Proceedings of the 4th Conference of Mathematical Society of Moldova
CMSM4'2017, June 28-July 2, 2017, Chisinau, Republic of Moldova

The Lyapunov quantities and the $GL(2,\mathbb{R})$-invariant center conditions for a class of bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree

Stanislav Ciubotaru, Iurie Calin

Abstract

For the bidimensional polynomial systems of differential equations with nonlinearities of the fourth degree the recurent equations for determination of the Lyapunov quantities were established. Moreover, the general form of Lyapunov quantities for the mentioned systems were obtained. For a class of such systems the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the existence of center are given.

Keywords: Polynomial differential systems, invariant, comitant, transvectant, Lyapunov quantities, center conditions.

1 Definitions and notations

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \tag{1}$$

where $P_i(x,y),\ Q_i(x,y)$ are homogeneous polynomials of degree i in x and y with real coefficients.

©2017 by Stanislav Ciubotaru, Iurie Calin

The goal of this paper is to determine the invariant [1, 2] recurent formulas for construction the Lyapunov quantities for the system (1) and to establish the invariant center conditions for a class of these systems.

Definition. [3] *Let φ and ψ be homogeneous polynomials in coordinates of the vector $(x,\ y) \in \mathbb{R}^2$ of the degrees ρ_1 and ρ_2, respectively. The polynomial*

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}}$$

is called the transvectant of index j of polynomials φ and ψ.

If the polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants [1, 2] of the system (1), then the transvectant of the index $j \leq \min(\rho_1, \rho_2)$ is also a $GL(2,\mathbb{R})$-comitant of the system (1) [4].

$GL(2,\mathbb{R})$-comitants of the first degree with respect to coefficients of system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = 1, 4.$$

By using the comitants R_i and S_i $(i = 1, 4)$, and the notion of transvectant the following $GL(2,\mathbb{R})$-comitants and invariants of the system (1) were constructed:

$$I_2 = (R_1, R_1)^{(2)}, \quad K_1 = (S_4, R_1)^{(1)}, \quad K_2 = ((S_4, R_1)^{(2)}, R_1)^{(1)},$$
$$I_3 = (((S_4, R_1)^{(2)}, R_1)^{(1)}, (S_4, R_1)^{(2)})^{(1)},$$
$$I_4 = ((((R_4, R_1)^{(2)}, R_1)^{(2)}, R_1)^{(1)}, ((R_4, R_1)^{(2)}, R_1)^{(2)})^{(1)},$$
$$K_3 = (R_4, S_4)^{(3)}, \quad K_4 = (K_3^2, S_4)^{(3)}, \quad K_5 = ((K_3, S_4)^{(2)}, R_1)^{(2)},$$
$$I_5 = (((R_4, S_4)^{(2)}, R_1)^{(2)}, R_1)^{(2)}, \quad I_6 = (K_4, K_5)^{(1)}$$

2 Lyapunov quantities for systems (1) with $S_1 = 0, I_2 \neq 0$

We will consider the system (1) with the conditions $S_1 = 0, I_2 > 0$,

which has the center or focus at $(0,0)$. In these conditions system (1) can be reduced to the system

$$\frac{dx}{dt} = y + P_4(x,y), \qquad \frac{dy}{dt} = -x + Q_4(x,y), \tag{2}$$

and can be written [5] in the form

$$\frac{dx}{dt} = \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{5}\frac{\partial R_4}{\partial y} + \frac{4}{5}S_4 x, \ \frac{dy}{dt} = -\frac{1}{2}\frac{\partial R_1}{\partial x} - \frac{1}{5}\frac{\partial R_4}{\partial x} + \frac{4}{5}S_4 y, \tag{3}$$

where $R_1 = x^2 + y^2$.

Let us consider the formal power series of the form

$$F(x,y) = x^2 + y^2 + \sum_{j=3}^{\infty} F_j(x,y)$$

where for each j, $F_j(x,y)$ is a homogeneous polynomial of degree j, so that the derivative of $F(x,y)$ along the solutions of the system (2) (or (3)) satisfies

$$\frac{dF(x,y)}{dt} = \sum_{k=2}^{\infty} G_{2k}(x^2+y^2)^k,$$

where G_{2k} are the polynomials of the coefficients of the system (2), called *Lyapunov quantities* [6].

For establishing the center conditions for the system (2) we will determine Lyapunov quantities. Polynomials $F_j(x,y)$ and constants G_{2k} can be determined from the infinite dimensional system of differential equations in partial derivatives:

$$(3m+2)(F_{3m+2}, R_1)^{(1)} + (3m-1)W(F_{3m-1}) = \\ = \begin{cases} 0, & \text{for } m = 2l-1, \ l \in \mathbb{N}^*, \\ \\ G_{3m+2} R_1^{\frac{3m+2}{2}}, & \text{for } m = 2l, \ l \in \mathbb{N}^*, \end{cases} \tag{4}$$

where $F_2 = R_1$, $W(F_j) = (F_j, R_4)^{(1)} + \dfrac{4}{5}F_j S_4$.

From the system (4) it follows that only the homogeneous polynomials $F_{3m-1}(x,y)$, $m \in \mathbb{N}^*$ and the Lyapunov quantities G_{6l+2}, $l \in \mathbb{N}^*$ participate in solving the center-focus problem for the system (1). By solving consecutively the equations of system (4) the polynomials F_{3m-1} and respectively the Lyapunov quantities G_{6l+2}, are determined. The general form of the polynomials F_{3m-1}, $m \in \mathbb{N}^*$ and respectively, the general form of the Lyapunov quantities G_{6l+2}, $l \in \mathbb{N}^*$, are the following:

$$F_{3m+2} =$$

$$= \sum_{j=0}^{\left[\frac{3m+1}{2}\right]} \frac{(3m-1)\cdot(3m+2)!\cdot 2^{j+1}\cdot R_1^j \cdot [\![W(F_{3m-1}), \overbrace{R_1)^{(2)}, \ldots, R_1)^{(2)}}^{j}, R_1)^{(1)}}{(3m-2j+1)!\cdot \prod\limits_{i=0}^{j}\left((3m-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)},$$

$$G_{6l+2} = \frac{(6l-1)\cdot(6l+2)!\cdot 2^{3l+1}\cdot [\![W(F_{6l-1}), \overbrace{R_1)^{(2)}, \ldots, R_1)^{(2)}}^{3l+1}}{\prod\limits_{i=0}^{3l}\left((6l-2i+2)^2\cdot(R_1,R_1)^{(2)}\right)}, \qquad (5)$$

where $m \in \mathbb{N}^*$, $l \in \mathbb{N}^*$, $W(F_i) = (F_i, R_4)^{(1)} + \frac{4}{5}F_i S_4$.

Noted that when $m = 2l-1,\ l \in \mathbb{N}^*$, the respectively equations of the system (4) have a unique solution with respect to F_{3m+2}, i.e. in this case F_{3m+2} are determined unambiguously. In the case $m = 2l,\ l \in \mathbb{N}^*$, the solutions of respectively equations of the system (4) with respect to F_{3m+2} are determined with accuracy to a term of the form $CR_1^{\frac{3m+2}{2}}$, where C is an arbitrary real constant. This implies that Lyapunov quantities $G_{6l+2},\ l \in \mathbb{N}^*$, are not determined unambiguously.

3 The $GL(2,\mathbb{R})$-invariant center conditions for a class of systems (1) with $S_1 = 0$, $I_2 > 0$, $I_3 = I_4 = 0$

We will consider the system (3) (or (1)) with the conditions $S_1 = 0$,

$I_2 > 0$, which has the center or the focus at $(0, 0)$.

If $R_4 \equiv 0$, then the system (3) (or (1)) with $S_1 = 0$ and $I_2 > 0$ has the singular point of the center type at the origin of coordinates. In this case the system (3) has the invariant algebraic curve

$$H(x, y) = 32R_1 \cdot K_2 + 8I_2 \cdot K_1 - 5I_2^2 = 0$$

and the first integral

$$|H|^{\frac{2}{3}} \cdot |R_1|^{-1} = c_1,$$

where c_1 is a real constant [7].

If $S_4 \equiv 0$, then the system (3) (or (1)) with $S_1 = 0$ and $I_2 > 0$ has the singular point of the center type at the origin of coordinates. In this case the system (3) has the first integral:

$$5R_1 + 2R_4 = c_2,$$

where c_2 is a real constant.

For the system (1) with $S_1 = 0$, $I_2 > 0$ and $I_3 = I_4 = 0$ were established the $GL(2, \mathbb{R})$-invariant conditions for distinguishing between center and focus.

Theorem 1. *The system* (1) *with the conditions* $S_1 = 0$, $I_2 > 0$ *and* $I_3 = I_4 = 0$ *has the center in the origin of the coordinates if and only if the following conditions are fulfilled*

$$G_8 = G_{26} = G_{32} = G_{38} = 0,$$

where G_8, G_{26}, G_{32} *and* G_{38} *are Lyapunov quantities of respectively indices.*

Theorem 2. *The system* (1) *with the conditions* $S_1 = 0$, $I_2 > 0$ *and* $I_3 = I_4 = 0$ *has the center in the origin of the coordinates if and only if the following conditions are fulfilled*

$$I_5 = I_6 = 0.$$

Acknowledgments. This article was partially supported by the project 15.817.02.03F from SCSTD of ASM.

References

[1] K. S. Sibirsky. *Introduction to the Algebraic Theory of Invariants of Differential Equations*. Manchester University Press, 1988.

[2] N. I. Vulpe., *Polynomial bases of comitants of differential systems and their applications in qualitative theory*, Ştiinţa, Kishinev, 1986 (in Russian).

[3] G. B. Gurevich. *Foundations of the Theory of Algebraic Invariants*. Noordhoff, Groningen, 1964.

[4] D. Boularas, Iu. Calin, L. Timochouk, N. Vulpe. *T-comitants of quadratic systems: A study via the translation invariants*, Report 96-90, Delft University of Technology, Faculty of Technical Mathematics and Informatics, 1996, pp. 1–36.

[5] Iu. Calin. *On rational bases of $GL(2,\mathbb{R})$-comitants of planar polynomial systems of differential equations*. Buletinul Academiei de Stiinte a Republicii Moldova. Matematica, 2003, nr. 2(42), pp. 69–86.

[6] V. V. Amelkin, N. A. Lucashevich, A. P. Sadovski. *Nonlinear variation in systems of the second order*. Minsk, 1982 (in Russian).

[7] V. Baltag, Iu. Calin. *The transvectants and the integrals for Darboux systems of differential equations*. Buletinul Academiei de Stiinte a Republicii Moldova. Matematica, 2008, nr. 1(56), pp. 4–18.

Stanislav Ciubotaru[1], Iurie Calin[1,2]

[1]Institute of Mathematics and Computer Science of ASM
[2]Moldova State University, Chişinău
Email: stanislav.ciubotaru@yahoo.com, iucalin@yahoo.com