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Extreme Sombor spectral radius of bicyclic graphs

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Abstract. Let G be a simple graph and $d_G(x)$ be the degree of the vertex x in G. The Sombor index of graph G is denoted as $SO(G) = \sum_{xy \in E(G)} \sqrt{d_G^2(x) + d_G^2(y)}$. Yinzhen Mei et al determine the upper and lower bounds of the Sombor index of unicyclic graphs in [7]. In this paper we do it for bicyclic graphs.

1. Introduction

Let G be a simple graph, $V(G) = \{v_1, \ldots, v_n\}$ and $E(G)$ be its vertex set and edge set, respectively. Also let d_i be the degree of the vertex v_i in G.

Topological and chemical indices have received quite some attention in graph theory already and have various applications in mathematical chemistry. One of the newest of these indices is Sombor index which was introduced in 2021, by Ivan Gutman [3]. It is defined for a graph G as

$$SO(G) = \sum_{v_i v_j \in E(G)} \sqrt{d_i^2 + d_j^2}.$$

A lot of research was happened on the Sombor index, the reader can refer to [1, 2, 4, 5].

Then, Wang et al [8], based on the Sombor index, introduced the Sombor matrix, which is denoted by $S = S(G) = (S_{ij})$, where

$$S_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & v_i v_j \in E(G) \\ 0 & o.w. \end{cases}$$

The eigenvalues of the Sombor matrix $S(G)$ are denoted by $\mu_1(G) \geqslant \mu_2(G) \ldots \geqslant \mu_n(G)$.

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In [6], X. Li et al solved the conjecture about the upper and lower bounds of the ABC spectral for unicyclic graphs. Then Y. Mei et al did the same for Sombor spectral for unicyclic graphs [7]. In this paper, we will find the upper and lower bounds of Sombor spectral for bicyclic graphs.

2. Preliminary

Definition 2.1. Let G be a graph of order n. It is called *bicyclic* if it is a connected graph and has $n+1$ edges.

Lemma 2.2. [7] *Let $T \geqslant 0$ be an irreducible matrix. Then $\overline{R} \leqslant \mu_1(T) \leqslant R_{max}$ where $\overline{R}$ is the average value of row sums of T and R_{max} is the value of the largest row sum. Either equality holds if and only if the row sums are equal.*

Lemma 2.3. (Perron-Frobenius Theory)
Let $T \geqslant 0$ be an irreducible matrix with an eigenvalue θ_0. Suppose $t \in R$, $x \in R$, $x > 0$. If $Tx \leqslant tx$, then $t \geqslant \theta_0$.

Lemma 2.4. (Cauchy-Schwarz Inequality)
Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be positive real numbers. Then

$$\sum_{i=1}^{n} a_i b_i \leqslant \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2},$$

where the equality holds if and only if $a_i = kb_i, 1 \leqslant i \leqslant n$.

Theorem 2.5. *Let $f(x) = x^2 + x - (2n+1) + \frac{n^2+4n+12}{x}$ where $x \in [3, n-2]$ and $n \geqslant 7$. Then $f(x) \leqslant n^2 - 4n + 13$.*

Proof. Since $x \leqslant n-2$, so

$$\begin{aligned} f(x) &\leqslant x^2 + n - 2 - (2n+1) + \frac{n^2+4n+12}{x} \\ &= x^2 - (n+3) + \frac{n^2+4n+12}{x} := g(x). \end{aligned}$$

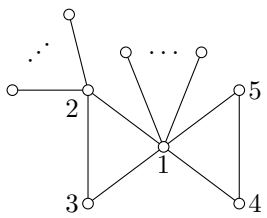
Now we will calculate the maximum of $g(x)$. We have

$$g'(x) = 2x - \frac{n^2+4n+12}{x^2} = 0 \quad \Rightarrow \quad x = (\frac{n^2+4n+12}{2})^{1/3}.$$

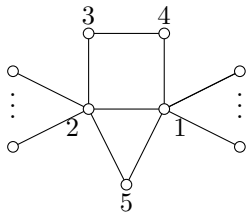
It is easy to see that

$$g(3) = \frac{n^2+n+30}{3}$$
$$g(n-2) = n^2-4n+7+\frac{24}{n-2} \leqslant n^2-4n+13$$
$$g((\frac{n^2+4n+12}{2})^{1/3}) = 3(\frac{n^2+4n+12}{2})^{2/3} - (n+3).$$

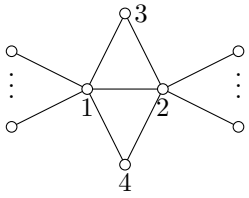
Consequently, $\max g(x) = n^2-4n+13$. Thus $f(x) \leqslant n^2-4n+13$ for $n \geqslant 7$. □



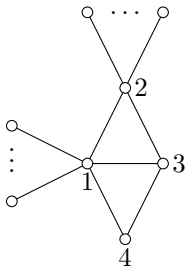
G_1



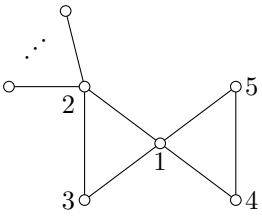
G_2



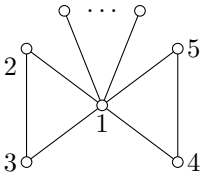
G_3



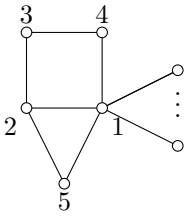
G_4



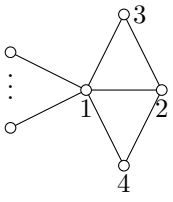
G_5



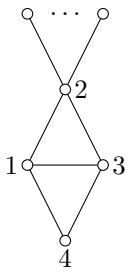
G_6



G_7



G_8



G_9

Theorem 2.6. *Let G be a bicyclic graph with n vertices and $uv \in E(G)$. If $d_u + d_v \geqslant n+1$, then G is one of the following graphs:*

Proof. Let C be the unique bicycle in G. We consider the following cases:

- Let $u, v \notin V(C)$.
  Hence, $N(u) \cap N(v) = \emptyset$ and $|N(u) \cap C| + |N(v) \cap C| \leqslant 1$. Since $|C| \geqslant 4$, so there exist at least 3 vertices in C such that they are not adjacent to u and v, which implies that $d_u + d_v \leqslant n-3$, which is a contradiction.

- Let $u, v \in V(C)$.
  If $|C| \geqslant 7$, then so there exist at least one vertex in C such that it is not adjacent to u and v, because $|N(u) \cap C| + |N(v) \cap C| \leqslant 6$. Therefore, $d_u + d_v \leqslant n-1$, which is a contradiction.
  If $|C| = 6$, then $d_u + d_v \leqslant n$, which is a contradiction.
  If $|C| = 5$, then $G = G_1$ or $G = G_2$ in the above figure.
  If $|C| = 4$, then $G = G_3$ or $G = G_4$ in the above figure.

- Let $u \in V(C)$ and $v \notin V(C)$.
  Hence, $N(u) \cap N(v) = \emptyset$, so $d_u + d_v \leqslant n$, which is a contradiction.

3. Main Theorem

Theorem 3.1. *Let G be a bicyclic graph of order n, then*

$$\mu_1(G) \geqslant 4\sqrt{2}\frac{(n+1)^2}{n}.$$

Proof. We know that $d_1 + \ldots + d_n = 2(n+1)$, since G is a bicyclic graph. By Lemma 2.2 and Cauchy-Schwarz Inequality, we have

$$\mu_1(G) \geqslant \frac{2}{n} \sum_{v_iv_j \in E(G)} \sqrt{d_i^2 + d_j^2} = \frac{\sqrt{2}}{n} \sum_{v_iv_j \in E(G)} \sqrt{2}\sqrt{d_i^2 + d_j^2}.$$

$$\sqrt{2}\sqrt{d_i^2 + d_j^2} \geqslant d_i + d_j,$$

$$n\sum_{i=1}^{n} d_i^2 \geqslant (\sum_{i=1}^{n} d_i)^2 = 4(n+1)^2.$$

Therefore,

$$\mu_1(G) \geqslant \frac{\sqrt{2}}{n} \sum_{v_iv_j \in E(G)} (d_i + d_j) = \frac{\sqrt{2}}{n} \sum_{i=1}^{n} d_i^2 \geqslant 4\sqrt{2}\frac{(n+1)^2}{n}.$$

□

Let G be a bicyclic graph of order $n \geqslant 7$. In the sequel, we try to find the upper bound for $\mu_1(G)$. We consider the following two cases:
1. If for every $uv \in E(G)$, $d_u + d_v \leqslant n$. (See Lemma 3.2)
2. If there exists $uv \in E(G)$ such that $d_u + d_v \geqslant n+1$. (See Lemmas 3.3 to 3.11)

Lemma 3.2. *Let G be a bicyclic graph and $|V(G)| = n$. Assume that for every $uv \in E(G)$, we have $d_u + d_v \leqslant n$, then*

$$\mu_1(G) \leqslant \sqrt{n^2 - 4n + 8}\sqrt{n+3}.$$

Proof. Since for every $uv \in E(G)$, we have $d_u + d_v \leqslant n$, so degree of every vertex of G is less than or equal to $n-2$. In the contrary, assume that there is $v \in V(G)$ such that $d_v \geqslant n-1$. Since there exists one vertex as u which is join to v and $d_u \geqslant 2$, so $d_u + d_v \geqslant n+1$, which is a contradiction. Hence, $d_v \leqslant n-2$, for every $v \in V(G)$.

Let $v_{j1}, v_{j2}, \ldots, v_{jd_i}$, be all adjacent vertices to the arbitrary vertex v_i. We know that $\sum_{l=1}^{n} d_l = 2(n+1)$, therefore

$$d_{j1} + d_{j2} + \ldots + d_{jd_i} + d_i \leqslant 2(n+1) - (n - d_i - 1),$$

which implies that $d_{j1} + d_{j2} + \ldots + d_{jd_i} \leqslant n+3$.

Now by definition and above discussion, we have

$$\begin{aligned}(SX)_i &= \sum_{v_iv_j \in E(G)} \sqrt{d_i^2 + d_j^2}\sqrt{d_j} \\ &\leqslant \sum_{v_iv_j \in E(G)} \sqrt{d_i^2 + (n-d_i)^2}\sqrt{d_j} \\ &= \sqrt{d_i^2 + (n-d_i)^2} \sum_{v_iv_j \in E(G)} \sqrt{d_j}.\end{aligned}$$

By Cauchy-Schwarz inequality, we know that

$$\begin{aligned}\sum_{v_iv_j \in E(G)} \sqrt{d_j} &\leqslant \sqrt{\sum_{v_iv_j \in E(G)} d_j \sum_{v_iv_j \in E(G)} 1} \\ &= \sqrt{\sum_{v_iv_j \in E(G)} d_j}\sqrt{\sum_{v_iv_j \in E(G)} 1} = \sqrt{\sum_{v_iv_j \in E(G)} d_j}\sqrt{d_i}.\end{aligned}$$

On the other hand, since $d_i \leqslant n-2$, so

$$\sqrt{d_i^2 + (n-d_i)^2} \leqslant \sqrt{n^2 - 4n + 8}.$$

Consequently,

$$\begin{aligned}(SX)_i &\leqslant \sqrt{n^2-4n+8}\sqrt{\sum_{v_iv_j\in E(G)} d_j}\sqrt{d_i}\\ &\leqslant \sqrt{n^2-4n+8}\sqrt{n+3}\sqrt{d_i}.\end{aligned}$$

Hence, by Lemma 2.3, $\mu_1(G) \leqslant \sqrt{n^2-4n+8}\sqrt{n+3}$. □

Lemma 3.3. *Let $G = G_1 \setminus \{G_5, G_6\}$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Proof. We have $d_3 = d_4 = d_5 = 2$, $d_6 = \ldots = d_n = 1$, $d_1, d_2 \leqslant n-2$ and $d_1 + d_2 = n+1$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2} + \sqrt{d_1^2+d_3^2}\sqrt{d_3} + \sqrt{d_1^2+d_4^2}\sqrt{d_4} + \sqrt{d_1^2+d_5^2}\sqrt{d_5}\\ &\qquad + (d_1-4)\sqrt{d_1^2+1}\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+d_1^2+d_5^2+(d_1-4)(d_1^2+1)}\\ &\qquad \cdot \sqrt{d_2+d_3+d_4+d_5+d_1-4}\\ &= \sqrt{4d_1^2+(n+1-d_1)^2+12+d_1^3-4d_1^2+d_1-4}\sqrt{n+3}\\ &= \sqrt{d_1^3+d_1^2-(2n+1)d_1+n^2+2n+9}\sqrt{n+3}\\ &= \sqrt{d_1}\sqrt{d_1^2+d_1-(2n+1)+\frac{n^2+2n+9}{d_1}}\sqrt{n+3}\\ &\leqslant \sqrt{d_1}\sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Also we have:

$$\begin{aligned}(SX)_2 &= \sum_{v_2v_j\in E(G)} \sqrt{d_2^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_2^2+d_1^2}\sqrt{d_1} + \sqrt{d_2^2+d_3^2}\sqrt{d_3} + (d_2-2)\sqrt{d_2^2+1}\\ &\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2+(d_2-2)(d_2^2+1)}\sqrt{d_1+d_3+d_2-2}\\ &= \sqrt{d_2^3+d_2^2-(2n+1)d_2+n^2+2n+3}\sqrt{n+1}\end{aligned}$$

$$= \sqrt{d_2}\sqrt{d_2^2 + d_2 - (2n+1) + \frac{n^2+2n+3}{d_2}}\sqrt{n+1}$$
$$\leqslant \sqrt{d_2}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Moreover, we have:

$$\begin{aligned}(SX)_3 &= \sum_{v_3v_j \in E(G)} \sqrt{d_3^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_3^2 + d_1^2}\sqrt{d_1} + \sqrt{d_3^2 + d_2^2}\sqrt{d_2} \\ &\leqslant \sqrt{d_3^2 + d_1^2 + d_3^2 + d_2^2}\sqrt{d_1 + d_2} \\ &\leqslant \sqrt{2}\sqrt{n^2-4n+8}\sqrt{n+1} \\ &\leqslant \sqrt{d_3}\sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

As well as, we have

$$\begin{aligned}(SX)_4 &= \sum_{v_4v_j \in E(G)} \sqrt{d_4^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_4^2 + d_1^2}\sqrt{d_1} + \sqrt{d_4^2 + d_5^2}\sqrt{d_5} \\ &\leqslant \sqrt{d_4^2 + d_1^2 + d_4^2 + d_5^2}\sqrt{d_1 + d_5} \\ &= \sqrt{d_1^2 + 12}\sqrt{d_1 + 2} \leqslant \sqrt{(n-2)^2 + 12}\sqrt{n-2+2} \\ &= \sqrt{n^2-4n+16}\sqrt{n} \leqslant \sqrt{d_4}\sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i = 5$ we have:

$$(SX)_5 \leqslant \sqrt{d_5}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Also we have:

$$\begin{aligned}(SX)_6 &= \sum_{v_6v_j \in E(G)} \sqrt{d_6^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_6^2 + d_1^2}\sqrt{d_1} = \sqrt{1 + d_1^2}\sqrt{d_1} \leqslant \sqrt{1 + (n-2)^2}\sqrt{n-2} \\ &= \sqrt{n^2-4n+5}\sqrt{n-2} \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 7$ we have:

$$(SX)_i \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}. \qquad \square$$

Lemma 3.4. *Let $G = G_2 \setminus \{G_7\}$, then*

$$\mu_1(G) \leqslant \sqrt{n^2 - 4n + 13}\sqrt{n+2}.$$

Proof. We know that $d_3 = d_4 = d_5 = 2$, $d_6 = \ldots = d_n = 1$, $d_1, d_2 \leqslant n-3$ and $d_1 + d_2 = n + 1$. Therefore,

$$\begin{aligned}
(SX)_1 &= \sum_{v_1 v_j \in E(G)} \sqrt{d_1^2 + d_j^2}\sqrt{d_j} \\
&= \sqrt{d_1^2 + d_2^2}\sqrt{d_2} + \sqrt{d_1^2 + d_4^2}\sqrt{d_4} + \sqrt{d_1^2 + d_5^2}\sqrt{d_5} + (d_1 - 3)\sqrt{d_1^2 + 1} \\
&\leqslant \sqrt{d_1^2 + d_2^2 + d_1^2 + d_4^2 + d_1^2 + d_5^2 + (d_1 - 3)(d_1^2 + 1)}\sqrt{d_2 + d_4 + d_5 + d_1 - 3} \\
&= \sqrt{d_1^3 + d_1^2 - (2n+1)d_1 + n^2 + 2n + 6}\sqrt{n+2} \\
&= \sqrt{d_1}\sqrt{d_1^2 + d_1 - (2n+1) + \frac{n^2 + 2n + 6}{d_1}}\sqrt{n+2} \\
&\leqslant \sqrt{d_1}\sqrt{n^2 - 4n + 13}\sqrt{n+2}.
\end{aligned}$$

Similarly to the above we have

$$(SX)_2 \leqslant \sqrt{d_2}\sqrt{n^2 - 4n + 13}\sqrt{n+2}.$$

Also we have:

$$\begin{aligned}
(SX)_3 &= \sum_{v_3 v_j \in E(G)} \sqrt{d_3^2 + d_j^2}\sqrt{d_j} \\
&= \sqrt{d_3^2 + d_2^2}\sqrt{d_2} + \sqrt{d_3^2 + d_4^2}\sqrt{d_4} \\
&\leqslant \sqrt{d_3^2 + d_2^2 + d_3^2 + d_4^2}\sqrt{d_2 + d_4} \\
&\leqslant \sqrt{(n-3)^2 + 12}\sqrt{n-1} \\
&\leqslant \sqrt{d_3}\sqrt{n^2 - 4n + 13}\sqrt{n+2}.
\end{aligned}$$

Similarly to the above we have

$$(SX)_4 \leqslant \sqrt{d_4}\sqrt{n^2 - 4n + 13}\sqrt{n+2}.$$

For $(SX)_5$ we can see:

$$\begin{aligned}
(SX)_5 &= \sum_{v_5 v_j \in E(G)} \sqrt{d_5^2 + d_j^2}\sqrt{d_j} \\
&= \sqrt{d_5^2 + d_1^2}\sqrt{d_1} + \sqrt{d_5^2 + d_2^2}\sqrt{d_2} \\
&\leqslant \sqrt{d_5^2 + d_1^2 + d_5^2 + d_2^2}\sqrt{d_1 + d_2}
\end{aligned}$$

$$\leqslant \sqrt{2(n-3)^2+8}\sqrt{n+1}$$
$$\leqslant \sqrt{d_5}\sqrt{n^2-4n+13}\sqrt{n+2}.$$

As well as we can see:

$$\begin{aligned}(SX)_6 &= \sum_{v_6v_j\in E(G)} \sqrt{d_6^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_6^2+d_1^2}\sqrt{d_1} \leqslant \sqrt{(n-3)^2+1}\sqrt{n-3}\\ &\leqslant \sqrt{n^2-4n+13}\sqrt{n+2}.\end{aligned}$$

Similarly for $i \geqslant 7$ we have

$$(SX)_i \leqslant \sqrt{n^2-4n+13}\sqrt{n+2}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-4n+13}\sqrt{n+2}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+2}.$$ □

Lemma 3.5. *Let $G = G_3 \setminus \{G_8\}$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Proof. We know that $d_3 = d_4 = 2$, $d_5 = \ldots = d_n = 1$, $d_1, d_2 \leqslant n-2$ and $d_1 + d_2 = n+2$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2} + \sqrt{d_1^2+d_3^2}\sqrt{d_3} + \sqrt{d_1^2+d_4^2}\sqrt{d_4} + (d_1-3)\sqrt{d_1^2+1}\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+(d_1-3)(d_1^2+1)}\sqrt{d_2+d_3+d_4+d_1-3}\\ &= \sqrt{d_1^3+d_1^2-(2n+3)d_1+n^2+4n+9}\sqrt{n+3}\\ &= \sqrt{d_1}\sqrt{d_1^2+d_1-(2n+3)+\frac{n^2+4n+9}{d_1}}\sqrt{n+3}\\ &\leqslant \sqrt{d_1}\sqrt{d_1^2+d_1-(2n+1)+\frac{n^2+4n+9}{d_1}}\sqrt{n+3}\\ &\leqslant \sqrt{d_1}\sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, we have:

$$(SX)_2 \leqslant \sqrt{d_2}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Also we can see:

$$\begin{aligned}(SX)_3 &= \sum_{v_3v_j\in E(G)} \sqrt{d_3^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_3^2+d_1^2}\sqrt{d_1}+\sqrt{d_3^2+d_2^2}\sqrt{d_2}\\ &\leqslant \sqrt{d_3^2+d_1^2+d_3^2+d_2^2}\sqrt{d_1+d_2}\\ &\leqslant \sqrt{2(n-2)^2+8}\sqrt{n+2}\\ &\leqslant \sqrt{d_3}\sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, we have:

$$(SX)_4 \leqslant \sqrt{d_4}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Moreover,

$$\begin{aligned}(SX)_5 &= \sum_{v_5v_j\in E(G)} \sqrt{d_5^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_5^2+d_1^2}\sqrt{d_1} = \sqrt{d_1^2+1}\sqrt{d_1} \leqslant \sqrt{(n-2)^2+1}\sqrt{n-2}\\ &\leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 6$ we have:

$$(SX)_i \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}. \qquad \square$$

Lemma 3.6. *Let $G = G_4 \setminus \{G_9\}$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Proof. We know that $d_3 = 3$, $d_4 = 2$, $d_5 = \ldots = d_n = 1$, $d_1, d_2 \leqslant n-2$ and $d_1 + d_2 = n+1$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2}+\sqrt{d_1^2+d_3^2}\sqrt{d_3}+\sqrt{d_1^2+d_4^2}\sqrt{d_4}+(d_1-3)\sqrt{d_1^2+1}\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+(d_1-3)(d_1^2+1)}\sqrt{d_2+d_3+d_4+d_1-3}\\ &= \sqrt{d_1^3+d_1^2-(2n+1)d_1+n^2+2n+11}\sqrt{n+3}\end{aligned}$$

$$\leqslant \sqrt{d_1}\sqrt{d_1^2+d_1-(2n+1)+\frac{n^2+2n+11}{d_1}}\sqrt{n+3}$$
$$\leqslant \sqrt{d_1}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Also we have:

$$(SX)_2 = \sum_{v_2v_j\in E(G)}\sqrt{d_2^2+d_j^2}\sqrt{d_j}$$
$$= \sqrt{d_2^2+d_1^2}\sqrt{d_1}+\sqrt{d_2^2+d_3^2}\sqrt{d_3}+(d_2-2)\sqrt{d_2^2+1}$$
$$\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2+(d_2-2)(d_2^2+1)}\sqrt{d_1+d_3+d_2-2}$$
$$\leqslant \sqrt{d_2}\sqrt{d_2^2+d_2-(2n+1)+\frac{n^2+2n+8}{d_2}}\sqrt{n+2}$$
$$\leqslant \sqrt{d_2}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Moreover, we have:

$$(SX)_3 = \sum_{v_3v_j\in E(G)}\sqrt{d_3^2+d_j^2}\sqrt{d_j}$$
$$= \sqrt{d_3^2+d_1^2}\sqrt{d_1}+\sqrt{d_3^2+d_2^2}\sqrt{d_2}+\sqrt{d_3^2+d_4^2}\sqrt{d_4}$$
$$\leqslant \sqrt{d_3^2+d_1^2+d_3^2+d_2^2+d_3^2+d_4^2}\sqrt{d_1+d_2+d_4}$$
$$\leqslant \sqrt{2(n-2)^2+31}\sqrt{n+3} \leqslant \sqrt{d_3}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

As well as

$$(SX)_4 = \sum_{v_4v_j\in E(G)}\sqrt{d_4^2+d_j^2}\sqrt{d_j}$$
$$= \sqrt{d_4^2+d_1^2}\sqrt{d_1}+\sqrt{d_4^2+d_3^2}\sqrt{d_3}$$
$$\leqslant \sqrt{d_4^2+d_1^2+d_4^2+d_3^2}\sqrt{d_1+d_3}$$
$$\leqslant \sqrt{(n-2)^2+17}\sqrt{n+1}$$
$$\leqslant \sqrt{d_4}\sqrt{n^2-4n+13}\sqrt{n+3}.$$

Also we can see:

$$(SX)_5 = \sum_{v_5v_j\in E(G)}\sqrt{d_5^2+d_j^2}\sqrt{d_j}$$
$$= \sqrt{d_5^2+d_1^2}\sqrt{d_1} \leqslant \sqrt{(n-2)^2+1}\sqrt{n-2}$$
$$\leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Similarly, for $i \geqslant 6$ we have:

$$(SX)_i \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}. \qquad \square$$

Lemma 3.7. *Let $G = G_5$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Proof. We know that $d_1 = 4$, $d_2 = n-3$, $d_3 = d_4 = d_5 = 2$ and $d_6 = \ldots = d_n = 1$. Hence,

$$\begin{aligned}
(SX)_1 &= \sum_{v_1v_j \in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j} \\
&= \sqrt{d_1^2+d_2^2}\sqrt{d_2} + \sqrt{d_1^2+d_3^2}\sqrt{d_3} + \sqrt{d_1^2+d_4^2}\sqrt{d_4} + \sqrt{d_1^2+d_5^2}\sqrt{d_5} \\
&\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+d_1^2+d_5^2}\sqrt{d_2+d_3+d_4+d_5} \\
&= \sqrt{n^2-6n+85}\sqrt{n+3} \leqslant \sqrt{d_1}\sqrt{n^2-4n+13}\sqrt{n+3}.
\end{aligned}$$

Also we have:

$$\begin{aligned}
(SX)_2 &= \sum_{v_2v_j \in E(G)} \sqrt{d_2^2+d_j^2}\sqrt{d_j} \\
&= \sqrt{d_2^2+d_1^2}\sqrt{d_1} + \sqrt{d_2^2+d_3^2}\sqrt{d_3} + \sqrt{d_2^2+1}(n-5) \\
&\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2+(d_2^2+1)(n-5)}\sqrt{d_1+d_3+(n-5)} \\
&\leqslant \sqrt{2(n-3)^2+16+4+((n-3)^2+1)(n-3)}\sqrt{n+1} \\
&= \sqrt{n-3}\sqrt{(n-3)^2+2(n-3)+1+\frac{20}{n-3}}\sqrt{n+1} \\
&\leqslant \sqrt{d_2}\sqrt{n^2-4n+13}\sqrt{n+3}.
\end{aligned}$$

As well as

$$\begin{aligned}
(SX)_3 &= \sum_{v_3v_j \in E(G)} \sqrt{d_3^2+d_j^2}\sqrt{d_j} \\
&= \sqrt{d_3^2+d_1^2}\sqrt{d_1} + \sqrt{d_3^2+d_2^2}\sqrt{d_2} \\
&\leqslant \sqrt{d_3^2+d_1^2+d_3^2+d_2^2}\sqrt{d_1+d_2} \\
&= \sqrt{n^2-6n+33}\sqrt{n+1} \leqslant \sqrt{d_3}\sqrt{n^2-4n+13}\sqrt{n+3}.
\end{aligned}$$

Moreover

$$\begin{aligned}(SX)_4 &= \sum_{v_4v_j\in E(G)} \sqrt{d_4^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_4^2+d_1^2}\sqrt{d_1}+\sqrt{d_4^2+d_5^2}\sqrt{d_5}\\ &\leqslant \sqrt{d_4^2+d_1^2+d_4^2+d_5^2}\sqrt{d_1+d_5}=2\sqrt{42}.\end{aligned}$$

Similarly we can see $(SX)_5 \leqslant 2\sqrt{42}$.

$$\begin{aligned}(SX)_6 &= \sum_{v_6v_j\in E(G)} \sqrt{d_6^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_6^2+d_2^2}\sqrt{d_2}=\sqrt{n^2-6n+10}\sqrt{n-3}\\ &\leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 7$ we have:

$$(SX)_i \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}.$$

Consequently, we have $SX \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-4n+13}\sqrt{n+3}. \qquad \square$$

Lemma 3.8. *Let $G = G_6$, then*

$$\mu_1(G) \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}.$$

Proof. We know that $d_1 = n-1$, $d_2 = d_3 = d_4 = d_5 = 2$ and $d_6 = \ldots = d_n = 1$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2}+\sqrt{d_1^2+d_3^2}\sqrt{d_3}+\sqrt{d_1^2+d_4^2}\sqrt{d_4}+\sqrt{d_1^2+d_5^2}\sqrt{d_5}\\ &\qquad +\sqrt{d_1^2+1}(n-5)\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+d_1^2+d_5^2+(d_1^2+1)(n-5)}\\ &\qquad \sqrt{d_2+d_3+d_4+d_5+n-5}\\ &\leqslant \sqrt{4(n-1)^2+16+((n-1)^2+1)(n-1)}\sqrt{n+3}\\ &= \sqrt{n-1}\sqrt{(n-1)^2+4(n-1)+1+\frac{16}{n-1}}\sqrt{n+3}\\ &\leqslant \sqrt{d_1}\sqrt{(n-1)^2+4(n-1)+5}\sqrt{n+3}\\ &= \sqrt{d_1}\sqrt{n^2+2n+2}\sqrt{n+3}.\end{aligned}$$

Also we have:

$$\begin{aligned}(SX)_2 &= \sum_{v_2v_j\in E(G)} \sqrt{d_2^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_2^2+d_1^2}\sqrt{d_1}+\sqrt{d_2^2+d_3^2}\sqrt{d_3}\\ &\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2}\sqrt{d_1+d_3}\\ &\leqslant \sqrt{n^2-2n+13}\sqrt{n+1}\\ &\leqslant \sqrt{d_2}\sqrt{n^2+2n+2}\sqrt{n+3}.\end{aligned}$$

Similarly, for $3 \leqslant i \leqslant 5$ we have:

$$(SX)_i \leqslant \sqrt{d_i}\sqrt{n^2+2n+2}\sqrt{n+3}.$$

And

$$\begin{aligned}(SX)_6 &= \sum_{v_6v_j\in E(G)} \sqrt{d_6^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_6^2+d_1^2}\sqrt{d_1} = \sqrt{n^2-2n+2}\sqrt{n-1}\\ &\leqslant \sqrt{n^2+2n+2}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 7$ we have:

$$(SX)_i \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}. \qquad \square$$

Lemma 3.9. *Let $G = G_7$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-n+4}\sqrt{n+2}.$$

Proof. We know that $d_1 = n-2$, $d_2 = 3$, $d_3 = d_4 = d_5 = 2$ and $d_6 = \ldots = d_n = 1$.

Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2}+\sqrt{d_1^2+d_4^2}\sqrt{d_4}+\sqrt{d_1^2+d_5^2}\sqrt{d_5}+\sqrt{d_1^2+1}(n-5)\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_4^2+d_1^2+d_5^2+(d_1^2+1)(n-5)}\sqrt{d_2+d_4+d_5+n-5}\\ &\leqslant \sqrt{3(n-2)^2+17+((n-2)^2+1)(n-2)}\sqrt{n+2}\\ &= \sqrt{n-2}\sqrt{(n-2)^2+3(n-2)+1+\frac{17}{n-2}}\sqrt{n+2}\\ &\leqslant \sqrt{d_1}\sqrt{(n-2)^2+3(n-2)+6}\sqrt{n+2}\\ &= \sqrt{d_1}\sqrt{n^2-n+4}\sqrt{n+2}.\end{aligned}$$

Also we have:

$$\begin{aligned}(SX)_2 &= \sum_{v_2v_j\in E(G)} \sqrt{d_2^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_2^2+d_1^2}\sqrt{d_1}+\sqrt{d_2^2+d_3^2}\sqrt{d_3}+\sqrt{d_2^2+d_5^2}\sqrt{d_5}\\ &\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2+d_2^2+d_5^2}\sqrt{d_1+d_3+d_5}\\ &\leqslant \sqrt{n^2-4n+39}\sqrt{n+2}\\ &\leqslant \sqrt{d_2}\sqrt{n^2-n+4}\sqrt{n+2}.\end{aligned}$$

As well as

$$\begin{aligned}(SX)_3 &= \sum_{v_3v_j\in E(G)} \sqrt{d_3^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_3^2+d_2^2}\sqrt{d_2}+\sqrt{d_3^2+d_4^2}\sqrt{d_4}\\ &\leqslant \sqrt{d_3^2+d_2^2+d_3^2+d_4^2}\sqrt{d_2+d_4}=\sqrt{105}.\end{aligned}$$

Moreover

$$\begin{aligned}(SX)_4 &= \sum_{v_4v_j\in E(G)} \sqrt{d_4^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_4^2+d_1^2}\sqrt{d_1}+\sqrt{d_4^2+d_3^2}\sqrt{d_3}\\ &\leqslant \sqrt{d_4^2+d_1^2+d_4^2+d_3^2}\sqrt{d_1+d_3}\\ &\leqslant \sqrt{n^2-4n+16}\sqrt{n}\\ &\leqslant \sqrt{d_4}\sqrt{n^2-n+4}\sqrt{n+2}.\end{aligned}$$

Also we can see:

$$\begin{aligned}(SX)_5 &= \sum_{v_5v_j\in E(G)} \sqrt{d_5^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_5^2+d_1^2}\sqrt{d_1}+\sqrt{d_5^2+d_2^2}\sqrt{d_2}\\ &\leqslant \sqrt{d_5^2+d_1^2+d_5^2+d_2^2}\sqrt{d_1+d_2}\\ &\leqslant \sqrt{n^2-4n+21}\sqrt{n+1}\\ &\leqslant \sqrt{d_5}\sqrt{n^2-n+4}\sqrt{n+2}.\end{aligned}$$

And

$$\begin{aligned}(SX)_6 &= \sum_{v_6v_j\in E(G)} \sqrt{d_6^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_6^2+d_1^2}\sqrt{d_1} = \sqrt{n^2-4n+5}\sqrt{n-2}\\ &\leqslant \sqrt{n^2-n+4}\sqrt{n+2}.\end{aligned}$$

Similarly, for $i \geqslant 7$ we have:

$$(SX)_i \leqslant \sqrt{n^2-n+4}\sqrt{n+2}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-n+4}\sqrt{n+2}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2-n+4}\sqrt{n+2}. \qquad \square$$

Lemma 3.10. *Let $G = G_8$, then*

$$\mu_1(G) \leqslant \sqrt{n^2+n+5}\sqrt{n+3}.$$

Proof. We know that $d_1 = n-1$, $d_2 = 3$, $d_3 = d_4 = 2$ and $d_5 = \ldots = d_n = 1$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j\in E(G)} \sqrt{d_1^2+d_j^2}\sqrt{d_j}\\ &= \sqrt{d_1^2+d_2^2}\sqrt{d_2}+\sqrt{d_1^2+d_3^2}\sqrt{d_3}+\sqrt{d_1^2+d_4^2}\sqrt{d_4}+\sqrt{d_1^2+1}(n-4)\\ &\leqslant \sqrt{d_1^2+d_2^2+d_1^2+d_3^2+d_1^2+d_4^2+(d_1^2+1)(n-4)}\sqrt{d_2+d_3+d_4+n-4}\\ &\leqslant \sqrt{3(n-1)^2+17+((n-1)^2+1)(n-1)}\sqrt{n+3}\\ &= \sqrt{n-1}\sqrt{(n-1)^2+3(n-1)+1+\frac{17}{n-1}}\sqrt{n+3}\\ &\leqslant \sqrt{d_1}\sqrt{(n-1)^2+3(n-1)+7}\sqrt{n+3}\\ &= \sqrt{d_1}\sqrt{n^2+n+5}\sqrt{n+3}.\end{aligned}$$

Also we have:

$$\begin{aligned}(SX)_2 &= \sum_{v_2v_j\in E(G)} \sqrt{d_2^2+d_j^2}\sqrt{d_j} \\ &= \sqrt{d_2^2+d_1^2}\sqrt{d_1}+\sqrt{d_2^2+d_3^2}\sqrt{d_3}+\sqrt{d_2^2+d_4^2}\sqrt{d_4} \\ &\leqslant \sqrt{d_2^2+d_1^2+d_2^2+d_3^2+d_2^2+d_4^2}\sqrt{d_1+d_3+d_4} \\ &= \sqrt{n^2-2n+36}\sqrt{n+3} \\ &\leqslant \sqrt{d_2}\sqrt{n^2+n+5}\sqrt{n+3}.\end{aligned}$$

As well as

$$\begin{aligned}(SX)_3 &= \sum_{v_3v_j\in E(G)} \sqrt{d_3^2+d_j^2}\sqrt{d_j} \\ &= \sqrt{d_3^2+d_1^2}\sqrt{d_1}+\sqrt{d_3^2+d_2^2}\sqrt{d_2} \\ &\leqslant \sqrt{d_3^2+d_1^2+d_3^2+d_2^2}\sqrt{d_1+d_2} \\ &= \sqrt{n^2-2n+18}\sqrt{n+2} \\ &\leqslant \sqrt{d_3}\sqrt{n^2+n+5}\sqrt{n+3}.\end{aligned}$$

Similarly $(SX)_4 \leqslant \sqrt{d_4}\sqrt{n^2+n+5}\sqrt{n+3}$. And

$$\begin{aligned}(SX)_5 &= \sum_{v_5v_j\in E(G)} \sqrt{d_5^2+d_j^2}\sqrt{d_j} \\ &= \sqrt{d_5^2+d_1^2}\sqrt{d_1} = \sqrt{n^2-2n+2}\sqrt{n-1} \\ &\leqslant \sqrt{n^2+n+5}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 6$ we have:

$$(SX)_i \leqslant \sqrt{n^2+n+5}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2+n+5}\sqrt{n+3}X$ and so

$$\mu_1(G) \leqslant \sqrt{n^2+n+5}\sqrt{n+3}. \qquad \square$$

Lemma 3.11. *Let $G = G_9$, then*

$$\mu_1(G) \leqslant \sqrt{n^2-2n+7}\sqrt{n+3}.$$

Proof. We know that $d_1 = d_3 = 3$, $d_2 = n-2$, $d_4 = 2$ and $d_5 = \ldots = d_n = 1$. Hence,

$$\begin{aligned}(SX)_1 &= \sum_{v_1v_j \in E(G)} \sqrt{d_1^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_1^2 + d_2^2}\sqrt{d_2} + \sqrt{d_1^2 + d_3^2}\sqrt{d_3} + \sqrt{d_1^2 + d_4^2}\sqrt{d_4} \\ &\leqslant \sqrt{d_1^2 + d_2^2 + d_1^2 + d_3^2 + d_1^2 + d_4^2}\sqrt{d_2 + d_3 + d_4} \\ &= \sqrt{n^2 - 4n + 44}\sqrt{n+3} \\ &\leqslant \sqrt{d_1}\sqrt{n^2 - 2n + 7}\sqrt{n+3}.\end{aligned}$$

Similarly we have

$$(SX)_3 \leqslant \sqrt{d_3}\sqrt{n^2 - 2n + 7}\sqrt{n+3}.$$

Also we have

$$\begin{aligned}(SX)_2 &= \sum_{v_2v_j \in E(G)} \sqrt{d_2^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_2^2 + d_1^2}\sqrt{d_1} + \sqrt{d_2^2 + d_3^2}\sqrt{d_3} + \sqrt{d_2^2 + 1}(n-4) \\ &\leqslant \sqrt{d_2^2 + d_1^2 + d_2^2 + d_3^2 + (d_2^2+1)(n-4)}\sqrt{d_1 + d_3 + n - 4} \\ &\leqslant \sqrt{2(n-2)^2 + 18 + ((n-2)^2+1)(n-2)}\sqrt{n+2} \\ &= \sqrt{n-2}\sqrt{(n-2)^2 + 2(n-2) + 1 + \frac{18}{n-2}}\sqrt{n+2} \\ &\leqslant \sqrt{d_2}\sqrt{(n-2)^2 + 2(n-2) + 7}\sqrt{n+2} \\ &= \sqrt{d_2}\sqrt{n^2 - 2n + 7}\sqrt{n+2}.\end{aligned}$$

As well as

$$\begin{aligned}(SX)_4 &= \sum_{v_4v_j \in E(G)} \sqrt{d_4^2 + d_j^2}\sqrt{d_j} \\ &= \sqrt{d_4^2 + d_1^2}\sqrt{d_1} + \sqrt{d_4^2 + d_3^2}\sqrt{d_3} \\ &\leqslant \sqrt{d_4^2 + d_1^2 + d_4^2 + d_3^2}\sqrt{d_1 + d_3} = \sqrt{156}.\end{aligned}$$

And

$$\begin{aligned}(SX)_5 &= \sum_{v_5v_j\in E(G)} \sqrt{d_5^2+d_j^2}\sqrt{d_j} \\ &= \sqrt{d_5^2+d_2^2}\sqrt{d_2} = \sqrt{n^2-4n+5}\sqrt{n-2} \\ &\leqslant \sqrt{n^2-2n+7}\sqrt{n+3}.\end{aligned}$$

Similarly, for $i \geqslant 6$ we have:

$$(SX)_i \leqslant \sqrt{n^2-2n+7}\sqrt{n+3}.$$

Therefore, we have $SX \leqslant \sqrt{n^2-2n+7}\sqrt{n+3}X$ and so $\mu_1(G) \leqslant \sqrt{n^2-2n+7}\sqrt{n+3}$. □

Theorem 3.12. *If G is a bicyclic graph of order $n \geqslant 7$, then*

$$\mu_1(G) \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}.$$

Proof. By Lemmas 3.2 to 3.11, it is clear. □

Corollary 3.13. *Let G be a bicyclic graph of order $n \geqslant 7$, then*

$$4\sqrt{2}\frac{(n+1)^2}{n} \leqslant \mu_1(G) \leqslant \sqrt{n^2+2n+2}\sqrt{n+3}.$$

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