),\left(x_1^{(h,\bar{\lambda})},Y_1^{(h,\bar{\lambda})},1\right)\right]\right)=$$

$$=p\left(x_j^{(\ell)},h\right)p^*\left(x_1^{(h,\bar{\lambda})}\right),\ \forall h\in\bigcup_{i=1}^{m}X_i\setminus Z_{j+1}^{(\ell)},\ \lambda=\overline{1,r^{(x_j^{(\ell)})}},\ \bar{\lambda}=\overline{1,r^{(h)}},$$

where $\iota=f_j^{(\ell)}(h)$ is the fallback position of the next outer state h when the current outer state is $(x_j^{(\ell)},Y_j^{(\ell)},j)$ and $\ell_\iota\in\{1,2,\ldots,r\}$ is the index of the related final sequence of outer states for which the fallback happens.

According to the definition, the position $f_j^{(\ell)}(h)$ is the fallback position in the transition from the state $(x_j^{(\ell)},Y_j^{(\ell)},j)$ to the state $h\in\bigcup_{i=1}^{m}X_i\setminus Z_{j+1}^{(\ell)}$ because the inequality $0\le f_j^{(\ell)}(h)\le j$ holds and $\iota=f_j^{(\ell)}(h)$ is the maximal integer that satisfies

$$(x_1^{(\ell_\iota)},x_2^{(\ell_\iota)},\ldots,x_\iota^{(\ell_\iota)})=(x_{j-\iota+2}^{(\ell)},x_{j-\iota+3}^{(\ell)},\ldots,x_j^{(\ell)},h)$$

for an index $\ell_\iota\in\{1,2,\ldots,r\}$. This means that, when the state h occurs instead of $x_{j+1}^{(\ell)}$ while the system is in the state $(x_j^{(\ell)},Y_j^{(\ell)},j)$, a shortcut happens and the critical sequence $(x_1^{(\ell)},x_2^{(\ell)},\ldots,x_j^{(\ell)})$ is replaced by the shorter one $(x_1^{(\ell_\iota)},x_2^{(\ell_\iota)},\ldots,x_\iota^{(\ell_\iota)})$.

Now, it is required only to write the formulas for the transitions inside the inner processes. Therefore, for $\forall h\in V$ and for $j=\overline{1,m}$, $\ell=\overline{1,r}$ and $\lambda=\overline{1,r^{(h)}}$, taking into account the notation $h_j^{(\ell)}=(h,Y_j^{(\ell)},j)$, we have:

$$p\left(\left[h_j^{(\ell)},\left(u,Y_0^{(h,0)},0\right)\right],\left[h_j^{(\ell)},\left(v,Y_0^{(h,0)},0\right)\right]\right)=$$

$$=p(u,v),\ \forall u,v\in V^{(h)}\setminus X_1^{(h)},$$

$$p\left(\left[h_j^{(\ell)},\left(u,Y_0^{(h,0)},0\right)\right],\left[h_j^{(\ell)},\left(x_1^{(h,\lambda)},Y_1^{(h,\lambda)},1\right)\right]\right)=$$

$$=p\left(u,x_1^{(h,\lambda)}\right),\ \forall u\in V^{(h)}\setminus X_1^{(h)},$$

$$p\left(\left[h_j^{(\ell)},\left(x_k^{(h,\lambda)},Y_k^{(h,\lambda)},k\right)\right],\left[h_j^{(\ell)},\left(v,Y_0^{(h,0)},0\right)\right]\right)=$$

$$=p\left(x_k^{(h,\lambda)},v\right),\ \forall v\in V^{(h)}\setminus\bigcup_{i=1}^{m}X_i^{(h)},\ k=\overline{1,m^{(h)}},$$

$$p\left(\left[h_j^{(\ell)},\left(x_k^{(h,\lambda)},Y_k^{(h,\lambda)},k\right)\right],\left[h_j^{(\ell)},\left(x_{k+1}^{(h,\lambda)},Y_{k+1}^{(h,\lambda)},k+1\right)\right]\right)=$$

$$=p\left(x_k^{(h,\lambda)},x_{k+1}^{(h,\lambda)}\right),k=\overline{1,m^{(h)}-1},$$

$$p\left(\left[h_j^{(\ell)},\left(x_k^{(h,\lambda)},Y_k^{(h,\lambda)},k\right)\right],\left[h_j^{(\ell)},\left(v,Y_\kappa^{(h,\lambda_\kappa^{(h)})},\kappa\right)\right]\right)=$$

$$=p\left(x_k^{(h,\lambda)},v\right),\ \forall v\in\bigcup_{i=1}^{m}X_i^{(h)}\setminus Z_{k+1}^{(h,\lambda)},k=\overline{1,m^{(h)}-1},$$

where $\kappa=f_k^{(h,\lambda)}(v)$ is the fallback position of the next inner state v when the current inner state is $(x_k^{(h,\lambda)},Y_k^{(h,\lambda)},k)$ and $\lambda_\kappa^{(h)}\in\{1,2,\ldots,r^{(h)}\}$ is the index of the related final sequence of inner states for which the fallback happens. The remaining transitions have the probability equal to 0.

Therefore, we have converted the nested model into a Markov process with multiple final sequences of states. The number of states is

$$\omega_{new}\le[\omega+r(m-1)]\cdot[\omega_{max}+r_{max}(m_{max}-1)]$$

and the final sequences of states are

$$X_{new}^{(\ell,\lambda)}=\left(\left[\left(x_m^{(\ell)},Y_m^{(\ell)},m\right),\left(x_{m^{(x_m^{(\ell)})}}^{(\lambda)},Y_{m^{(x_m^{(\ell)})}}^{(x_m^{(\ell)},\lambda)},m^{(x_m^{(\ell)})}\right)\right]\right),\ \ell=\overline{1,r},\lambda=\overline{1,r^{(x_m^{(\ell)})}}.$$

Therefore, there are

$$r_{new}=\sum_{\ell=1}^{r}r^{(x_m^{(\ell)})}\le r\cdot r_{max}$$

final sequences of states and the length of each of them is equal to $m_{new}=1$.

We can easily note that the evolution time has not been changed during this conversion. As a result, the evolution time of the nested model is equal to the evolution time of the obtained system with multiple final sequences of states. According to Section 8.5, the evolution time is a homogeneous linear recurrence of order

$$\omega_{new} + m_{new} r_{new} \le [\omega + r(m-1)] \cdot [\omega_{max} + r_{max}(m_{max} - 1)] + r \cdot r_{max}.$$

In conclusion, Algorithm 4.4 can be applied to determine the main probabilistic characteristics of the evolution time T.

In particular, when all inner processes share the same set of states, the same final sequence of states and the same initial and transition probabilities, the evolution times of all inner processes have the same expectation E. As in Section 10.4, the evolution time T of the nested model is:

$$T = T_L + T^{(0)} + T^{(1)} + \ldots + T^{(T_L)},$$

where T_L represents the evolution time of the outer process and $T^{(t)}$, $t = \overline{0, T_L}$, are the evolution times of the inner processes corresponding to activated outer states at the steps $t = 0,1,2,\ldots,T_L$. Therefore, the expectation of the evolution time T is

$$\mathbb{E}(T) = \mathbb{E}(T_L) + (1 + \mathbb{E}(T_L))E = E + (1+E)\mathbb{E}(T_L),$$

where E and $\mathbb{E}(T_L)$ can be computed using the previously developed algorithms.

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ALEXANDRU LAZARI

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