The Stability Conditions of Unperturbed Motion Governed by Differential System with Nonlinearities of Fifth Degree

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Abstract

According to Lyapunov's first method of stability of unperturbed motion, there were determined the stability conditions for the differential system with nonlinearities of fifth degree.

Keywords: Differential systems, stability of unperturbed motion, center-affine transformation, characteristic equation.

1 Introduction

We will examine the two-dimensional differential system with nonlinearities of fifth degree of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5, \\ \frac{dy}{dt} &= cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5, \end{aligned} \tag{1}$$

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p$ are real arbitrary coefficients. Lyapunov proved that, for such systems, it can always be found a center-affine transformation, so that the system (1) contains a critical and a non-critical equation, that is

$$\begin{aligned} \frac{dx}{dt} &= ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5, \\ \frac{dy}{dt} &= cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5. \end{aligned} \tag{2}$$

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We analyze the non-critical equation

$$cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5 = 0. \quad (3)$$

Because the characteristic equation of the system (2) has one zero root and the other ones are real and negative, we have

$$I_1 = a_\alpha^\alpha = d < 0. \quad (4)$$

From the last relation, we express y

$$y = -\frac{c}{d}x - \frac{k}{d}x^5 - 5\frac{l}{d}x^4y - 10\frac{m}{d}x^3y^2 - 10\frac{n}{d}x^2y^3 - 5\frac{o}{d}xy^4 - \frac{p}{d}y^5. \quad (5)$$

We seek y as a holomorphic function of x. Then we can write

$$\begin{gathered} y = -\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + \\ + B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + B_{13}x^{13} + B_{14}x^{14} + \\ + B_{15}x^{15} + B_{16}x^{16} + ... \end{gathered} \quad (6)$$

Substituting (6) into (5), we have

$$\begin{gathered} -\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \\ + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + B_{13}x^{13} + B_{14}x^{14} + B_{15}x^{15} + \\ + B_{16}x^{16} + ... = -\frac{c}{d}x - \frac{k}{d}x^5 - 5\frac{l}{d}x^4(-\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + \\ + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + \\ + B_{13}x^{13} + B_{14}x^{14} + B_{15}x^{15} + B_{16}x^{16} + ...) - 10\frac{m}{d}x^3(-\frac{c}{d}x + B_2x^2 + \\ + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + \end{gathered}$$

$$+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^2-10\frac{n}{d}x^2(-\frac{c}{d}x+$$
$$+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+$$
$$+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^3-$$
$$-5\frac{o}{d}x(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+$$
$$+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^4-\frac{p}{d}(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^5.$$

This implies that

$$B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+$$
$$+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...=$$
$$=-\frac{1}{d^6}(d^5k-5cd^4l+10c^2d^3m-10c^3d^2n+5c^4do-c^5p)x^5-$$
$$-\frac{5B_5}{d^5}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)x^9-\frac{5}{d^5}[2B_5^2d(d^3m-$$
$$-3cd^2n+3c^2do-c^3p)+B_9(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{13}-$$
$$-\frac{5}{d^5}[2B_5^3d^2(d^2n-2cdo+c^2p)+4B_5B_9d(d^3m-3cd^2n+3c^2do-c^3p)+$$
$$+B_{13}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{17}-$$
$$-\frac{5}{d^5}[B_5^4d^3(do-cp)+6B_5^2B_9d^2(d^2n-2cdo+c^2p)+4B_5B_{13}d(d^3m-$$
$$-3cd^2n+3c^2do-c^3p)+2B_9^2d(d^3m-3cd^2n+3c^2do-c^3p)+B_{17}(d^4l-$$
$$-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{21}-$$
$$-\frac{5}{d^5}[B_5^5d^4p+20B_5^3B_9d^3(do-cp)+30d^2B_5(B_5B_{13}+B_9^2)(d^2n-$$
$$-2cdo+c^2p)+$$
$$+20d(B_5B_{17}+B_9B_{13})(d^3m-3cd^2n+3c^2do-c^3p)+$$
$$+5B_{21}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{25}+...$$

From this identity, we have

$$B_2 = B_3 = B_4 = 0,$$

$$B_5 = -\frac{1}{d^6}(d^5k - 5cd^4l + 10c^2d^3m - 10c^3d^2n + 5c^4do - c^5p),$$

$$B_6 = B_7 = B_8 = 0,$$

$$B_9 = -\frac{5B_5}{d^5}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p),$$

$$B_{10} = B_{11} = B_{12} = 0,$$

$$B_{13} = -\frac{5}{d^5}[2B_5^2d(d^3m - 3cd^2n + 3c^2do - c^3p) + B_9(d^4l - 4cd^3m +$$
$$+6c^2d^2n - 4c^3do + c^4p)], \quad B_{14} = B_{15} = B_{16} = 0,$$

$$B_{17} = -\frac{5}{d^5}[2B_5^3d^2(d^2n - 2cdo + c^2p) + 4B_5B_9d(d^3m - 3cd^2n +$$
$$+3c^2do - c^3p) + B_{13}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)],$$

$$B_{18} = B_{19} = B_{20} = 0,$$

$$B_{21} = -\frac{5}{d^5}[B_5^4d^3(do - cp) + 6B_5^2B_9d^2(d^2n - 2cdo + c^2p) +$$
$$+4B_5B_{13}d(d^3m - 3cd^2n + 3c^2do - c^3p) + 2B_9^2d(d^3m -$$
$$-3cd^2n + 3c^2do - c^3p) + B_{17}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)],$$

$$B_{22} = B_{23} = B_{24} = 0,$$

$$B_{25} = -\frac{5}{d^5}[B_5^5d^4p + 20B_5^3B_9d^3(do - cp) + 30d^2B_5(B_5B_{13} + B_9^2)(d^2n -$$
$$-2cdo + c^2p) + 20d(B_5B_{17} + B_9B_{13})(d^3m - 3cd^2n + 3c^2do - c^3p) +$$
$$+5B_{21}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)], ... \tag{7}$$

Substituting (6) into the right-hand sides of the critical differential equations (2), we get the following identity

$$ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5 = A_2x^2 + A_3x^3 +$$

$$+A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + A_{11}x^{11} + A_{12}x^{12} +$$

$$+A_{13}x^{13}+A_{14}x^{14}+A_{15}x^{15}+A_{16}x^{16}+...,$$

or in detailed form

$$ex^5+5fx^4(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+$$
$$+B_{15}x^{15}+B_{16}x^{16}+...)+10gx^3(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+$$
$$+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+$$
$$+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^2+$$
$$+10hx^2(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+$$
$$+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^3+5ix(-\frac{c}{d}x+$$
$$+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+$$
$$+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^4+j(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+$$
$$+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+$$
$$+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^5=A_2x^2+A_3x^3+$$
$$+A_4x^4+A_5x^5+A_6x^6+A_7x^7+A_8x^8+A_9x^9+A_{10}x^{10}+$$
$$+A_{11}x^{11}+A_{12}x^{12}+A_{13}x^{13}+A_{14}x^{14}+A_{15}x^{15}+A_{16}x^{16}+...$$

From here we obtain

$$A_2=A_3=A_4=0,$$
$$A_5=\frac{1}{d^5}(d^5e-5cd^4f+10c^2d^3g-10c^3d^2h+5c^4di-c^5j),$$
$$A_6=A_7=A_8=0,$$
$$A_9=\frac{5}{d^4}B_5(d^4f-4cd^3g+6c^2d^2h-4c^3di+c^4j),$$
$$A_{10}=A_{11}=A_{12}=0,$$
$$A_{13}=\frac{5}{d^4}[2B_5^2d(d^3g-3cd^2h+3c^2di-c^3j)+B_9(d^4f-4cd^3g+6c^2d^2h-$$

$$-4c^3di + c^4j)], \quad A_{14} = A_{15} = A_{16} = 0,$$
$$A_{17} = \frac{5}{d^4}[(2B_5^3d^2(d^2h - 2cdi + c^2j) + 4B_5B_9d(d^3g - 3cd^2h +$$
$$+3c^2di - c^3j) + B_{13}(d^4f - 4cd^3g + 6c^2d^2h - 4c^3di + c^4j)],$$
$$A_{18} = A_{19} = A_{20} = 0,$$
$$A_{21} = \frac{5}{d^4}[B_5^4d^3(di - cj) + 6B_5^2B_9d^2(d^2h - 2cdi + c^2j) +$$
$$+2d(2B_5B_{13} + B_9^2)(d^3g - 3cd^2h + 3c^2di - c^3j) + B_{17}(d^4f - \quad (8)$$
$$-4cd^3g + 6c^2d^2h - 4c^3di + c^4j)],$$
$$A_{22} = A_{23} = A_{24} = 0,$$
$$A_{25} = \frac{1}{d^4}[B_5^5d^4j + 20B_5^3B_9d^3(di - cj) + 30d^2B_5(B_5B_{13} +$$
$$+B_9^2)(d^2h - 2cdi + c^2j) + 20d(B_5B_{17} + B_9B_{13})(d^3g -$$
$$-3cd^2h + 3c^2di - c^3j) + 5B_{21}(d^4f - 4cd^3g + 6c^2d^2h -$$
$$-4c^3di + c^4j)], ...$$

Let us introduce the following notations:

$$T = d^5k - 5cd^4l + 10c^2d^3m - 10c^3d^2n + 5c^4do - c^5p;$$
$$C = d^5e - 5cd^4f + 10c^2d^3g - 10c^3d^2h + 5c^4di - c^5j;$$
$$D = d^4f - 4cd^3g + 6c^2d^2h - 4c^3di + c^4j; \quad (9)$$
$$E = d^3g - 3cd^2h + 3c^2di - c^3j;$$
$$F = d^2h - 2cdi + c^2j; \quad G = di - cj.$$

Then, taking into account (7), (8), and (9), we obtain

$$A_5 = \frac{C}{d^5}; \quad B_5 = -\frac{T}{d^6}; \quad A_9 = \frac{5}{d^4}B_5D,$$
$$A_{13} = \frac{5}{d^4}[2B_5^2dE + B_9D],$$
$$A_{17} = \frac{5}{d^4}[(2B_5^3d^2F + 4B_5B_9dE + B_{13}D];$$

$$A_{21} = \frac{5}{d^4}[B_5^4 d^3 G + 6B_5^2 B_9 d^2 F + 2d(2B_5 B_{13} + B_9^2)E + B_{17} D];$$
$$A_{25} = \frac{1}{d^4}[B_5^5 d^4 j + 20B_5^3 B_9 d^3 G + 30d^2 B_5 (B_5 B_{13} + B_9^2)F + \quad (10)$$
$$+20d(B_5 B_{17} + B_9 B_{13})E + 5B_{21} D], ...$$

Using the Lyapunov theorem [1], and the expressions (10), consi--dering that $I_1 = d < 0$, we get

Theorem 1. *The stability of unperturbed motion in the system of perturbed motion* (2) *is described by fourteen possible cases:*

I. $C > 0$, *then the unperturbed motion is stable*;

II. $C < 0$, *then the unperturbed motion is unstable*;

III. $C = 0$, $TD > 0$, *then the unperturbed motion is stable*;

IV. $C = 0$, $TD < 0$, *then the unperturbed motion is unstable*;

V. $C = D = 0$, $T \neq 0$, $E < 0$, *then the unperturbed motion is unstable*;

VI. $C = D = 0$, $T \neq 0$, $E > 0$, *then the unperturbed motion is stable*;

VII. $C = D = E = 0$, $TF > 0$, *then the unperturbed motion is stable*;

VIII. $C = D = E = 0$, $TF < 0$, *then the unperturbed motion is unstable*;

IX. $C = D = E = F = 0$, $T \neq 0$, $G > 0$, *then the unperturbed motion is stable*;

X. $C = D = E = F = 0$, $T \neq 0$, $G < 0$, *then the unperturbed motion is unstable*;

XI. $C = D = E = F = G = 0$, $Tj > 0$, *then the unperturbed motion is stable*;

XII. $C = D = E = F = G = 0$, $Tj < 0$, *then the unperturbed motion is unstable*;

XIII. $C = T = 0$, *then the unperturbed motion is stable*;

XIV. $e = f = g = h = i = j = 0$, *then the unperturbed motion is stable.*

In the last two cases, according to the Lyapunov theorem, the unperturbed motion belongs to some continuous series of stabilized motions. In this case, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotically stable [2] *in Cases* I, III, VI, VII, IX, *and* XI. *The expressions* C, D, E, F, G, T *are from* (9).

Proof. According to Lyapunov's theorem [1], we analyze the coeffcients of the series (10).

Asume that $A_5 < 0$, then from (10) we get $\frac{C}{d^5} < 0$. Taking into account that $d < 0$, it follows $C > 0$. So we get case I, and similarly we get case II.

Suppose in (9) that $T \neq 0$. Then, from (10), we have $B_5 \neq 0$.

If $A_5 = 0$ ($C = 0$), and $A_9 = \frac{5}{d^4} B_5 D = -\frac{5}{d^{10}} TD$, then the stability or the instability of unperturbed motion is determined by the sign of expression TD. Then, using Lyapunov's theorem, we proved the Cases III and IV.

If $A_5 = A_9 = 0$ ($C = D = 0$), then $A_{13} = \frac{10}{d^{15}} T^2 E$. Taking into account that $d < 0$, the stability or the instability of unperturbed motion is determined by the sign of expression E. Then, using Lyapunov's theorem, we proved the Cases V and VI.

If $A_5 = A_9 = A_{13} = 0$ ($C = D = E = 0$), then $A_{17} = -\frac{10}{d^{20}} T^3 F$, and the stability or the instability of unperturbed motion is determined by the sign of expression TF. Then, using Lyapunov's theorem, we proved the Cases VII and VIII.

If $A_5 = A_9 = A_{13} = A_{17} = 0$ ($C = D = E = F = 0$), then $A_{21} = \frac{5}{d^{25}} T^4 G$. Taking into account that $d < 0$, the stability or the instability of unperturbed motion is determined by the sign of expression G. Then, using Lyapunov's theorem, we proved the Cases IX and X.

If $A_5 = A_9 = A_{13} = A_{17} = A_{21} = 0$ ($C = D = E = F = G = 0$), then $A_{25} = -\frac{1}{d^{30}} T^5 j$, and the stability or the instability of unperturbed motion is determined by the sign of expression Tj. Then, using Lyapunov's theorem, we proved the Cases XI and XII.

If $C = T = 0$, then all A_k ($k \geq 5$) are equal to zero. Then, using

Lyapunov's theorem, we proved the Case XIII.

If $T \neq 0$ a nd $C = D = E = F = G = j = 0$ ($e = f = g = h = i = j = 0$), then we obtain the Case XIV. Theorem 1 is proved.

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